Runge-Kutta Time Discretization of Fully Nonlinear Parabolic Problems

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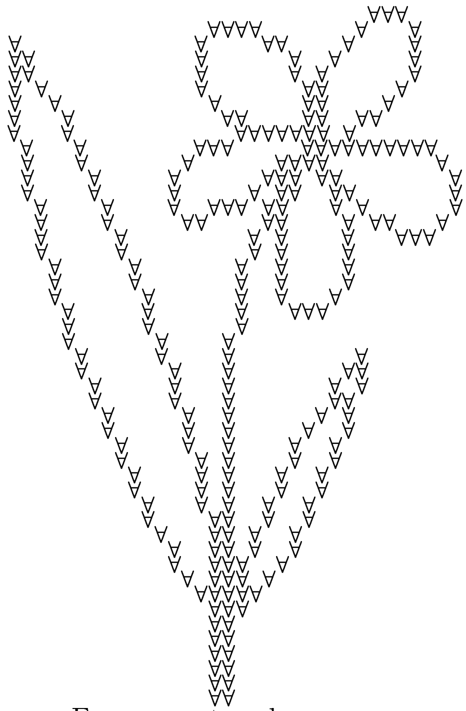

For support and company

Many thanks to

My parents

My brother

Alexander and Barbara

My friends and colleagues

The members of the institute

Gerhard and Günther

Cesar and Cesáreo

Abstract

In the present work we study Runge-Kutta time discretizations of a fully nonlinear parabolic evolution equation on a Banach space

$$u'(t) = F(t, u(t)), \quad t > 0, \qquad u(0) = u_0.$$

The analysis is based on the framework of sectorial operators and analytic semigroups and extends ideas from the semilinear case to the fully nonlinear one.

Our interest in such an abstract initial value problem is caused by the fact that a wide range of fully nonlinear parabolic initial-boundary value problems which arise as mathematical models for processes involving nonlinear diffusion or heat conduction can be cast in this form.

The purpose of this work is a twofold. In the first place, it provides existence and convergence results for Runge-Kutta methods under certain smoothness assumptions on the analytical solution of the evolution equation and stability requirements on the method. We prove that the order of convergence is essentially determined by the stage order of the method. Secondly, the long-term behaviour of Runge-Kutta time discretizations for fully nonlinear equations is studied. We show that the geometric properties in the neighbourhood of an equilibrium point are well captured by Runge-Kutta discretizations. For illustrating the theoretical results, several numerical examples are presented.

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Introduction

In the present work we study Runge-Kutta time discretizations of a fully nonlinear parabolic evolution equation on a Banach space

$$u'(t) = F(t, u(t)), \quad t > 0, \qquad u(0) = u_0. \tag{1}$$

The analysis which relies on the framework of sectorial operators and analytic semigroups is motivated by the theory of semilinear parabolic equations. Our interest in such an abstract initial value problem is caused by the fact that a wide range of fully nonlinear parabolic initial-boundary value problems which arise as mathematical models for processes involving nonlinear diffusion or heat conduction can be cast in the form (1). As example, we mention the following one-dimensional problem for a real-valued function $U(x,t)$ depending on the position $x \in [0,1]$ and the time $t \geq 0$ that describes the shock displacement in a detonation phenomenon

$$\begin{aligned}
&\partial_t U(x,t) = \log\left(\frac{\exp\big(a\, U(x,t)\, \partial_{xx} U(x,t)\big) - 1}{a\, \partial_{xx} U(x,t)}\right) - \frac{1}{2}\big(\partial_x U(x,t)\big)^2,\\
&\partial_x U(0,t) = \partial_x U(1,t) = 0, \qquad U(x,0) = U_0(x), \qquad 0 < x < 1, \quad t > 0.
\end{aligned}$$

The purpose of this work is a twofold. In the first place, it provides existence and convergence results for Runge-Kutta methods under certain smoothness assumptions on the analytical solution of (1) and stability requirements on the method. We prove that the order of convergence is essentially determined by the stage order of the method. Secondly, the long-term behaviour of Runge-Kutta time discretizations for fully nonlinear equations is studied. We show that the geometric properties in the neighbourhood of an equilibrium point are well captured by Runge-Kutta discretizations. For illustrating the theoretical results, several numerical examples are presented.

In the past years, the interest in nonlinear evolution equations of parabolic type has increased a lot. According to that, the knowledge on the existence and regularity theory and on the numerics of nonlinear parabolic problems has grown considerably. Therefore, with regard to the abstract Banach space setting of analytic semigroups, important contributions for the analysis of the continuous problem are the monographs Henry [15], Pazy [29] and Lunardi [25]. In [15, 29] semilinear problems are studied, and in [25] a comparatively simple framework for fully nonlinear problems is developed that extends concepts for semilinear equations to fully nonlinear ones. Alike, there are various works studying the convergence and qualitative behaviour of time discretizations for nonlinear parabolic equations. For Runge-Kutta methods applied to semilinear or quasilinear problems we just mention LUBICH & OSTERMANN [21], LUBICH & OSTERMANN [22] and NAKAGUCHI & YAGI [26]. Further, we refer to the works LUBICH & OSTERMANN [23], AKRIVIS ET AL. [2] and LE ROUX [19]. There, linearly-implicit and multistep discretizations of quasilinear problems are analyzed. However, for the fully nonlinear case not much is known so far. In this context, we are aware of the following two references only: GONZÁLEZ, OSTERMANN, PALENCIA & THALHAMMER [11] and the recent paper OSTERMANN & THALHAMMER [27].

The present work is in strong connection with [11, 27] and likewise based on the analytical framework given in LUNARDI [25]. In [11] variable stepsize backward Euler discretizations for fully nonlinear parabolic problems of the form

$$u'(t) = F(u(t)), \quad t > 0, \qquad u(0) = u_0,$$

are studied. The present work generalizes the convergence results for smooth solutions and the results on the dynamical behaviour of backward Euler discretizations near asymptotically stable equilibrium points given there to strongly stable Runge-Kutta methods. Furthermore, we construct the local unstable and stable manifold of the Runge-Kutta discretization for a stationary solution of saddle point type. The fundamental concepts for deriving these results are as follows. By linearizing (1) around the initial value u_0, we obtain a formally semilinear equation

$$u'(t) = Au(t) + G(t, u(t)), \quad t > 0, \qquad A = D_v F(0, u_0), \tag{2}$$

involving the sectorial operator A and the nonlinear map $G(t, u(t)) = F(t, u(t)) - Au(t)$. We emphasize that Runge-Kutta methods are invariant under this linearization. Especially for the backward Euler solution u_n of (1), we thus get

$$\frac{u_{n+1} - u_n}{h} = Au_{n+1} + G(t_{n+1}, u_{n+1}).$$

By solving this recursion, we receive the representation

$$\begin{aligned} u_n = (I - hA)^{-n} u_0 &+ h \sum_{\nu=0}^{n-1} (I - hA)^{-(n-\nu)} \big(g(t_{\nu+1}) - g(t_n) \big) \\ &+ h \sum_{\nu=0}^{n-1} (I - hA)^{-(n-\nu)} g(t_n). \end{aligned}$$

Within a space of (α, α)-Hölder continuous sequences, this modification of the discrete variation-of-constants formula has important applications. It is used in order to construct the numerical solution by a fixed point iteration. The needed stability results for strongly stable Runge-Kutta methods applied to (2) are found in various works as for example [3, 4, 9, 11, 18, 20, 21, 24, 26, 28].

The recent paper [27] uses a different approach for deriving alternative convergence results for Runge-Kutta time discretizations of (1). In order to keep the paper free from technicalities, in [27], the proofs were performed only for methods satisfying $R(\infty) = 0$. For the general case the omitted details are completed in the present work. This second approach avoids the consideration of the complicated (α, α)-Hölder norms encountered in the first one and is based on linearizing the right-hand side of (1) along the exact solution

$$u'(t) = A(t)u(t) + G(t, u(t)), \quad t > 0, \qquad A(t) = D_v F(t, u(t)), \tag{3}$$

with $G(t, u(t)) = F(t, u(t)) - A(t)u(t)$. The Runge-Kutta solution is constructed by a fixed point iteration relying on a discrete variation-of-constants formula. Stability estimates for (3) are provided by González & Palencia [12].

In Part I we state the basic assumptions on the nonlinearity F in (1) and the required stability properties on the Runge-Kutta method, and we define the fundamental spaces of (α, α)-Hölder continuous sequences. Further, we give several examples for nonlinear initial-boundary value problems that fit into our analytical framework, and, in view of the theoretical results on convergence and dynamical behaviour, numerical experiments are presented. In Part II the afore mentioned results on existence and convergence are derived in detail. As expected, for strongly stable Runge-Kutta method of order p and stage order q the order of convergence on finite time intervals is at least $r = \min\{p, q+1\}$ whenever the analytical solution is sufficiently smooth, i.e. we have

$$\|u_n - u(t_n)\|_{\mathcal{D}} \le Ch^r, \quad 0 \le t_n \le T, \qquad r = \min\{p, q+1\}.$$

In Part III we investigate the question whether the qualitative behaviour of fully nonlinear evolution equations near stationary solutions of saddle point type is captured correctly in the time discretization. At first, we consider the special case of an asymptotically stable equilibrium point where the Runge-Kutta solution decays exponentially fast towards the equilibrium point. For the general situation of a saddle point we then construct the discrete local manifolds. Supplementary definitions and remarks on the analytical framework and the numerical method are given in Part IV. There, we furthermore collect the auxiliary results needed in Parts II and III. This is done in order to keep the main parts free from technicalities.

Part I.

Abstract Banach space setting and examples

1. Analytical framework and discretization

We commence in Section 1.1 by summarizing the basic assumptions employed throughout on the abstract initial value problem and on the numerical method. Moreover, we introduce in Section 1.2 the fundamental function spaces and the corresponding spaces of sequences. The assumptions on the evolution equation as well as the definitions of the function spaces base on LUNARDI [25], and the presentation of Runge-Kutta methods follows HAIRER & WANNER [14].

1.1. Basic assumptions on evolution equation and numerical method

In Section 1.1.1 we state the precise requirements on the continuous problem, and in Section 1.1.2 we then give the stability assumptions on the associated Runge-Kutta scheme. For the definition of sectorial operators we refer to Chapter A, and a short overview on strongly stable Runge-Kutta methods is found in Chapter C.

1.1.1. Evolution equation

Our field of interest is the study of the nonlinear initial value problem

$$u'(t) = F(t, u(t)), \quad 0 < t \leq T, \qquad u(0) = u_0, \tag{1.1}$$

given on a Banach space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$. The initial value u_0 is chosen in an open subset $\mathcal{O} \subset \mathcal{D}$ of the densely embedded Banach space $(\mathcal{D}, \|\cdot\|_{\mathcal{D}}) \subset \mathcal{X}$. The requirements on the map F arising on the right-hand side of (1.1) are the following.

Assumption 1.1. *The map $F : [0,T] \times \mathcal{O} \to \mathcal{X} : (t,v) \mapsto F(t,v)$ has the following properties.*

(i) F is continuous with respect to the variable t on $[0,T]$ and Fréchet differentiable with respect to v on $[0,T] \times \mathcal{O}$.

(ii) There exists some $\alpha \in (0,1)$ such that for each $v^* \in \mathcal{O}$ there are constants $R, K > 0$ and $L > 0$ with the property

$$\begin{aligned} \|F(t,v) - F(s,v)\|_{\mathcal{X}} + \|D_vF(t,v) - D_vF(s,v)\|_{\mathcal{X}\leftarrow\mathcal{D}} &\leq K|t-s|^{\alpha}, \\ \|D_vF(t,v) - D_vF(t,w)\|_{\mathcal{X}\leftarrow\mathcal{D}} &\leq L\|v-w\|_{\mathcal{D}}, \end{aligned} \tag{1.2}$$

for all $s, t \in [0,T]$ and $v, w \in \mathcal{O}$ with $\|v - v^*\|_{\mathcal{D}} \le R$ and $\|w - v^*\|_{\mathcal{D}} \le R$.

(iii) For every $t^* \in [0,T]$ and $v^* \in \mathcal{O}$ the derivative $D_v F(t^*, v^*)$ is a sectorial operator in $\mathcal{X}$, and its graph norm is equivalent to the norm in $\mathcal{D}$, i.e. the relation

$$C^{-1}\|w\|_{\mathcal{D}} \le \|w\|_{\mathcal{X}} + \|D_v F(t^*, v^*)w\|_{\mathcal{X}} \le C\|w\|_{\mathcal{D}}$$

holds for all elements $w \in \mathcal{D}$ with constant $C > 0$ depending on (t^*, v^*).

Our assumptions on (1.1) are mainly that of Lunardi [25, Chapter 8], but we further suppose that $\mathcal{D}$ is dense in $\mathcal{X}$. Thus, the requirement

$$F(0, u_0) \in \overline{\mathcal{D}}$$

arising there, for example in the local existence and uniqueness result Theorem 8.1.1, is always satisfied.

Convention. In the situation of Assumption 1.1, for some $(t^*, v^*) \in [0,T] \times \mathcal{O}$, we set

$$A = D_v F(t^*, v^*).$$

We can choose a real number $\delta \in \boldsymbol{R}$ such that the spectrum of the sectorial operator A_δ

$$A_\delta = \delta I + A$$

is contained in the negative complex plane

$$\Re(\lambda) < 0 \qquad \text{for all} \quad \lambda \in \sigma(A_\delta).$$

Hence, A_δ is invertible and the graph norm of A_δ is equivalent to the norm $\|A_\delta \cdot\|_{\mathcal{X}}$. Moreover, for a constant $C > 0$, it follows

$$C^{-1}\|\cdot\|_{\mathcal{D}} \le \|A_\delta \cdot\|_{\mathcal{X}} \le C\|\cdot\|_{\mathcal{D}}.$$

From now on, for notational simplicity, we often put

$$\delta = 0,$$

since then the following relation

$$\|x\|_{\mathcal{D}(A)} = \|Ax\|_{\mathcal{X}} \qquad \text{for } x \in \mathcal{D}(A),$$

defines a norm on $\mathcal{D}(A)$.

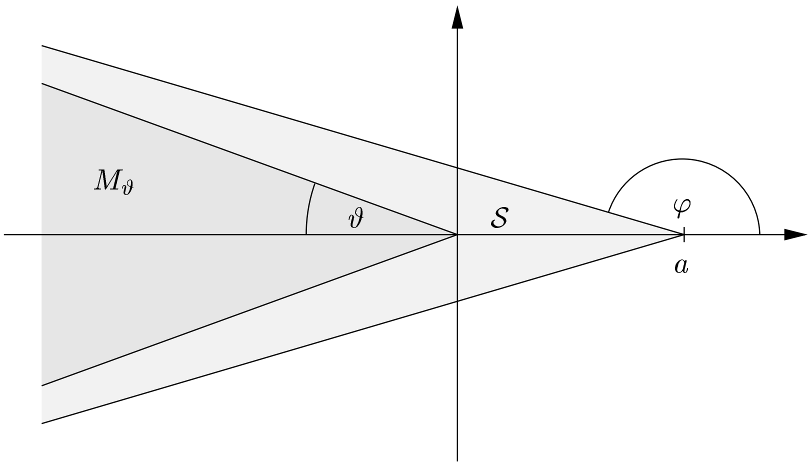

Figure 1.1.: Stability region M_ϑ of the Runge-Kutta method

1.1.2. Numerical method

In the sequel, we consider a s-stage Runge-Kutta method with real coefficients

$$\mathcal{O}\!\iota = (a_{ij})_{1\le i,j\le s}, \qquad b = (b_1, \dots, b_s)^T \qquad \text{and} \qquad c = (c_1, \dots, c_s)^T \in [0,1]^s$$

applied to (1.1). For a constant stepsize $h > 0$ this yields a sequence

$$\underline{u} = (u_n)_{0\le nh\le T}.$$

The single components u_n of $\underline{u}$ are approximations to the value of the exact solution u of (1.1) at $t_n = nh$ and given by the scheme

$$\begin{aligned} U_n' &= F(t_n + ch, U_n), \qquad U_n = \mathbb{1}u_n + h\mathcal{O}\!\iota U_n', \\ u_{n+1} &= u_n + hb^T U_n', \qquad 0 < (n+1)h \le T, \end{aligned} \tag{1.3}$$

for some initial value u_0. Here, we employ the vector notation

$$U_n = (U_{n1}, \dots, U_{ns})^T \qquad \text{and} \qquad U_n' = (U_{n1}', \dots, U_{ns}')^T$$

for the stages of the method. Further, we set $\mathbb{1} = (1, \dots, 1)^T \in \boldsymbol{R}^s$ and

$$F(t_n + ch, U_n) = \big(F(t_n + c_1h, U_{n1}), \dots, F(t_n + c_sh, U_{ns})\big)^T.$$

In order to simplify the notation, we do not use Kronecker product notation.

For an element $(t^*, v^*) \in [0,T] \times \mathcal{O}$ that will be specified later, due to Assumption 1.1, the derivative $D_vF(t^*, v^*)$ is a sectorial operator. Thus, we can choose a real number $a \in \boldsymbol{R}$ and an angle $\varphi \in (\pi/2, \pi)$ such that the spectrum of $D_vF(t^*, v^*)$ is contained in

$$\mathcal{S} = \{\lambda \in \boldsymbol{C} : |\arg(\lambda - a)| \ge \varphi\},$$

see Definition A.2. We assume that the Runge-Kutta method (1.3) is strongly $A(\vartheta)$-stable on a sector

$$M_\vartheta = \{z \in \boldsymbol{C} : |\arg(-z)| \leq \vartheta\}$$

with $\vartheta > \pi - \varphi$, see Definition C.2. This situation is illustrated in Figure 1.1.

1.2. Spaces of Hölder continuous functions and sequences

In the sequel, we define the spaces of α-Hölder and (α,α)-Hölder continuous functions and the corresponding sequence spaces which are fundamental for our further analysis. An overview on these definitions is given in Table 1.1 and Table 1.2.

1.2.1. Function spaces

We consider spaces of functions that are defined on an arbitrary interval $I \subset \boldsymbol{R}$ and take values in some Banach space $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$. Fix $\alpha \in (0,1)$.

As usually, $\mathcal{C}(I,\mathcal{Y})$ denotes the space of all continuous functions from I to $\mathcal{Y}$. $\mathcal{B}(I,\mathcal{Y})$, the space of bounded functions, is endowed with the norm

$$\|f\|_{\mathcal{B}(I,\mathcal{Y})} = \operatorname*{ess\,sup}_{t\in I} \|f(t)\|_{\mathcal{Y}} \qquad \text{for } f \in \mathcal{B}(I,\mathcal{Y}).$$

The function space $\mathcal{C}^\alpha(I,\mathcal{Y})$ of α-Hölder continuous functions consists of all continuous and bounded functions $f : I \to \mathcal{Y}$ such that the quantity

$$[f]_{\mathcal{C}^\alpha(I,\mathcal{Y})} = \sup_{s,t\in I,\ s<t} (t-s)^{-\alpha} \|f(t) - f(s)\|_{\mathcal{Y}}$$

remains finite. The norm on $\mathcal{C}^\alpha(I,\mathcal{Y})$ is defined through

$$\|f\|_{\mathcal{C}^\alpha(I,\mathcal{Y})} = \|f\|_{\mathcal{B}(I,\mathcal{Y})} + [f]_{\mathcal{C}^\alpha(I,\mathcal{Y})}.$$

For intervals of the form $I = (0,T]$, the space $\mathcal{C}^\alpha_\alpha(I,\mathcal{Y})$ of (α,α)-Hölder continuous functions contains all bounded functions f which are α-Hölder continuous on each interval $[\varepsilon, T]$, $\varepsilon > 0$, and satisfy

$$[f]_{\mathcal{C}^\alpha_\alpha(I,\mathcal{Y})} = \sup_{0<\varepsilon<T} \varepsilon^\alpha [f]_{\mathcal{C}^\alpha([\varepsilon,T],\mathcal{Y})} < \infty.$$

This function space is endowed with the norm

$$\|f\|_{\mathcal{C}^\alpha_\alpha(I,\mathcal{Y})} = \|f\|_{\mathcal{B}(I,\mathcal{Y})} + [f]_{\mathcal{C}^\alpha_\alpha(I,\mathcal{Y})}.$$

Let $\mathcal{H}$ be one of the symbols $\mathcal{B}, \mathcal{C}^\alpha$ or $\mathcal{C}^\alpha_\alpha$, and let I denote either the positive or the negative half line. Then, for a real number $\omega \in \boldsymbol{R}$, the space $\mathcal{H}^\omega(I,\mathcal{Y})$ of functions with exponential decay or growth, respectively, is defined by

$$\mathcal{H}^\omega(I,\mathcal{Y}) = \left\{f : I \longrightarrow \mathcal{Y} : e^{(\cdot)\omega} f(\cdot) \in \mathcal{H}(I,\mathcal{Y})\right\}$$

and endowed with the norm

$$\|f\|_{\mathcal{H}^\omega(I,\mathcal{Y})} = \left\|e^{(\cdot)\omega} f(\cdot)\right\|_{\mathcal{H}(I,\mathcal{Y})}.$$

The definitions of these function spaces are collected in Table 1.1.

Notice that for a fixed time interval I the dependence of the function spaces on I is often omitted. Instead of $\|\cdot\|_{\mathcal{C}^\alpha_\alpha(I,\mathcal{Y})}$ we then employ the abbreviation $\|\cdot\|_{\mathcal{C}^\alpha_\alpha(\mathcal{Y})}$.

In the following, we give several examples of α-Hölder and (α,α)-Hölder continuous functions taking values in $\boldsymbol{R}$. We show below that for $0 < \varepsilon \le s < t$ and arbitrary $0 < \alpha < 1$, one can easily derive the estimate

$$|\log(t) - \log(s)| \le \alpha^{-1}\varepsilon^{-\alpha}(t-s)^\alpha.$$

However, even though the logarithm possesses the weakly singular behaviour that characterizes a (α,α)-Hölder continuous function, it is not an appropriate example for a map belonging to $\mathcal{C}^\alpha_\alpha((0,\infty),\boldsymbol{R})$ as the boundedness requirement is not fulfiled. To overcome this, we hence consider the composition of sine and logarithm. Moreover, we give a simple example for a function which is continuous on $[0,\infty)$ and (α,α)-Hölder continuous on $(0,\infty)$ but does not belong to $\mathcal{C}^\alpha([0,\infty),\boldsymbol{R})$.

Examples. 1. The logarithm, denoted by

$$\log : (0,\infty) \longrightarrow \boldsymbol{R} : t \longmapsto \log(t),$$

is continuous on $(0,\infty)$ and bounded on each interval $[\varepsilon, T]$ for $0 < \varepsilon < T$.

Fix some $0 < \alpha < 1$ and choose real numbers $s, t \in \boldsymbol{R}$ such that $0 < \varepsilon \le s < t \le T$. Then, with the help of the identity

$$\log(t) - \log(s) = \int_s^t \tau^{-1}\,\mathrm{d}\tau$$

and the relation $\tau^{-1} = \tau^{-\alpha}\tau^{-1+\alpha} \le \varepsilon^{-\alpha}\tau^{-1+\alpha}$, we receive the estimate

$$|\log(t) - \log(s)| \le \varepsilon^{-\alpha}\int_s^t \tau^{-1+\alpha}\,\mathrm{d}\tau \le \varepsilon^{-\alpha}\int_0^{t-s} \tau^{-1+\alpha}\,\mathrm{d}\tau = \alpha^{-1}\varepsilon^{-\alpha}(t-s)^\alpha \tag{1.4}$$

which implies the bounds

$$[\log]_{\mathcal{C}^\alpha([\varepsilon,T],\boldsymbol{R})} \le \alpha^{-1}\varepsilon^{-\alpha} \qquad \text{and} \qquad [\log]_{\mathcal{C}^\alpha_\alpha((0,T],\boldsymbol{R})} \le \alpha^{-1}.$$

For arbitrary $0 < \alpha < 1$ and $0 < \varepsilon < T$ we thus have $\log \in \mathcal{C}^\alpha([\varepsilon, T], \boldsymbol{R})$. However, due to the fact that the logarithm is unbounded on $(0, T]$, it follows $\log \notin \mathcal{C}^\alpha_\alpha((0, T], \boldsymbol{R})$.

2. The composition

$$\sin \circ \log : (0, \infty) \longrightarrow \boldsymbol{R} : t \longmapsto \sin(\log(t))$$

is continuous and bounded on $(0, \infty)$. Due to the validity of

$$|\sin(x) - \sin(y)| \leq \int_0^1 |\cos(\xi x + (1 - \xi)y)| \cdot |x - y| \, \mathrm{d}\xi \leq |x - y|, \qquad x, y \in \boldsymbol{R},$$

by using (1.4), we get

$$|\sin(\log(t)) - \sin(\log(s))| \leq |\log(t) - \log(s)| \leq \alpha^{-1} \varepsilon^{-\alpha} (t - s)^\alpha$$

for $0 < \varepsilon \leq s < t$. From these deliberations follows

$$\sin \circ \log \in \mathcal{C}^\alpha([\varepsilon, \infty), \boldsymbol{R}) \qquad \text{and} \qquad \sin \circ \log \in \mathcal{C}^\alpha_\alpha((0, \infty), \boldsymbol{R})$$

for every $0 < \alpha < 1$ and $\varepsilon > 0$.

3. The function $f : [0, \infty) \to \boldsymbol{R}$, defined through

$$f(t) = \frac{\sin(\log(t))}{\log(t)} \qquad \text{for} \quad t \neq 0 \quad \text{and} \quad t \neq 1,$$

as well as $f(0) = 0$ and $f(1) = 1$, is continuous and bounded on $[0, \infty)$. Let $L_g > 0$ denote the Lipschitz constant of the map

$$g(x) = \frac{\sin(x)}{x}, \qquad x \in \boldsymbol{R}.$$

Then, the following estimate holds for all $0 < \varepsilon \leq s < t$

$$|f(t) - f(s)| \leq \alpha^{-1} L_g \, \varepsilon^{-\alpha} (t - s)^\alpha,$$

wherefore we have $f \in \mathcal{C}([0, \infty), \boldsymbol{R}) \cap \mathcal{C}^\alpha_\alpha((0, \infty), \boldsymbol{R})$ for every $0 < \alpha < 1$. Nevertheless, it holds $f \notin \mathcal{C}^\alpha([0, \infty), \boldsymbol{R})$.

1.2.2. Sequence spaces

For some fixed stepsize $h > 0$ and arbitrary intervals $I \subset \boldsymbol{R}$, set $t_n = nh$ and let

$$\mathcal{N} = \{t_n = nh : n \in \boldsymbol{Z} \text{ such that } t_n \in I\}.$$

In the sequel, we consider sequences $\underline{f} = (f_n)_{t_n \in \mathcal{N}}$ in $\mathcal{Y}$.

The space $\mathcal{B}(\mathcal{N},\mathcal{Y})$ of bounded sequences is endowed with the norm

$$\|\underline{f}\|_{\mathcal{B}(\mathcal{N},\mathcal{Y})} = \sup_{t_n\in\mathcal{N}} \|f_n\|_{\mathcal{Y}}.$$

The discretization of $\mathcal{C}^\alpha(I,\mathcal{Y})$, denoted by $\mathcal{C}^\alpha(\mathcal{N},\mathcal{Y})$, contains all bounded sequences such that the quantity

$$[\underline{f}]_{\mathcal{C}^\alpha(\mathcal{N},\mathcal{Y})} = \sup_{t_m,t_n\in\mathcal{N},t_m<t_n} (t_n-t_m)^{-\alpha}\|f_n-f_m\|_{\mathcal{Y}}$$

is finite. The norm of $\mathcal{C}^\alpha(\mathcal{N},\mathcal{Y})$ is given by

$$\|\underline{f}\|_{\mathcal{C}^\alpha(\mathcal{N},\mathcal{Y})} = \|\underline{f}\|_{\mathcal{B}(\mathcal{N},\mathcal{Y})} + [\underline{f}]_{\mathcal{C}^\alpha(\mathcal{N},\mathcal{Y})}.$$

Let $I=(0,T]$ and for fixed $h>0$ define N through $t_N \le T < t_{N+1}$. We set

$$\mathcal{N} = \{t_n = nh : n\in\boldsymbol{N}, 0<t_n\le t_N\} \quad \text{and} \quad \mathcal{N}_1 = \{t_n = nh : n\in\boldsymbol{N}, t_{n_1}\le t_n\le t_N\}$$

for some $n_1\in\boldsymbol{N}$ with $0<t_{n_1}<t_N$. Then, the space $\mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{Y})$ of all (α,α)-Hölder sequences is defined through

$$\mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{Y}) = \big\{\underline{f}\in\mathcal{B}(\mathcal{N},\mathcal{Y}) : f\in\mathcal{C}^\alpha(\mathcal{N}_1,\mathcal{Y}) \text{ for all } 0<t_{n_1}<t_N, [\underline{f}]_{\mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{Y})}<\infty\big\}$$

and endowed with the seminorm $[\cdot]_{\mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{Y})}$ given by

$$[\underline{f}]_{\mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{Y})} = \sup_{0<t_{n_1}<t_N} t_{n_1}^\alpha [\underline{f}]_{\mathcal{C}^\alpha(\mathcal{N}_1,\mathcal{Y})}.$$

The norm of $\mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{Y})$ is defined through

$$\|\underline{f}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{Y})} = \|\underline{f}\|_{\mathcal{B}(\mathcal{N},\mathcal{Y})} + [\underline{f}]_{\mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{Y})}.$$

For $\omega\in\boldsymbol{R}$ the space of sequences with exponential decay or growth is defined through

$$\mathcal{H}^\omega(\mathcal{N},\mathcal{Y}) = \big\{\underline{f} = (f_n)_{t_n\in\mathcal{N}} : e^{(\cdot)\omega}\underline{f} = \big(e^{t_n\omega}f_n\big)_{t_n\in\mathcal{N}} \in \mathcal{H}(\mathcal{N},\mathcal{Y})\big\},$$

where either $\mathcal{H}=\mathcal{B},\mathcal{C}^\alpha$ or $\mathcal{H}=\mathcal{C}^\alpha_\alpha$, and endowed with the norm

$$\|\underline{f}\|_{\mathcal{H}^\omega(\mathcal{N},\mathcal{Y})} = \big\|e^{(\cdot)\omega}\underline{f}\big\|_{\mathcal{H}(\mathcal{N},\mathcal{Y})}.$$

Whenever $\mathcal{N}$ is a bounded set, it follows at once that any sequence in $\mathcal{Y}$ belongs to $\mathcal{H}^\omega(\mathcal{N},\mathcal{Y})$. The definitions of the sequence spaces are reviewed in Table 1.2.

As mentioned in the preceding section, we occasionally dismiss the dependence of the function and sequence spaces on I and $\mathcal{N}$, respectively.

$$\mathcal{C}(I,\mathcal{Y}) = \big\{ f : I \longrightarrow \mathcal{Y} : f \text{ continuous} \big\}$$

$$\mathcal{B}(I,\mathcal{Y}) = \big\{ f : I \longrightarrow \mathcal{Y} : \|f\|_{\mathcal{B}(I,\mathcal{Y})} < \infty \big\}$$
$$\|f\|_{\mathcal{B}(I,\mathcal{Y})} = \text{ess sup}\{\|f(t)\|_{\mathcal{Y}} : t \in I\}$$

$$[f]_{\mathcal{C}^\alpha(I,\mathcal{Y})} = \sup_{s,t\in I, s<t} (t-s)^{-\alpha} \|f(t) - f(s)\|_{\mathcal{Y}}$$
$$\mathcal{C}^\alpha(I,\mathcal{Y}) = \big\{ f \in \mathcal{C}(I,\mathcal{Y}) \cap \mathcal{B}(I,\mathcal{Y}) : [f]_{\mathcal{C}^\alpha(I,\mathcal{Y})} < \infty \big\}$$
$$\|f\|_{\mathcal{C}^\alpha(I,\mathcal{Y})} = \|f\|_{\mathcal{B}(I,\mathcal{Y})} + [f]_{\mathcal{C}^\alpha(I,\mathcal{Y})}$$

$$[f]_{\mathcal{C}^\alpha_\alpha((0,T],\mathcal{Y})} = \sup_{0<\varepsilon<T} \varepsilon^\alpha [f]_{\mathcal{C}^\alpha([\varepsilon,T],\mathcal{Y})}$$
$$\begin{aligned}\mathcal{C}^\alpha_\alpha((0,T],\mathcal{Y}) = \big\{ f \in \mathcal{B}((0,T],\mathcal{Y}) : \forall\, 0 < \varepsilon < T : f \in \mathcal{C}^\alpha([\varepsilon,T],\mathcal{Y}),& \\ [f]_{\mathcal{C}^\alpha_\alpha((0,T],\mathcal{Y})} < \infty \big\}&\end{aligned}$$
$$\|f\|_{\mathcal{C}^\alpha_\alpha((0,T],\mathcal{Y})} = \|f\|_{\mathcal{B}((0,T],\mathcal{Y})} + [f]_{\mathcal{C}^\alpha_\alpha((0,T],\mathcal{Y})}$$

$$\mathcal{B}^\omega(I,\mathcal{Y}) = \big\{ f : I \longrightarrow \mathcal{Y} : e^{(\cdot)\omega} f(\cdot) \in \mathcal{B}(I,\mathcal{Y}) \big\}$$
$$\|f\|_{\mathcal{B}^\omega(I,\mathcal{Y})} = \big\| e^{(\cdot)\omega} f(\cdot) \big\|_{\mathcal{B}(I,\mathcal{Y})}$$

$$\mathcal{C}^{\alpha,\omega}((-\infty,0],\mathcal{Y}) = \big\{ f : (-\infty,0] \longrightarrow \mathcal{Y} : e^{(\cdot)\omega} f(\cdot) \in \mathcal{C}^\alpha((-\infty,0],\mathcal{Y}) \big\}$$
$$\|f\|_{\mathcal{C}^{\alpha,\omega}((-\infty,0],\mathcal{Y})} = \big\| e^{(\cdot)\omega} f(\cdot) \big\|_{\mathcal{C}^\alpha((-\infty,0],\mathcal{Y})}$$

$$\mathcal{C}^{\alpha,\omega}_\alpha((0,\infty),\mathcal{Y}) = \big\{ f : (0,\infty) \longrightarrow \mathcal{Y} : e^{(\cdot)\omega} f(\cdot) \in \mathcal{C}^\alpha_\alpha((0,\infty),\mathcal{Y}) \big\}$$
$$\|f\|_{\mathcal{C}^{\alpha,\omega}_\alpha((0,\infty),\mathcal{Y})} = \big\| e^{(\cdot)\omega} f(\cdot) \big\|_{\mathcal{C}^\alpha_\alpha((0,\infty),\mathcal{Y})}$$

Table 1.1.: Function spaces

$$\mathcal{N} = \{t_n = nh : n \in \boldsymbol{Z}, t_n \in I\} \text{ for } I \subset \boldsymbol{R} \text{ interval}$$
$$\mathcal{B}(\mathcal{N},\mathcal{Y}) = \left\{\underline{f} = (f_n)_{t_n\in\mathcal{N}} : \|\underline{f}\|_{\mathcal{B}(\mathcal{N},\mathcal{Y})} < \infty\right\}$$
$$\|\underline{f}\|_{\mathcal{B}(\mathcal{N},\mathcal{Y})} = \sup_{t_n\in\mathcal{N}} \|f_n\|_{\mathcal{Y}}$$

$$[\underline{f}]_{\mathcal{C}^\alpha(\mathcal{N},\mathcal{Y})} = \sup_{t_m,t_n\in\mathcal{N},t_m<t_n} (t_n - t_m)^{-\alpha}\|f_n - f_m\|_{\mathcal{Y}}$$
$$\mathcal{C}^\alpha(\mathcal{N},\mathcal{Y}) = \left\{\underline{f} = (f_n)_{t_n\in\mathcal{N}} \in \mathcal{B}(\mathcal{N},\mathcal{Y}) : [\underline{f}]_{\mathcal{C}^\alpha(\mathcal{N},\mathcal{Y})} < \infty\right\}$$
$$\|\underline{f}\|_{\mathcal{C}^\alpha(\mathcal{N},\mathcal{Y})} = \|\underline{f}\|_{\mathcal{B}(\mathcal{N},\mathcal{Y})} + [\underline{f}]_{\mathcal{C}^\alpha(\mathcal{N},\mathcal{Y})}$$

$$\mathcal{N} = \{t_n = nh : n \in \boldsymbol{N}, 0 < t_n \le t_N\}, \quad \mathcal{N}_1 = \{t_n = nh : n \in \boldsymbol{N}, t_{n_1} \le t_n \le t_N\}$$
$$[\underline{f}]_{\mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{Y})} = \sup_{0<t_{n_1}<t_N} t_{n_1}^\alpha [\underline{f}]_{\mathcal{C}^\alpha(\mathcal{N}_1,\mathcal{Y})}$$
$$\mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{Y}) = \left\{\underline{f} \in \mathcal{B}(\mathcal{N},\mathcal{Y}) : \forall\, 0 < t_{n_1} < t_N : \underline{f} \in \mathcal{C}^\alpha(\mathcal{N}_1,\mathcal{Y}), [\underline{f}]_{\mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{Y})} < \infty\right\}$$
$$\|\underline{f}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{Y})} = \|\underline{f}\|_{\mathcal{B}(\mathcal{N},\mathcal{Y})} + [\underline{f}]_{\mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{Y})}$$

$$\mathcal{B}^\omega(\mathcal{N},\mathcal{Y}) = \left\{\underline{f} = (f_n)_{t_n\in\mathcal{N}} : e^{(\cdot)\omega}\underline{f} = \left(e^{t_n\omega} f_n\right)_{t_n\in\mathcal{N}} \in \mathcal{B}(\mathcal{N},\mathcal{Y})\right\}$$
$$\|\underline{f}\|_{\mathcal{B}^\omega(\mathcal{N},\mathcal{Y})} = \left\|e^{(\cdot)\omega}\underline{f}\right\|_{\mathcal{B}(\mathcal{N},\mathcal{Y})}$$

$$\mathcal{C}^{\alpha,\omega}(\mathcal{N},\mathcal{Y}) = \left\{\underline{f} = (f_n)_{t_n\in\mathcal{N}} : e^{(\cdot)\omega}\underline{f} \in \mathcal{C}^\alpha(\mathcal{N},\mathcal{Y})\right\}$$
$$\|\underline{f}\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{N},\mathcal{Y})} = \left\|e^{(\cdot)\omega}\underline{f}\right\|_{\mathcal{C}^\alpha(\mathcal{N},\mathcal{Y})}$$

$$\mathcal{C}^{\alpha,\omega}_\alpha(\mathcal{N},\mathcal{Y}) = \left\{\underline{f} = (f_n)_{t_n\in\mathcal{N}} : e^{(\cdot)\omega}\underline{f} \in \mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{Y})\right\}$$
$$\|\underline{f}\|_{\mathcal{C}^{\alpha,\omega}_\alpha(\mathcal{N},\mathcal{Y})} = \left\|e^{(\cdot)\omega}\underline{f}\right\|_{\mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{Y})}$$

Table 1.2.: Spaces of sequences

2. Nonlinear initial-boundary value problems and numerical experiments

In the present chapter, we are concerned with applications of the abstract Banach space setting from Chapter 1 to nonlinear initial-boundary value problems. We study several one-dimensional initial-boundary value problems involving second order partial differential equations of parabolic type that are used as mathematical models for processes involving nonlinear detonation phenomena, heat conduction or diffusion. More precisely, we consider the following applications. In Section 2.2 we give a nonlinear model from detonation theory that describes the displacement of a shock, see Brauner et al. [6] and Lunardi [25, Section 8.5.1]. In Section 2.3 we then consider an example from combustion theory which is found in Belleni-Morante & McBride [5, Section 6.7]. Finally, a semilinear problem with moving boundary is studied in Section 2.4. The Stefan problem describing the melting of ice is of this form, see Taylor [33, Section 15.4].

In each section, we show that the presented example can be written as an abstract initial value problem of the form

$$u'(t) = F(t, u(t)), \quad t > 0, \qquad u(0) \text{ given}, \tag{2.1}$$

and that the actual right-hand side F fulfils the basic requirements of Assumption 1.1. As an illustration, for special initial states, we discretize the initial-boundary value problem with respect to space and time, and we demonstrate how the approximate solution evolves in time. Besides, in Sections 2.2 and 2.3, we determine the qualitative behaviour near steady state solutions. In view of our theoretical results on the convergence and the long-term behaviour of Runge-Kutta time discretizations for (2.1), we carry out several numerical experiments.

For our purposes, it is useful to consider afore in Section 2.1 a general one-dimensional initial-boundary value problem comprising a nonlinear partial differential equation

$$\partial_t U(x,t) = f\big(t, U(x,t), \partial_x U(x,t), \partial_{xx} U(x,t)\big), \qquad 0 < x < 1, \quad t > 0,$$

subject to initial and boundary conditions. We write this equation in abstract form (2.1), and we investigate under which requirements such an evolution equation can be cast into our analytical framework. Thereby, the presentation follows Lunardi [25, Section 8.5].

2.1. A general initial-boundary value problem in abstract form

The fully nonlinear problems considered in Sections 2.2 and 2.3 have in common that we are looking for a real-valued function U depending on the position $x \in [0,1]$ and the time $t \geq 0$ that satisfies a partial differential equation of the form

$$\partial_t U(x,t) = f\big(t, U(x,t), \partial_x U(x,t), \partial_{xx} U(x,t)\big), \qquad 0 < x < 1, \quad t > 0, \tag{2.2a}$$

and furthermore a given initial state

$$U(x,0) = U_0(x), \qquad 0 < x < 1. \tag{2.2b}$$

Additionally, we impose certain boundary conditions on U such as *homogeneous Dirichlet boundary conditions*

$$U(0,t) = 0 \qquad \text{and} \qquad U(1,t) = 0 \qquad \text{for} \quad t > 0, \tag{2.2c}$$

or *homogeneous Neumann boundary conditions*

$$\partial_x U(0,t) = 0 \qquad \text{and} \qquad \partial_x U(1,t) = 0 \qquad \text{for} \quad t > 0. \tag{2.2d}$$

The right-hand side of the partial differential equation (2.2a) is determined by the map

$$f : \boldsymbol{R}_{\geq 0} \times \boldsymbol{R}^3 \longrightarrow \boldsymbol{R} : (t, \zeta, \eta, \vartheta) \longmapsto f(t, \zeta, \eta, \vartheta).$$

Here, for notational simplicity, we assume that f is defined for all elements in $\boldsymbol{R}_{\geq 0} \times \boldsymbol{R}^3$. In order to write the initial-boundary value problem (2.2) in abstract form, we set

$$u(t) = U(\cdot, t) \quad \text{and} \quad \big(F(t,v)\big)(x) = f(t, v(x), \partial_x v(x), \partial_{xx} v(x)) \quad \text{for } v : [0,1] \to \boldsymbol{R}$$

wherewith we formally receive the following initial value problem

$$u'(t) = F(t, u(t)), \quad t > 0, \qquad u(0) \text{ given.} \tag{2.3}$$

In the sequel, we study the question whether (2.3) with right-hand side

$$F : [0,\infty) \times \mathcal{O} \longrightarrow \mathcal{X} : (t,v) \longmapsto F(t,v), \qquad \mathcal{O} \subset \mathcal{D},$$

fits into our framework for suitable choices of the underlying function spaces $\mathcal{D}$ and $\mathcal{X}$. Roughly spoken, the desired regularity properties of the solution and the boundary conditions (2.2c) or (2.2d), e.g., have an effect on the choice of $\mathcal{D}$. According to Assumption 1.1, we have to show that F has certain smoothness properties and that its derivative $D_v F(t,v) : \mathcal{D} \to \mathcal{X}$, for $t \geq 0$ and $v \in \mathcal{O}$ given by

$$\begin{aligned}\big(D_v F(t,v)\big)(x) = \partial_\zeta f(t, v(x), \partial_x v(x), \partial_{xx} v(x))\, I \\ + \partial_\eta f(t, v(x), \partial_x v(x), \partial_{xx} v(x))\, \partial_x \\ + \partial_\vartheta f(t, v(x), \partial_x v(x), \partial_{xx} v(x))\, \partial_{xx}, \qquad 0 \leq x \leq 1,\end{aligned} \tag{2.4}$$

fulfils the following requirements for suitably chosen Banach spaces $\mathcal{D} \subset \mathcal{X}$ and the open subset $\mathcal{O} \subset \mathcal{D}$.

(i) $D_vF(t,v)$ is Hölder continuous in t and locally Lipschitz continuous in v.

(ii) $D_vF(t,v)$ is a sectorial operator in $\mathcal{X}$.

As the regularity of F and D_vF is mainly determined by the smoothness properties of f, we investigate the first requirement if needed in the subsequent sections.

For the second requirement we have to show that the second order differential operator (2.4) defines a sectorial operator. For this reason, we now consider a general boundary value problem in a domain $\Omega \subset \boldsymbol{R}^n$

$$\mathcal{A}(x,\partial)v(x) = 0, \quad x \in \Omega, \qquad \mathcal{B}(x,\partial)v(x) = 0, \quad x \in \partial\Omega,$$

involving a partial differential operator $\mathcal{A}(\cdot,\partial)$ of second order and the first order differential operator $\mathcal{B}(\cdot,\partial)$ describing the boundary conditions, and we investigate the question under which assumptions the *realization* of $\mathcal{A}(\cdot,\partial)$ in some Banach space $\mathcal{X}$

$$A : \mathcal{D}(A) \longrightarrow \mathcal{X} : v \longmapsto Av = \mathcal{A}(\cdot,\partial)v \tag{2.5}$$

is a sectorial operator and thus generates an analytic semigroup. Thereby, the domain of A is chosen in such a way that any element $v \in \mathcal{D}(A)$ immediately fulfils the actual boundary conditions. Roughly spoken, the linear operator A defined through (2.5) is sectorial in the function space of integrable or uniformly continuous functions, respectively, whenever $\mathcal{A}(\cdot,\partial)$ is a strongly elliptic operator on $\boldsymbol{R}^n$ with sufficiently smooth coefficients. This result is still valid for the case $\Omega \neq \boldsymbol{R}^n$ under homogeneous Dirichlet or Neumann boundary conditions, e.g., provided that the boundary of Ω is smooth enough. In the following, we state the precise requirements on Ω and the differential operators $\mathcal{A}(\cdot,\partial)$ and $\mathcal{B}(\cdot,\partial)$, and we briefly review the generation results for the Banach spaces $L^p(\Omega)$ where $1 < p < \infty$ and $\mathcal{UC}(\overline{\Omega})$. The specified statements and further results can be found in Lunardi [25, Chapter 3]. For convenience, we afore recall some standard notation and the definitions of the involved function spaces, see also Adams [1].

In the meantime, we denote by $\Omega \subset \boldsymbol{R}^n$ a nonempty open subset of the n-dimensional Euclidian space $\boldsymbol{R}^n$. Besides, for some multi-index $\alpha = (\alpha_1,\dots,\alpha_n) \in \boldsymbol{N}_0^n$, for vectors $\xi = (\xi_1,\dots,\xi_n) \in \boldsymbol{R}^n$ and the Nabla-operator $\partial = (\partial_{x_1},\dots,\partial_{x_n})$ comprising the partial derivatives, we set

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \qquad \xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n} \qquad \text{and} \qquad \partial^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}.$$

For any integer $p \geq 1$ the function space $L^p(\Omega)$ contains all real-valued and measurable functions v, defined a.e. on Ω, such that $|v|^p$ is integrable over Ω. The linear space $L^p(\Omega)$ is endowed with the usual norm

$$\|v\|_p = \left(\int_\Omega |v(\xi)|^p \,\mathrm{d}\xi\right)^{1/p}$$

and becomes a Banach space by the convention that two elements are considered as equivalent if they are equal a.e. in Ω. The space $L^\infty(\Omega)$ of all measurable and essentially bounded functions $v : \Omega \to \boldsymbol{R}$ is a Banach space with

$$\|v\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega}|v(x)| = \inf\{c > 0 : |v(x)| \le c \text{ a.e. on } \Omega\}.$$

The vector space $\mathcal{C}^k(\Omega)$ consists of all k times continuously differentiable functions, and the space $\mathcal{C}_0^\infty(\Omega)$ contains all functions in

$$\mathcal{C}^\infty(\Omega) = \bigcap_{k=0}^{\infty} \mathcal{C}^k(\Omega)$$

with compact support in Ω. The space $\mathcal{C}^k(\overline{\Omega})$ comprises all k times continuously differentiable functions v in Ω with derivatives up to order k bounded and continuously extendable up to the boundary. The function space $\mathcal{C}^k(\overline{\Omega})$ is a Banach space with the norm

$$\|v\|_{\mathcal{C}^k(\overline{\Omega})} = \sum_{|\alpha|\le k} \big\|\partial^\alpha v\big\|_\infty.$$

Obversely, the space $\mathcal{UC}^k(\overline{\Omega})$ of all k times continuously differentiable functions with derivatives up to order k bounded and uniformly continuously extendable up to $\partial\Omega$ is a Banach space with the norm $\|\cdot\|_{\mathcal{C}^k(\overline{\Omega})}$. For integers $k \ge 1$ we denote by $\mathcal{W}^{k,p}(\Omega)$ the Sobolev space

$$\mathcal{W}^{k,p}(\Omega) = \big\{v \in L^p(\Omega) : \forall\alpha \in \boldsymbol{N}_0^n, |\alpha| \le k : \partial^\alpha v \in L^p(\Omega)\big\}$$
$$\|v\|_{k,p} = \Big(\sum_{|\alpha|\le k} \big\|\partial^\alpha v\big\|_p^p\Big)^{1/p}$$

comprising all functions in $L^p(\Omega)$ for which the distributional derivatives up to order k belong to $L^p(\Omega)$ again. The space $\mathcal{W}_0^{k,p}(\Omega)$ is defined as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $\mathcal{W}^{k,p}(\Omega)$.

Let Ω denote either the n-dimensional Euclidian space $\boldsymbol{R}^n$ or an open subset of $\boldsymbol{R}^n$ with uniformly twice continuously differentiable boundary

$$\Omega = \boldsymbol{R}^n \qquad \text{or} \qquad \Omega \subset \boldsymbol{R}^n \text{ open with } \partial\Omega \text{ uniformly } \mathcal{C}^2,$$

see Edmunds & Evans [8, Definition V.4.1]. We consider a differential operator of second order

$$\mathcal{A}(\cdot,\partial) = \sum_{|\alpha|\le 2} a_\alpha \partial^\alpha, \qquad \alpha \in \boldsymbol{N}_0^n,$$

with uniformly continuous and bounded coefficients $a_\alpha : \overline{\Omega} \to \boldsymbol{R}$. Our basic assumption on $\mathcal{A}(\cdot, \partial)$ is that the *uniform ellipticity condition*

$$\sum_{|\alpha|=2} a_\alpha(x)\,\xi^\alpha \geq \kappa|\xi|^2, \qquad x \in \overline{\Omega}, \quad \xi \in \boldsymbol{R}^n, \tag{2.6}$$

is satisfied for some $\kappa > 0$. Whenever $\Omega \neq \boldsymbol{R}^n$, the boundary conditions are described by means of the first order differential operator

$$\mathcal{B}(\cdot, \partial) = \sum_{|\beta|\leq 1} b_\beta \partial^\alpha, \qquad \beta \in \boldsymbol{N}_0^n.$$

We suppose that the coefficients $b_\beta : \overline{\Omega} \to \boldsymbol{R}$ are continuously differentiable in Ω with bounded and up to $\partial\Omega$ extendable uniformly continuous first derivatives and that the *uniform non-tangentiability condition*

$$\inf_{x\in\partial\Omega} \left| \sum_{|\beta|=1} b_\beta(x)\nu(x)^\beta \right| > 0 \tag{2.7}$$

holds. Here, $\nu(x) \in \boldsymbol{R}^n$ denotes the exterior unit normal vector to the boundary $\partial\Omega$ at the point $x \in \partial\Omega$.

Under the above assumptions on Ω and the differential operators $\mathcal{A}(\cdot, \partial)$ and $\mathcal{B}(\cdot, \partial)$, the linear operator A defined by (2.5) is sectorial for the following choices of $\mathcal{X}$ and $\mathcal{D}(A)$. For indicating the boundary conditions described by $\mathcal{B}(\cdot, \partial)$, we employ the notation A_1.

(i) For every $1 < p < \infty$ the linear operator

$$A : \mathcal{W}^{2,p}(\boldsymbol{R}^n) \longrightarrow L^p(\boldsymbol{R}^n) : v \longmapsto \mathcal{A}(\cdot, \partial)v$$

is sectorial in $L^p(\boldsymbol{R}^n)$ and said to be the *realization of* $\mathcal{A}(\cdot, \partial)$ *in* $L^p(\boldsymbol{R}^n)$. If $\Omega \neq \boldsymbol{R}^n$ is bounded, the operator

$$A_1 : \left\{ v \in \mathcal{W}^{2,p}(\Omega) : \mathcal{B}(\cdot, \partial)v\big|_{\partial\Omega} = 0 \right\} \longrightarrow L^p(\Omega) : v \longmapsto \mathcal{A}(\cdot, \partial)v$$

is sectorial. It is called the *realization of* $\mathcal{A}(\cdot, \partial)$ *in* $L^p(\boldsymbol{R}^n)$ *with oblique boundary conditions in* $L^p(\Omega)$.

(ii) For denoting the ball around $a \in \Omega$ of radius $r > 0$ contained in Ω, we use the abbreviation $B_r(a) = \{x \in \Omega : \|x - a\|_{\boldsymbol{R}^n} < r\}$. Moreover, we henceforth set

$$\mathcal{W}(\Omega) = \bigcap_{p\geq 1} \mathcal{W}^{2,p}_{\text{loc}}(\Omega) \qquad \text{where}$$

$$\mathcal{W}^{2,p}_{\text{loc}}(\Omega) = \left\{ v : \Omega \to \boldsymbol{R} : \forall\, a \in \Omega\ \forall\, r > 0 : v\big|_{B_r(a)} \in \mathcal{W}^{2,p}(B_r(a)) \right\}.$$

Then, for the function space $\mathcal{UC}(\overline{\Omega})$ of uniformly continuous functions the following generation results are valid. The densely defined linear operators

$$\begin{aligned} A &: \big\{v \in \mathcal{W}(\boldsymbol{R}^n) : v, \mathcal{A}(\cdot,\partial)v \in \mathcal{UC}(\boldsymbol{R}^n)\big\} \longrightarrow \mathcal{UC}(\boldsymbol{R}^n), \\ A_1 &: \Big\{v \in \mathcal{W}(\Omega) : v, \mathcal{A}(\cdot,\partial)v \in \mathcal{UC}(\overline{\Omega}), \mathcal{B}(\cdot,\partial)v\big|_{\partial\Omega} = 0\Big\} \longrightarrow \mathcal{UC}(\overline{\Omega}) \end{aligned}$$

are sectorial. Moreover, the set $\mathcal{UC}^2(\boldsymbol{R}^n)$ is dense in $\mathcal{D}(A)$, and, in particular, it holds $\mathcal{D}(A) = \mathcal{UC}^2(\boldsymbol{R})$ for $n = 1$.

In the sequel, we specialize the above results to the one-dimensional case. Here, Ω denotes either the real line or the open unit interval

$$\Omega = \boldsymbol{R} \qquad \text{or} \qquad \Omega = (0,1).$$

The second order differential operator $\mathcal{A}(\cdot,\partial_x)$ with uniformly continuous and bounded coefficients $a_\alpha : \overline{\Omega} \to \boldsymbol{R}$ for $0 \le \alpha \le 2$ is of the form

$$\mathcal{A}(\cdot,\partial_x) = a_0 I + a_1\partial_x + a_2\partial_{xx},$$

and the uniform ellipticity condition (2.6) simplifies to

$$a_2(x) \ge \kappa > 0 \qquad \text{for all } x \in \overline{\Omega}.$$

If $\Omega = (0,1)$, we assume that the coefficients $b_\beta : \overline{\Omega} = [0,1] \to \boldsymbol{R}$, $0 \le \beta \le 1$, of the first order differential operator

$$\mathcal{B}(\cdot,\partial_x) = b_0 I + b_1\partial_x$$

have bounded and uniformly continuous first derivatives. Clearly, the uniform non-tangentiability condition (2.7) is fulfiled for homogeneous Dirichlet or Neumann boundary conditions. Especially for $n = 1$, we receive the following generation result in the space of uniformly continuous functions

$$A : \mathcal{UC}^2(\boldsymbol{R}) \longrightarrow \mathcal{UC}(\boldsymbol{R}).$$

Furthermore, we have

$$\begin{aligned} A_0 &: \big\{v \in \mathcal{C}^2([0,1]) : v(0) = v(1) = 0\big\} \longrightarrow \mathcal{C}([0,1]), \\ A_1 &: \big\{v \in \mathcal{C}^2([0,1]) : \partial_x v(0) = \partial_x v(1) = 0\big\} \longrightarrow \mathcal{C}([0,1]), \end{aligned} \tag{2.8}$$

since $\mathcal{UC}^k([0,1]) = \mathcal{C}^k([0,1])$ for all $k \in \boldsymbol{N}_0$.

Now, we apply the above results to the operator $D_vF(t,v)$ defined by (2.4). Let the symbol $\mathcal{H}$ denote either L^p with $1 < p < \infty$ or $\mathcal{UC}$ and assume that the subset $\mathcal{O} \subset \mathcal{H}(0,1)$ is chosen such that the uniform ellipticity condition

$$\partial_\vartheta f(t, v(x), \partial_x v(x), \partial_{xx} v(x)) \ge \kappa > 0, \qquad 0 \le x \le 1, \tag{2.9}$$

is satisfied for all elements $v \in \mathcal{O}$. Then, it follows from the previous considerations that for any $t \geq 0$ and $v \in \mathcal{O}$ the realization of $D_v F(t,v)$ in $\mathcal{H}(\boldsymbol{R})$ with homogeneous Dirichlet or Neumann boundary conditions in $\mathcal{H}(0,1)$ is a sectorial operator.

We close this section with a final remark on the notions of elliptic and parabolic equations. Let

$$\mathcal{A}(\cdot,\partial) = \sum_{|\alpha|\leq 2} a_\alpha \partial^\alpha u$$

be a second order differential operator in $\boldsymbol{R}^n$ with real-valued coefficients a_α. According to Triebel [34, Chapter III.14], we call $\mathcal{A}(\cdot,\partial)u$ *elliptic* if

$$\sum_{|\alpha|=2} a_\alpha \xi^\alpha \neq 0 \qquad \text{for all } 0 \neq \xi \in \boldsymbol{R}^n.$$

In the $(n+1)$-dimensional space $\boldsymbol{R}^{n+1}$ any differential expression of the form

$$\partial_t u - \widetilde{\mathcal{A}}(\cdot,\partial)u \qquad \text{where} \quad \widetilde{\mathcal{A}}(\cdot,\partial) = \sum_{\substack{|\alpha|\leq 2 \\ \alpha=(\alpha_1,\dots,\alpha_n,0)}} a_\alpha \partial^\alpha u$$

is said to be *parabolic* whenever $\widetilde{\mathcal{A}}(\cdot,\partial)u$ is elliptic in $\boldsymbol{R}^n$.

2.2. Displacement of a shock

The following one-dimensional initial-boundary value problem arises in the study of the shock displacement in a detonation phenomenon

$$\begin{aligned} &\partial_t U(x,t) = \log\left(\frac{\exp\big(a\,U(x,t)\,\partial_{xx}U(x,t)\big)-1}{a\,\partial_{xx}U(x,t)}\right) - \frac{1}{2}\big(\partial_x U(x,t)\big)^2, \\ &\partial_x U(0,t) = \partial_x U(1,t) = 0, \qquad U(x,0) = U_0(x), \qquad 0<x<1, \quad t>0. \end{aligned} \tag{2.10}$$

Here, a denotes a positive constant. The real-valued function U depends on the position $x \in [0,1]$ and on the time $t \geq 0$ and corresponds to the displacement of the shock. Only positive solutions have a physical meaning.

In the first place, we justify that this problem written in abstract form fits into our analytical framework. For this, we define the function f on $\boldsymbol{R}_{>0} \times \boldsymbol{R}^2$ through

$$f(\zeta,\eta,\vartheta) = \begin{cases} \log\left(\frac{\exp(a\zeta\vartheta)-1}{a\vartheta}\right) - \frac{1}{2}\eta^2 & \text{if } \vartheta \neq 0, \\ \log(\zeta) - \frac{1}{2}\eta^2 & \text{if } \vartheta = 0. \end{cases}$$

Then, f is analytic and its partial derivatives equal

$$\begin{aligned}
&\partial_\zeta f(\zeta,\eta,\vartheta) = \frac{a\vartheta\, e^{a\zeta\vartheta}}{e^{a\zeta\vartheta}-1} \qquad \text{if } \vartheta \neq 0,\\
&\partial_\eta f(\zeta,\eta,\vartheta) = -\,\eta,\\
&\partial_\vartheta f(\zeta,\eta,\vartheta) = \frac{1}{e^{a\zeta\vartheta}-1}\left(a\zeta\, e^{a\zeta\vartheta} - \frac{e^{a\zeta\vartheta}-1}{\vartheta}\right) \qquad \text{if } \vartheta \neq 0.
\end{aligned}$$

Moreover, the following relations

$$\lim_{\vartheta\to 0} \partial_\zeta f(\zeta,\eta,\vartheta) = \frac{1}{\zeta} \qquad \text{and} \qquad \lim_{\vartheta\to 0} \partial_\vartheta f(\zeta,\eta,\vartheta) = \frac{a\zeta}{2}$$

are valid. The function $g : \boldsymbol{R} \longrightarrow \boldsymbol{R}$ defined through

$$g(z) = (z-1)\, e^z + 1$$

strictly monotonously increases on the positive half line $\boldsymbol{R}_{>0}$ and strictly monotonously decreases on $\boldsymbol{R}_{<0}$, since $g'(z) = ze^z$. Together with the identity $g(0) = 0$, this implies

$$g(z) > 0 \quad \text{for } z > 0 \qquad \text{and} \qquad g(z) < 0 \quad \text{for } z < 0.$$

Notice that the following identity holds

$$\partial_\vartheta f(\zeta,\eta,\vartheta) = \frac{g(a\zeta\vartheta)}{\vartheta(e^{a\zeta\vartheta}-1)}.$$

By considering the cases $\vartheta > 0$, $\vartheta = 0$ and $\vartheta < 0$ seperately

$$\begin{aligned}
&\vartheta > 0 \quad\Longrightarrow\quad z = a\zeta\vartheta > 0,\ g(z) > 0,\ e^z > 1 \quad\Longrightarrow\quad \partial_\vartheta f(\zeta,\eta,\vartheta) = \frac{g(z)}{\vartheta(e^z-1)} > 0,\\
&\vartheta = 0 \quad\Longrightarrow\quad \partial_\vartheta f(\zeta,\eta,\vartheta) = \frac{a\zeta}{2} > 0,\\
&\vartheta < 0 \quad\Longrightarrow\quad z = a\zeta\vartheta < 0,\ g(z) < 0,\ e^z < 1 \quad\Longrightarrow\quad \partial_\vartheta f(\zeta,\eta,\vartheta) = \frac{g(z)}{\vartheta(e^z-1)} > 0,
\end{aligned}$$

we consequently receive the relation

$$\partial_\vartheta f(\zeta,\eta,\vartheta) > 0 \qquad \text{for } \zeta > 0. \tag{2.11}$$

In order to write the nonlinear initial-boundary value problem (2.10) as an abstract initial value problem, we choose the function space

$$\mathcal{X} = \mathcal{C}([0,1])$$

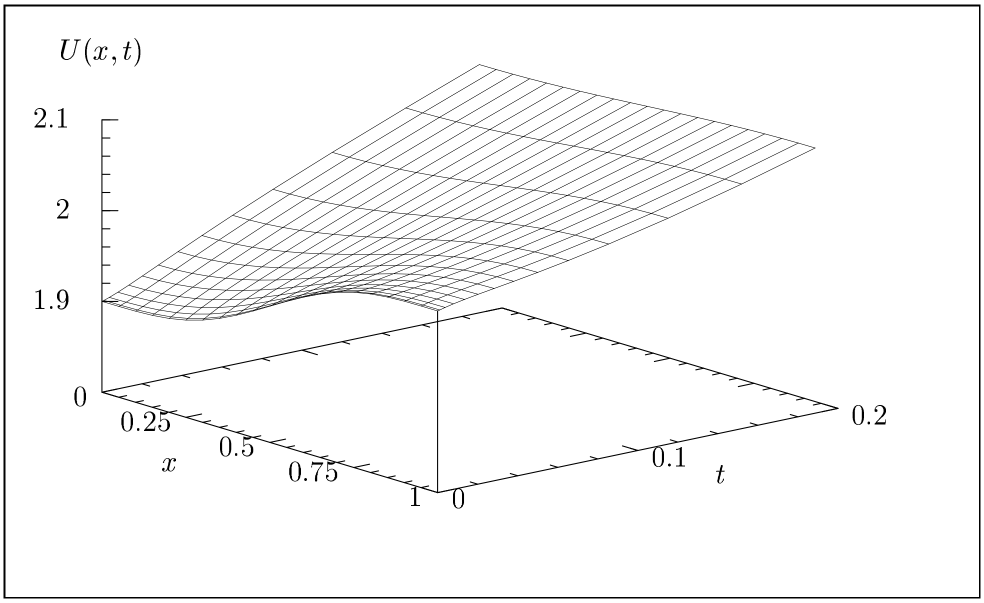

Figure 2.1.: Numerical solution for $U_0(x) = -2x^7 + 7x^6 - 8.4x^5 + 3.5x^4 + 1.9$

as underlying Banach space, and we further set

$$\mathcal{D} = \left\{ v \in \mathcal{C}^2([0,1]) : \partial_x v(0) = \partial_x v(1) = 0 \right\}.$$

Then, the map F defined through

$$F(v) = f(v, \partial_x v, \partial_{xx} v) \qquad \text{for} \quad v \in \mathcal{O} = \left\{ v \in \mathcal{D} : v > 0 \text{ on } [0,1] \right\}$$

fulfils the smoothness requirements of Assumption 1.1. Due to the fact that (2.11) implies

$$\partial_\vartheta f(v, \partial_x v, \partial_{xx} v) > 0 \qquad \text{whenever } v \in \mathcal{O},$$

for any $v \in \mathcal{O}$ the derivative $D_v F(v) : \mathcal{D} \to \mathcal{X}$, given by

$$D_v F(v) = \partial_\zeta f(v, \partial_x v, \partial_{xx} v)\, I + \partial_\eta f(v, \partial_x v, \partial_{xx} v)\, \partial_x + \partial_\vartheta f(v, \partial_x v, \partial_{xx} v)\, \partial_{xx},$$

is a sectorial operator in $\mathcal{X}$, see (2.8) and below. Therefore, the following evolution equation for $u(t) = U(\cdot, t)$

$$u'(t) = F(u(t)), \quad t > 0, \qquad u(0) \text{ given}, \tag{2.12}$$

fits into our analytical framework for arbitrary initial values in $\mathcal{O}$.

In the sequel, we study the qualitative behaviour of (2.10) in the neighbourhood of a special steady state solution, namely the constant solution

$$\overline{U} = 1$$

which corresponds to the detonation wave. For this purpose, we linearize (2.10) or (2.12), respectively, around the equilibrium point $\overline{U}$. Due to the fact that

$$\begin{aligned} D_v F(\overline{U}) &= \partial_\zeta f(\overline{U},0,0)\, I + \partial_\eta f(\overline{U},0,0)\, \partial_x + \partial_\vartheta f(\overline{U},0,0)\, \partial_{xx} \\ &= \frac{1}{\overline{U}}\, I + \frac{a\overline{U}}{2}\, \partial_{xx} = I + \frac{a}{2}\, \partial_{xx}, \end{aligned}$$

this yields the partial differential equation

$$\partial_t \widetilde{U}(x,t) = \frac{a}{2}\, \partial_{xx}\widetilde{U}(x,t) + \widetilde{U}(x,t) - 1.$$

By setting $V(x,t) = \widetilde{U}(x,t) - 1$, the equilibrium point $\overline{U}$ is transformed to the origin, and we finally obtain the following initial-boundary value problem

$$\begin{aligned} &\partial_t V(x,t) = \frac{a}{2}\, \partial_{xx} V(x,t) + V(x,t), \qquad 0 < x < 1, \quad t > 0, \\ &\partial_x V(0,t) = \partial_x V(1,t) = 0, \quad t > 0, \qquad V(x,0) = V_0(x), \quad 0 < x < 1, \end{aligned}$$

or, equivalently, the linear initial value problem

$$v'(t) = Av(t), \quad t > 0, \qquad v(0) \text{ given},$$

involving the sectorial operator

$$A : \big\{ v \in \mathcal{C}^2([0,1]) : \partial_x v(0) = \partial_x v(1) = 0 \big\} \longrightarrow \mathcal{C}([0,1]) : v \longmapsto \Big(\frac{a}{2}\, \partial_{xx} + I \Big) v.$$

The spectrum of A just consists of the real eigenvalues $(\lambda_n)_{n=0}^{\infty}$ where

$$\lambda_n = 1 - \frac{a}{2}\, n^2 \pi^2, \qquad n \geq 0,$$

see also EDMUNDS & EVANS [8, Chapter XI.2]. Hence, the part of $\sigma(A)$ contained in the positive complex plane is not empty. This implies that the stationary solution $\overline{U} = 1$ is unstable, see Definition 6.4. Moreover, the spectrum of A does not intersect the imaginary axis whenever the parameter $a > 0$ fulfils

$$a \neq \frac{2}{n^2 \pi^2} \qquad \text{for all } n \geq 1.$$

We remark that for a function $y(t)$ which does not depend on the position x the initial-boundary value problem (2.10) simplifies to

$$y'(t) = \log(y(t)), \quad t > 0, \qquad y(0) \text{ given}, \tag{2.13}$$

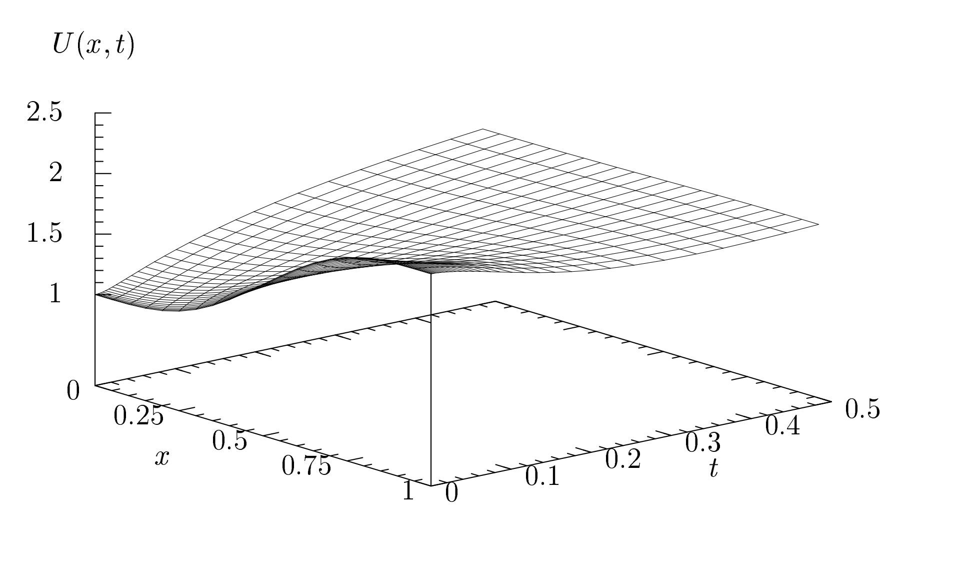

Figure 2.2.: Numerical solution for $V_0(x) = -20x^7 + 70x^6 - 84x^5 + 35x^4 + 1$

wherefore we obtain the following simple equation

$$z'(t) = z(t), \quad t > 0,$$

for the linearization. For the latter, it follows at once that any initial value $z_0 \in \boldsymbol{R}$ leads to a solution which is defined backward in time and converges to the origin as the time goes to $-\infty$, i.e. it holds

$$z(t) = e^t z_0, \quad t \leq 0, \qquad \text{and} \qquad \lim_{t \to -\infty} z(t) = 0.$$

However, there is a blow up for positive times whenever $z_0 \neq 0$. For the nonlinear ordinary differential equation (2.13) a similar behaviour is observed for initial values in $\{y_0 \in \boldsymbol{R},\, y_0 > 0\}$, and, equally, this is the case for the original initial-boundary value problem (2.10) if the initial state is in the set

$$\mathcal{U} = \{U_0 : [0,1] \to \boldsymbol{R} \text{ constant},\, U_0 > 0\},$$

the *locally unstable manifold* of the stationary solution $\overline{U} = 1$.

For our numerical experiment we discretized the initial-boundary value problem (2.10) in space by standard finite differences on an equidistant grid with meshwidth $\Delta x = 10^{-3}$. The resulting first order system was integrated up to the time T by means of the backward Euler method. In addition, we applied the variable stepsize implementation RADAU5 of

h	1/10	1/20	1/40	1/80	1/160	1/320
$\kappa_h(U_0)$	2.312	2.040	1.668	1.379	1.209	1.126
$\kappa_h(V_0)$	1.609	1.427	1.256	1.139	1.072	1.037
$\kappa_h(2)$	1.032	1.016	1.008	1.004	1.002	1.001

Table 2.1.: Numerical convergence order κ_h for U_0 and V_0 from (2.14) and $U_0 = 2$

the 3-stage Radau IIA method from Hairer & Wanner [14] in order to obtain a more precise approximation to the exact solution of (2.10). This approximation is needed for computing the numerical convergence order κ_h in a standard way from the quotients of the errors. In order to indicate the dependence of κ_h on the actual initial value U_0 we employ the notation $\kappa_h(U_0)$.

In view of our convergence result for the backward Euler method from Theorem 5.3, we chose the initial values

$$U_0(x) = -2x^7 + 7x^6 - 8.4x^5 + 3.5x^4 + 1.9, \tag{2.14a}$$

$$V_0(x) = -20x^7 + 70x^6 - 84x^5 + 35x^4 + 1. \tag{2.14b}$$

The progression of the numerical solution obtained by Radau5 is displayed in Figure 2.1 and 2.2, respectively. For the stepsizes

$$h = h_0/2^k, \qquad 0 \leq k \leq 6,$$

with initial time-step $h_0 = 1/5$, we performed the backward Euler time integration up to $T = 1$. The numerically observed convergence order approaches one as h is refined. This can be seen from Table 2.1. Note that U_0 is positive on the interval $[0, 1]$ and that its derivatives up to order 3 vanish at $x = 0$ and $x = 1$. Thus, U_0 lies in $\mathcal{D}$ and further fulfils $F(U_0) = 0$. An application of Lunardi [25, Theorem 8.1.1] to the initial value problem

$$u''(t) = F'(u(t))u'(t), \quad t > 0, \qquad u'(0) = F(U_0),$$

ensures that u'' is bounded on $\mathcal{X}$. The analogous arguments are valid for V_0. Consequently, Theorem 5.3 is applicable in both cases.

As mentioned above, for constant initial values the exact solution of (2.10) does not depend on the position wherefore the partial differential equation in (2.10) reduces to the ordinary differential equation (2.13). For the initial value $U_0 = 2$ we solved the original problem numerically, and we determined the numerical convergence order, see Table 2.1.

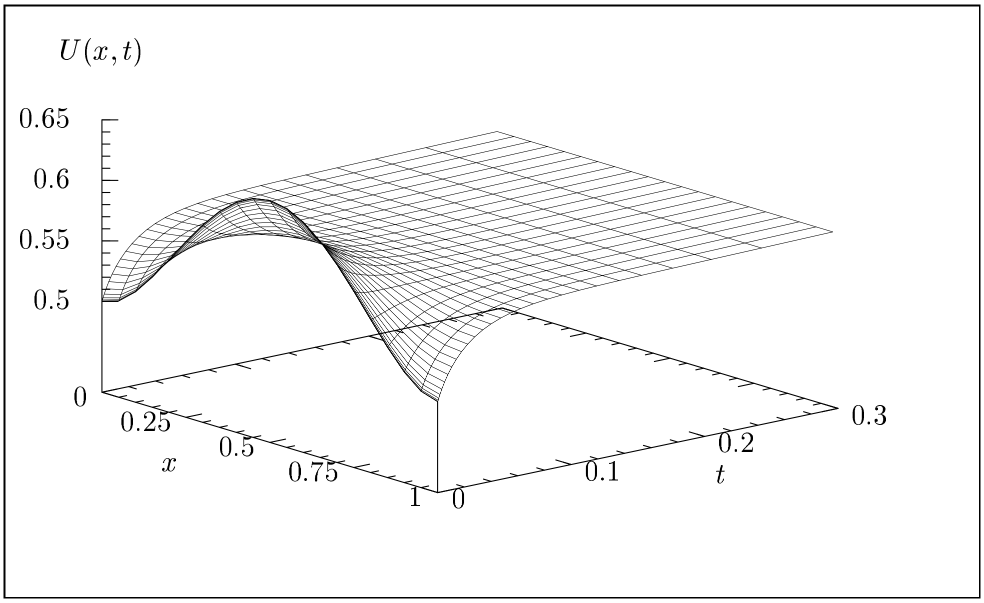

Figure 2.3.: Numerical solution for $U_0(x) = 1/2 + 2x^2(1-x)^2$

2.3. Combustion of a solid fuel

We denote by $U : [0,1] \times [0,\infty) \to \boldsymbol{R} : (x,t) \mapsto U(x,t)$ the temperature of a burning solid fuel as for example of coal. For a given initial state U_0 the evolution of U is mathematically modelled by the following nonlinear initial-boundary value problem

$$\begin{aligned} &\partial_t U(x,t) = \partial_x \Big(k\big(\partial_x U(x,t)\big) \partial_x U(x,t) \Big) + \varphi(U(x,t)), \\ &\partial_x U(0,t) = \partial_x U(1,t) = 0, \qquad U(x,0) = U_0(x), \qquad 0 < x < 1, \quad t > 0. \end{aligned} \tag{2.15}$$

The positive diffusion coefficient k depends on the values of $\partial_x U$, and the real-valued function φ varies with the temperature U. We suppose that k is twice differentiable with bounded second derivative and that φ is differentiable with locally Lipschitz continuous first derivative. Then, the function

$$f : \boldsymbol{R}^3 \longrightarrow \boldsymbol{R} : (\zeta, \eta, \vartheta) \longmapsto k'(\eta)\, \eta\, \vartheta + k(\eta)\, \vartheta + \varphi(\zeta)$$

is differentiable with partial derivatives

$$\begin{aligned} \partial_\zeta f(\zeta,\eta,\vartheta) &= \varphi'(\zeta), \\ \partial_\eta f(\zeta,\eta,\vartheta) &= k''(\eta)\, \eta\, \vartheta + 2k'(\eta)\, \vartheta, \\ \partial_\vartheta f(\zeta,\eta,\vartheta) &= k'(\eta)\, \eta + k(\eta), \end{aligned}$$

and, provided that k satisfies the uniform ellipticity condition

$$\partial_\vartheta f(\zeta,\eta,\vartheta) = k'(\eta)\,\eta + k(\eta) \geq \kappa > 0 \qquad \text{for all } \eta \in \boldsymbol{R}, \tag{2.16}$$

the partial differential equation in (2.15) is of parabolic type. As in the preceding section, we choose the function spaces

$$\mathcal{X} = \mathcal{C}([0,1]) \qquad \text{and} \qquad \mathcal{D} = \big\{ v \in \mathcal{C}^2([0,1]) : \partial_x v(0) = \partial_x v(1) = 0 \big\}.$$

Then, with $u(t) = U(t,\cdot)$ and $F : \mathcal{D} \to \mathcal{X} : v \mapsto F(v) = f(v, \partial_x v, \partial_{xx} v)$, the nonlinear initial-boundary value problem (2.15) takes the form

$$u'(t) = F(u(t)), \quad t > 0, \qquad u(0) \text{ given}. \tag{2.17}$$

Under the above smoothness requirements on k and φ the derivative

$$\begin{aligned} D_v F(v) = \varphi'(v)\, I &+ \big(k''(\partial_x v)\, \partial_x v\, \partial_{xx} v + 2\, k'(\partial_x v)\, \partial_{xx} v \big) \partial_x \\ &+ \big(k'(\partial_x v)\, \partial_x v + k(\partial_x v) \big) \partial_{xx} \end{aligned} \tag{2.18}$$

exists for $v \in \mathcal{C}^2([0,1])$ and is Lipschitz continuous as a map from $\mathcal{D}$ to $\mathcal{X}$. Notice for example that for arbitrary elements $v, w \in \mathcal{D}$ and $z \in \mathcal{D}$ the bound

$$\begin{aligned} \big\| \big(\varphi'(v) - \varphi'(w)\big) z \big\|_{\mathcal{X}} &\leq \sup_{0 \leq x \leq 1} \big| \varphi'(v(x)) - \varphi'(w(x)) \big| \cdot \sup_{0 \leq x \leq 1} |z(x)| \\ &\leq L_{\varphi'} \sup_{0 \leq x \leq 1} |v(x) - w(x)| \cdot \|z\|_\infty \leq L_{\varphi'} \|v - w\|_\infty \|z\|_\infty \leq L_{\varphi'} \|v - w\|_{\mathcal{D}} \|z\|_{\mathcal{D}} \end{aligned}$$

is valid. Here, $L_{\varphi'}$ denotes the Lipschitz constant of φ'. Furthermore, we have

$$\begin{aligned} \big\| \big(k(\partial_x v) - k(\partial_x w)\big) \partial_{xx} z \big\|_{\mathcal{X}} &\leq \sup_{0 \leq x \leq 1} \big| k(\partial_x v(x)) - k(\partial_x w(x)) \big| \cdot \sup_{0 \leq x \leq 1} |\partial_{xx} z(x)| \\ &\leq L_k \|\partial_x (v - w)\|_\infty \|\partial_{xx} z\|_\infty \leq L_k \|v - w\|_{\mathcal{D}} \|z\|_{\mathcal{D}}. \end{aligned}$$

From similar considerations the Lipschitz continuity of $D_v F$ follows. Due to the generation result in $\mathcal{C}([0,1])$, the ellipticity condition (2.16) implies that $D_v F(v)$ is a sectorial operator in $\mathcal{X}$ for every $v \in \mathcal{D}$, see (2.8) and below. This shows that the right-hand side F of (2.17) fulfils Assumption 1.1.

For the subsequent considerations, we specify the diffusion coefficient k and the map φ arising in the right-hand side of (2.15) as follows

$$k(z) = z^2 + 1 \qquad \text{and} \qquad \varphi(z) = -z(z - 1/2)(z - 1) \qquad \text{for } z \in \boldsymbol{R}.$$

Clearly, k satisfies (2.16) with $\kappa = 1$. In the present situation, (2.15) has the constant steady state solutions

$$\overline{U} = 0, \qquad \overline{U} = 1/2 \qquad \text{and} \qquad \overline{U} = 1.$$

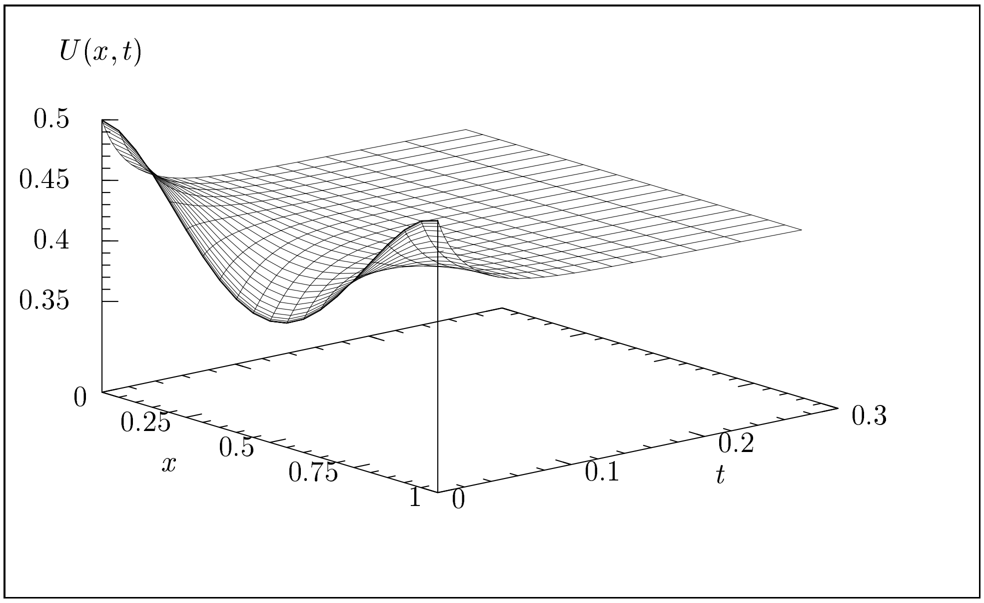

Figure 2.4.: Numerical solution for $V_0(x) = 1/2 - 2x^2(1-x)^2$

Due to the fact that $\varphi'(z) = 3z(1-z) - 1/2$, we have

$$D_v F(\overline{U}) = \varphi'(\overline{U})\, I + k(0)\, \partial_{xx} = \begin{cases} \partial_{xx} - 1/2\, I & \text{if } \overline{U} = 0 \text{ or } \overline{U} = 1, \\ \partial_{xx} + 1/4\, I & \text{if } \overline{U} = 1/2. \end{cases}$$

As linearization of (2.15) around the equilibrium point $\overline{U}$, we thus obtain

$$\begin{aligned} &\partial_t \widetilde{U}(x,t) = \partial_{xx} \widetilde{U}(x,t) + \varphi'(\overline{U})\big(\widetilde{U}(x,t) - \overline{U}\big), \\ &\partial_x \widetilde{U}(0,t) = \partial_x \widetilde{U}(1,t) = 0, \qquad \widetilde{U}(x,0) = \widetilde{U}_0(x), \qquad 0 < x < 1, \quad t > 0. \end{aligned}$$

The eigenvalues $(\lambda_n)_{n=0}^{\infty}$ of $D_v F(\overline{U})$ equal $\lambda_n = 1/4 - n^2\pi^2$ if $\overline{U} = 1/2$ and

$$\lambda_n = -1/2 - n^2\pi^2 \leq -1/2 < 0 \qquad \text{if} \quad \overline{U} = 0 \quad \text{or} \quad \overline{U} = 1.$$

Consequently, $\overline{U} = 1/2$ is unstable whereas the equilibrium points $\overline{U} = 0$ and $\overline{U} = 1$ are exponentially asymptotically stable. Roughly spoken, this means that any solution U of the initial-boundary value problem (2.15) with initial state U_0 lying nearby $\overline{U} = 1$, e.g., decays exponentially fast towards the equilibrium point

$$\big\|U(\cdot,t) - 1\big\|_{\mathcal{C}^2([0,1])} \leq C e^{-\omega t} \big\|U_0 - 1\big\|_{\mathcal{C}^2([0,1])}, \qquad t \geq 0,$$

with exponent $\omega = 1/2$, see also Theorem 7.1.

h	1/1	1/2	1/4	1/8	1/16	1/32	1/64
ω_h	0.406	0.446	0.471	0.485	0.492	0.496	0.498

Table 2.2.: Numerical contraction factor ω_h for U_0 and V_0 from (2.19)

In order to discretize (2.15) with initial value

$$U_0(x) = \frac{1}{2} + 2x^2(1-x)^2 \qquad \text{or} \qquad V_0(x) = \frac{1}{2} - 2x^2(1-x)^2, \tag{2.19}$$

respectively, in space and time, we applied standard finite differences on an equidistant grid with meshwidth $\Delta x = 10^{-3}$ and the backward Euler method with stepsize $h = 1/2^k$ for $0 \le k \le 6$. The integration was performed up to the time $T = 40$. Since

$$U_0'(x) = 4x(1-x)(1-2x) = -V_0(x),$$

U_0 and V_0 clearly satisfy the homogeneous Neumann boundary conditions. For the initial state U_0 we observe exponentially fast convergence towards $\overline{U} = 1$, whereas the numerical solution with initial value V_0 decays exponentially fast against the equilibrium point $\overline{U} = 0$. The numerical contraction factor ω_h which is the same for U_0 and V_0, approaches $\omega = 1/2$ as h is refined, see Table 2.2. The numerical solutions obtained by means of Radau5 are displayed in Figures 2.3 and 2.4.

2.4. Melting of ice

In the following, we consider the initial boundary value problem

$$\begin{aligned} &\partial_t V(y,t) = \partial_{yy} V(y,t) + \varphi\big(V(y,t), \partial_y V(y,t)\big), \quad 0 < y < b(t), \quad t > 0, \\ &V(0,t) = V(b(t),t) = 0, \quad t > 0, \qquad V(y,0) = V_0(y), \quad 0 < y < b(t). \end{aligned} \tag{2.20}$$

The actual position of the time-dependent boundary b is determined by the ordinary differential equation

$$\partial_t b(t) = \psi\big(b(t), V(b(t),t), \partial_y V(b(t),t)\big), \qquad t > 0,$$

with initial condition $b(0) = 1$.

In order to transform the problem (2.20) to the unit interval, we change the variables (y,t) to (x,t), and we replace V with

$$U(x,t) = V(y,t) \qquad \text{where} \quad y = b(t)x.$$

Then, the function $U : [0,1] \times [0,\infty) \to \boldsymbol{R}$ is related to V in the following way

$$\partial_t V(y,t) = \partial_t U(x,t) - \frac{x\,\partial_t b(t)}{b(t)}\,\partial_x U(x,t),$$
$$\partial_y V(y,t) = \frac{\partial_x U(x,t)}{b(t)} \qquad \text{and} \qquad \partial_{yy} V(y,t) = \frac{\partial_{xx} U(x,t)}{b(t)^2}.$$

Consequently, the map $(U,b) : [0,1] \times [0,\infty) \to \boldsymbol{R}^2 : (x,t) \to (U(x,t), b(t))$ solves a nonlinear initial-boundary value problem comprising the partial differential equations

$$\begin{aligned} \partial_t U(x,t) &= \frac{\partial_{xx} U(x,t)}{b(t)^2} + \varphi\left(U(x,t), \frac{\partial_x U(x,t)}{b(t)}\right) + \frac{x\,\partial_t b(t)}{b(t)}\,\partial_x U(x,t), \\ \partial_t b(t) &= \psi\left(b(t), U(1,t), \frac{\partial_x U(1,t)}{b(t)}\right), \qquad 0 < x < 1, \quad t > 0, \end{aligned} \tag{2.21a}$$

and the following boundary and initial conditions

$$U(0,t) = U(1,t) = 0, \quad U(x,0) = U_0(x), \quad b(0) = 1, \qquad 0 < x < 1, \quad t > 0. \tag{2.21b}$$

The Stefan problem which describes the melting of ice is of this form by choosing

$$\varphi = 0 \qquad \text{and} \qquad \psi(p,q,r) = -\beta r \quad \text{for some } \beta > 0.$$

For this special case, (2.21) leads to the following initial boundary value problem

$$\begin{aligned} &\partial_t U(x,t) = \frac{\partial_{xx} U(x,t)}{b(t)^2} + \frac{x\,\partial_t b(t)}{b(t)}\,\partial_x U(x,t), \qquad \partial_t b(t) = -\frac{\beta}{b(t)}\,\partial_x U(1,t), \\ &U(0,t) = U(1,t) = 0, \quad U(x,0) = U_0(x), \quad b(0) = 1, \qquad 0 < x < 1, \quad t > 0. \end{aligned} \tag{2.22}$$

This problem can be cast into our analytical framework by choosing

$$\mathcal{X} = C([0,1]) \times \boldsymbol{R} \qquad \text{and} \qquad \mathcal{D} = \left\{v \in C^2([0,1]) : v(0) = v(1) = 0\right\} \times \boldsymbol{R}.$$

Therewith, we get the following evolution equation for $\widetilde{u}(t) = (U(\cdot,t), b(t)) : [0,1] \to \boldsymbol{R}^2$

$$\widetilde{u}'(t) = F(\widetilde{u}(t)), \quad t > 0, \qquad \widetilde{u}(0) = (U_0, 1).$$

Here, for a function $\widetilde{v} = (v(\cdot), c) : [0,1] \to \boldsymbol{R}^2$ with $c \in \boldsymbol{R}$, $c \neq 0$, the right-hand side F is defined through

$$\big(F(\widetilde{v})\big)(x) = \begin{pmatrix} c^{-2}\partial_{xx} v(x) - \beta x c^{-2} \partial_x v(1) \partial_x v(x) \\ -\beta c^{-1} \partial_x v(1) \end{pmatrix}.$$

Due to the fact that the multiplication operator μ and the projection p, given by

$$\mu : \mathcal{C}([0,1]) \longrightarrow \mathcal{C}([0,1]) : v \longmapsto xv(x) \qquad \text{and} \qquad p : \mathcal{C}([0,1]) \longrightarrow \boldsymbol{R} : v \longmapsto v(1),$$

are linear and bounded, the map F satisfies Assumption 1.1 as long as $b(t)$ remains positive.

Part II.

Existence and convergence results

3. Preliminary remarks

The present part provides a detailed study of strongly stable Runge-Kutta methods for discretizing an abstract nonlinear initial value problem of the form

$$u'(t) = F(t, u(t)), \quad t > 0, \qquad u(0) \text{ given}, \tag{3.1}$$

in time. We develop existence and uniqueness results for the numerical solution, and we prove convergence on finite time intervals under certain smoothness assumptions on the analytical solution.

The results are presented for a general s-stage Runge-Kutta method, in compact vector notation written as

$$\begin{aligned} U_n' &= F(t_n + ch, U_n), \qquad F(t_n + ch, U_n) = \big(F(t_n + c_i h, U_{ni})\big)_{i=1}^{s}, \\ U_n &= \mathbb{1} u_n + h\mathcal{O}U_n', \qquad u_{n+1} = u_n + hb^T U_n', \end{aligned} \tag{3.2}$$

but we particularly focus on the backward Euler method which is given by the following recursion

$$\frac{u_{n+1} - u_n}{h} = F(t_{n+1}, u_{n+1}). \tag{3.3}$$

There, the proofs are less involved and thus the basic ideas become more transparently. In view of Section 5.2, it is useful to allow the initial value u_0 of the numerical method to be different from $u(0)$.

In Section 3.1 below we state the basic assumptions and we introduce some notation used throughout Part II. Then, we represent the defects of a Runge-Kutta method in terms of the exact solution, see Section 3.2. Chapter 4 contains the preliminary work for Chapter 5, the core of Part II, where we derive the main results on existence and convergence. Theorem 5.2 generalizes the convergence estimate for the backward Euler method given in GONZÁLEZ ET AL. [11, Theorem 3.3] to a general Runge-Kutta method of the form (3.2), and in Section 5.3.2 the existence and convergence result stated in OSTERMANN & THALHAMMER [27, Theorem 8] is proven in detail. Important auxiliary results for the present part are collected in Part IV.

3.1. Basic assumptions and notation

Henceforth, we always assume that the requirements of Section 1.1 on the initial value problem (3.1) and the numerical method (3.2) are satisfied.

Due to Assumption 1.1, for every $(t^*, v^*) \in [0,T] \times \mathcal{O}$, the graph norm of the sectorial operator $D_vF(t^*, v^*)$ is equivalent to the norm in $\mathcal{D}$. Thus, there exists a constant $C_\gamma > 0$ such that the following relation holds

$$C_\gamma^{-1} \|\cdot\|_{\mathcal{D}} \leq \|D_vF(t^*, v^*) \cdot \|_{\mathcal{X}} + \|\cdot\|_{\mathcal{X}} \leq C_\gamma \|\cdot\|_{\mathcal{D}}.$$

Further, we assume that the maximal stepsize $\overline{h}$ satisfies

$$\overline{h}a < 1,$$

where $a \in \boldsymbol{R}$ denotes the top of the sector containing the spectrum of the sectorial operator $D_vF(t^*, v^*)$. Due to the resolvent estimate (A.3), the inverse

$$\big(I - hD_vF(t^*, v^*)\big)^{-1}$$

is then well defined for all $0 < h \leq \overline{h}$.

In the sequel we introduce some useful sequence notation for the numerical and the analytical solution of (3.1). For some $h > 0$ and $N \in \boldsymbol{N}$ let

$$\underline{u} = (u_n)_{n=0}^N \qquad \text{and} \qquad \underline{\widehat{u}} = (\widehat{u}_n)_{n=0}^N = \big(u(t_n)\big)_{n=0}^N$$

denote the Runge-Kutta solution and the values of the exact solution u at the sequence $\underline{t} = (t_n)_{n=0}^N$ of time levels with $_n = nh$, and set

$$\underline{U} = \big(U_n\big)_{n=0}^{N-1} \qquad \text{and} \qquad \underline{\widehat{U}} = \big(\widehat{U}_n\big)_{n=0}^{N-1} \quad \text{where} \quad \widehat{U}_n = \big(\widehat{U}_{ni}\big)_{i=1}^s = \big(u(t_n + c_ih)\big)_{i=1}^s.$$

Analogously to this, the sequences containing the first derivatives of u at t_n or at $t_n + c_ih$, respectively, are denoted by

$$\underline{\widehat{u}}' = (\widehat{u}_n')_{n=0}^N = \big(u'(t_n)\big)_{n=0}^N \qquad \text{and} \qquad \underline{\widehat{U}}' = \big(\widehat{U}_n'\big)_{n=0}^{N-1} \quad \text{with} \quad \widehat{U}_n' = \big(u'(t_n + c_ih)\big)_{i=1}^s.$$

The errors of the numerical solution are defined through

$$\underline{e} = (e_n)_{n=0}^N = (u_n - \widehat{u}_n)_{n=0}^N,$$

and the errors of the stages U_n and U_n' of the Runge-Kutta method are given by

$$\underline{E} = (E_n)_{n=0}^{N-1} = \big(U_n - \widehat{U}_n\big)_{n=0}^{N-1} \qquad \text{and} \qquad \underline{E}' = (E_n')_{n=0}^{N-1} = \big(U_n' - \widehat{U}_n'\big)_{n=0}^{N-1}.$$

The exact solution of (3.1) does not satisfy the numerical scheme identically. Inserting u into (3.2) yields

$$\begin{aligned} \widehat{U}'_n &= F(t_n + ch, \widehat{U}_n), \qquad & F(t_n + ch, \widehat{U}_n) &= \big(F(t_n + c_i h, \widehat{U}_{ni})\big)_{i=1}^{s}, \\ \widehat{U}_n &= \mathbb{1}\,\widehat{u}_n + h\mathcal{Q}\widehat{U}'_n + D_n, \qquad & \widehat{u}_{n+1} &= \widehat{u}_n + hb^T\widehat{U}'_n + d_{n+1}, \end{aligned} \tag{3.4a}$$

which defines the defects

$$\underline{d} = (d_n)_{n=1}^{N} \qquad \text{and} \qquad \underline{D} = (D_n)_{n=0}^{N-1} \tag{3.4b}$$

of the Runge-Kutta method.

3.2. Representation of the defects

The following lemma shows that the regularity of the analytical solution of (3.1) determines the properties of the defects $\underline{d}$ and $\underline{D}$ given by (3.4). More precisely, it provides a representation for d_n involving the derivative of u of order $p+1$ and for D_{ni} in terms of $u^{(q+1)}$ where p and q denote the order and the stage order of the method.

Lemma 3.1. *Assume that the analytical solution u of (3.1) is sufficiently smooth on $[0,T]$. Suppose further that the Runge-Kutta solution u_n given by (3.2) exists for all $0 \le n \le N$ where $N \in \boldsymbol{N}$ is defined through $t_N \le T < t_{N+1}$. Then, the following statements hold. If the method is of order p, the relation*

$$d_{n+1} = \frac{h^{p+1}}{p!} \int_0^1 (1-\tau)^{p-1} \Big((1-\tau) u^{(p+1)}(t_n + \tau h) - p \sum_{j=1}^{s} b_j c_j^p u^{(p+1)}(t_n + \tau c_j h) \Big)\, \mathrm{d}\tau$$

is valid for $0 \le n \le N-1$. Moreover, the defects of the internal stages D_{ni} are given by

$$D_{ni} = \frac{h^{q+1}}{q!} \int_0^1 (1-\tau)^{q-1} \Big((1-\tau) c_i^{q+1} u^{(q+1)}(\tau_{ni}) - q \sum_{j=1}^{s} a_{ij} c_j^q u^{(q+1)}(\tau_{nj}) \Big)\, \mathrm{d}\tau,$$

where $\tau_{ni} = t_n + \tau c_i h$ for $1 \le i \le s$, whenever the stage order of the method is q, $0 \le n \le N-1$.

Proof. Let $h > 0$ and set $\tau_{ni} = t_n + \tau c_i h$ for $1 \le i \le s$. A Taylor series expansion yields

$$\begin{aligned} u(t_n + c_i h) &= \sum_{k=0}^{m} \frac{u^{(k)}(t_n)}{k!}\,(c_i h)^k + \frac{1}{m!} \int_0^1 (1-\tau)^m u^{(m+1)}(\tau_{ni}) (c_i h)^{m+1}\, \mathrm{d}\tau \\ u'(t_n + c_j h) &= \sum_{k=1}^{m} \frac{u^{(k)}(t_n)}{(k-1)!}\,(c_j h)^{k-1} + \frac{1}{(m-1)!} \int_0^1 (1-\tau)^{m-1} u^{(m+1)}(\tau_{nj}) (c_j h)^m\, \mathrm{d}\tau \end{aligned}$$

for all $1 \leq i, j \leq s$. From (3.4) we consequently receive the relation

$$\begin{aligned} d_{n+1} &= \widehat{u}_{n+1} - \widehat{u}_n - h \sum_{j=1}^{s} b_j \widehat{U}_{nj}' = \sum_{k=1}^{m} \Big(\frac{1}{k} - \sum_{j=1}^{s} b_j c_j^{k-1} \Big) \frac{u^{(k)}(t_n)}{(k-1)!}\, h^k \\ &+ \frac{h^{m+1}}{m!} \int_0^1 (1-\tau)^{m-1} \Big((1-\tau) u^{(m+1)}(t_n + \tau h) - m \sum_{j=1}^{s} b_j c_j^m u^{(m+1)}(\tau_{nj}) \Big)\, \mathrm{d}\tau \end{aligned} \tag{3.5a}$$

and furthermore the identity

$$\begin{aligned} D_{ni} &= \widehat{U}_{ni} - \widehat{u}_n - h \sum_{j=1}^{s} a_{ij} \widehat{U}_{nj}' = \sum_{k=1}^{m} \Big(\frac{c_i^k}{k} - \sum_{j=1}^{s} a_{ij} c_j^{k-1} \Big) \frac{u^{(k)}(t_n)}{(k-1)!}\, h^k \\ &+ \frac{h^{m+1}}{m!} \int_0^1 (1-\tau)^{m-1} \Big((1-\tau) c_i^{m+1} u^{(m+1)}(\tau_{ni}) - m \sum_{j=1}^{s} a_{ij} c_j^m u^{(m+1)}(\tau_{nj}) \Big)\, \mathrm{d}\tau. \end{aligned} \tag{3.5b}$$

If the Runge-Kutta method has order p and stage order q, by setting $m = p$ in (3.5a) and $m = q$ in (3.5b), respectively, only the expressions involving integrals remain and the statement of the lemma follows. $\square$

Remark 3.1. As a consequence of Lemma 3.1 we receive the following bounds for the defects, provided that the analytical solution $u : [0, T] \to \mathcal{D}$ of equation (3.1) is sufficiently smooth. As before, we define $N \in \boldsymbol{N}$ through $t_N \leq T < t_{N+1}$. Then, for $r = \min\{p, q+1\}$ it holds

$$\begin{aligned} &\|D_n\|_{\mathcal{D}^s} + \|d_{n+1}\|_{\mathcal{D}} \leq C h^r \big\| u^{(r)} \big\|_{\mathcal{B}([t_n, t_{n+1}], \mathcal{D})}, && 0 \leq n \leq N-1, \\ &\|d_{n+1}\|_{\mathcal{X}} \leq C h^{r+1} \big\| u^{(r+1)} \big\|_{\mathcal{B}([t_n, t_{n+1}], \mathcal{X})}, && 0 \leq n \leq N-1, \end{aligned}$$

with constants C depending on the method, see Relation (3.5). The corresponding estimates for the (α, α)-Hölder norms are

$$\begin{aligned} &\|\underline{D}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{N}_0, \mathcal{D}^s)} + \|\underline{d}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{N}, \mathcal{D})} \leq C h^r \big\| u^{(r)} \big\|_{\mathcal{C}_\alpha^\alpha([0,T], \mathcal{D})}, \\ &\|\underline{d}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{N}, \mathcal{X})} \leq C h^{r+1} \big\| u^{(r+1)} \big\|_{\mathcal{C}_\alpha^\alpha([0,T], \mathcal{X})}, \end{aligned}$$

where $\mathcal{N} = \{t_n = nh : n \in \boldsymbol{N}, 1 \leq n \leq N\}$ and $\mathcal{N}_0 = \{t_n = nh : n \in \boldsymbol{N}, 0 < n \leq N-1\}$.

4. Step-by-step approach

In the present chapter, we construct the numerical solution of (3.1) by fixed point iteration. For this, we first use in Section 4.1 a local approach based on the Runge-Kutta scheme (3.2), and then in Section 4.2 a global approach relying on a modification of the discrete variation-of-constants formula. However, since both approaches use the fact that the domain of the linearization is dense in the underlying Banach space $\mathcal{X}$, in general, they do not provide any information on the length of the time interval of existence. Nevertheless, the results of Section 4.1 are needed in Section 5.2 for proving an existence result which remains valid for finite times, and the ideas arising in Section 4.2 are of use in Chapter 7 for studying the dynamical behaviour of Runge-Kutta time discretizations near an equilibrium point.

4.1. Solving nonlinear equations by fixed point iteration

At first, we state a local existence result for the backward Euler method. In order to perform a single step in (3.3), we linearize the nonlinearity F on the right-hand side of (3.1), and we use a standard fixed point iteration. The corresponding result for general Runge-Kutta methods is given in Lemma 4.2 below.

Lemma 4.1. (González et al. [11, Lemma 5.2]) *Choose some $(t^*, v^*) \in [0,T] \times \mathcal{O}$ and $R_0 > 0$. Then there are constants $\tau_0 > 0$, $0 < \delta_0 < \varrho_0 \leq R_0$, and $\overline{h}_0 > 0$ such that the following holds. For times $t \in [0,T]$ with $|t + h - t^*| \leq \tau_0$, for initial values $v_0 \in \mathcal{O}$ with $\|v_0 - v^*\|_{\mathcal{D}} \leq \delta_0$, and for stepsizes $0 < h \leq \overline{h}_0$, the equation*

$$\frac{v - v_0}{h} = F(t + h, v) \tag{4.1}$$

possesses a unique solution $v \in \mathcal{O}$ such that $\|v - v^*\|_{\mathcal{D}} \leq \varrho_0$. The quantities $\tau_0, \delta_0, \varrho_0$ and $\overline{h}_0$ can be chosen uniformly for (t^*, v^*) belonging to a relatively compact subset of $[0,T] \times \mathcal{O}$.

Proof. Set $A = D_vF(t^*, v^*)$ and $G(t,v) = F(t,v) - Av$ for $v \in \mathcal{O}$, and assume that Condition (1.2) from Assumption 1.1 is satisfied for $R_0, K > 0$ and $L > 0$. Then, for the sectorial operator A the resolvent estimate

$$\big\|(\lambda I - A)^{-1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \frac{M}{|\lambda - a|} \qquad \text{if} \quad |\arg(\lambda - a)| < \varphi \tag{4.2}$$

holds with constants $a \in \boldsymbol{R}$, $\pi/2 < \varphi < \pi$ and $M > 0$, see Definition A.2. Moreover, the nonlinear map G is Lipschitz continuous with respect to v and fulfils the estimate

$$\|G(t+h,v) - G(t+h,w)\|_{\mathcal{X}} \le \big(K\tau_0^\alpha + L\varrho_0\big)\|v - w\|_{\mathcal{D}} \tag{4.3}$$

for all $|t+h-t^*| \le \tau_0$, $\|v - v^*\|_{\mathcal{D}} \le \varrho_0 \le R_0$, and $\|w - v^*\|_{\mathcal{D}} \le \varrho_0 \le R_0$, see Lemma E.1.

We solve (4.1) by the fixed point iteration

$$\begin{aligned} \varphi : \mathcal{V} = \{v \in \mathcal{O} : \|v - v^*\|_{\mathcal{D}} \le \varrho_0\} &\longrightarrow \mathcal{V} \\ v &\longmapsto \varphi(v) = (I - hA)^{-1}\big(v_0 + hG(t+h,v)\big). \end{aligned}$$

For this, we have to show that φ is contractive and that $\varphi(v) \in \mathcal{V}$ for

$$|t+h-t^*| \le \tau_0, \quad \|v_0 - v^*\|_{\mathcal{D}} \le \delta_0, \quad \|v - v^*\|_{\mathcal{D}} \le \varrho_0 \quad \text{and} \quad 0 < h \le \overline{h}_0,$$

if τ_0, δ_0, ϱ_0 and $\overline{h}_0$ are chosen sufficiently small.

Note that there exist constants $\overline{M}$ and $\overline{h}_0 > 0$ such that

$$\big\|h(I - hA)^{-1}\big\|_{\mathcal{D}\leftarrow\mathcal{X}} + \big\|(I - hA)^{-1}\big\|_{\mathcal{D}\leftarrow\mathcal{D}} \le \overline{M} \qquad \text{for } 0 < h \le \overline{h}_0. \tag{4.4}$$

With the help of the bound

$$\big\|(I - hA)^{-1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \frac{M}{|1 - ha|} \le \begin{cases} M(1 - \overline{h}_0 a)^{-1} & \text{if } a > 0, \\ M & \text{if } a \le 0, \end{cases}$$

see (4.2), this statement follows easily from the relations

$$\begin{aligned} \big\|h(I - hA)^{-1}\big\|_{\mathcal{D}\leftarrow\mathcal{X}} &\le C_\gamma \big\|hA(I - hA)^{-1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le C_\gamma\big(1 + \big\|(I - hA)^{-1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\big), \\ \big\|(I - hA)^{-1}\big\|_{\mathcal{D}\leftarrow\mathcal{D}} &\le C_\gamma^2 \big\|(I - hA)^{-1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}. \end{aligned}$$

(i) From the Estimates (4.3) and (4.4) we have for $|t+h-t^*| \le \tau_0$ and $v, w \in \mathcal{V}$

$$\begin{aligned} \|\varphi(v) - \varphi(w)\|_{\mathcal{D}} &\le \big\|h(I - hA)^{-1}\big\|_{\mathcal{D}\leftarrow\mathcal{X}} \|G(t+h,v) - G(t+h,w)\|_{\mathcal{X}} \\ &\le \overline{M}\big(K\tau_0^\alpha + L\varrho_0\big)\|v - w\|_{\mathcal{D}}, \end{aligned}$$

whenever $0 < h \le \overline{h}_0$. Thus, φ is contractive with factor $\overline{M}\big(K\tau_0^\alpha + L\varrho_0\big) \le 1/2$ for τ_0 and ϱ_0 sufficiently small.

(ii) Adding $\varphi(v^*)$ and applying (i) yields the following estimate for $v \in \mathcal{V}$

$$\|\varphi(v) - v^*\|_{\mathcal{D}} \leq \|\varphi(v) - \varphi(v^*)\|_{\mathcal{D}} + \|\varphi(v^*) - v^*\|_{\mathcal{D}} \leq \varrho_0/2 + \|\varphi(v^*) - v^*\|_{\mathcal{D}}.$$

Since $\overline{\mathcal{D}} = \mathcal{X}$, there exists some $\widetilde{x} \in \mathcal{D}$ such that

$$\overline{M}\big\|Av^* + G(t+h, v^*) - \widetilde{x}\big\|_{\mathcal{X}} \leq \varrho_0/4. \tag{4.5}$$

By using (4.4), we consequently receive for δ_0 and $0 < h \leq \overline{h}_0$ small enough

$$\begin{aligned}\|\varphi(v^*) - v^*\|_{\mathcal{D}} \leq{}& \big\|(I - hA)^{-1}\big\|_{\mathcal{D}\leftarrow\mathcal{D}}\|v_0 - v^*\|_{\mathcal{D}} + h\big\|(I - hA)^{-1}\big\|_{\mathcal{D}\leftarrow\mathcal{D}}\big\|\widetilde{x}\big\|_{\mathcal{D}} \\ &+ \big\|h(I - hA)^{-1}\big\|_{\mathcal{D}\leftarrow\mathcal{X}}\big\|Av^* + G(t+h, v^*) - \widetilde{x}\big\|_{\mathcal{X}} \\ \leq{}& \overline{M}\delta_0 + h\overline{M}\big\|\widetilde{x}\big\|_{\mathcal{D}} + \varrho_0/4 \leq \varrho_0,\end{aligned}$$

i.e. φ maps $\mathcal{V}$ to $\mathcal{V}$. Thus, the fixed point theorem of Banach provides the existence of a unique solution $v \in \mathcal{V}$ of equation (4.1).

The quantities R_0, K and L can be taken uniformly for (t^*, v^*) belonging to relatively compact subsets of $[0,T] \times \mathcal{O}$. Moreover, the equivalence of $\|\cdot\|_{\mathcal{D}}$ with the graph norm of $D_vF(t^*, v^*)$ is also uniform on such sets. Therefore, the statement of the lemma follows. □

The preceding lemma, applied to $(t^*, v^*) = (0, u(0))$ and $v_0 = u_n$, guarantees the existence of a unique backward Euler solution $u_{n+1} \in \mathcal{O}$ of (3.1), as long as the maximal stepsize $\overline{h}_0 > 0$ is small enough and $(t_n, u_n) \in [0,T] \times \mathcal{O}$ lies sufficiently close to $(0, u(0))$.

Notice that this step-by-step approach is not suited to construct the numerical solution on a finite time interval. If $\|u_0 - u(0)\|_{\mathcal{D}} \leq \delta_0$, after one step we have

$$\|u_1 - u(0)\|_{\mathcal{D}} \leq C\delta_0.$$

Owing to the dependence of the constant C on some element $\widetilde{x} \in \mathcal{D}$ nothing is known on the size of C, see (4.5) and formula below.

An application of the following result with $(t^*, v^*) = (0, u(0))$, $V_0 = \mathbb{1}\, u_n$ and $v_0 = u_n$ shows that a unique solution $u_{n+1} \in \mathcal{O}$ of the Runge-Kutta scheme (3.2) exists for sufficiently small stepsizes $\overline{h}_0 > 0$ and times t_n if $\|u_n - u(0)\|_{\mathcal{D}}$ is small enough. As before, we linearize F and use a fixed point iteration for the internal stages which then implies the desired result for the Runge-Kutta solution.

Lemma 4.2. *Choose $(t^*, v^*) \in [0,T] \times \mathcal{O}$ and $R_0 > 0$. Then there are constants $\tau_0 > 0$, $0 < \delta_0 < r_0 < \varrho_0 \leq R_0$ and $\overline{h}_0 > 0$ such that, for all $t \in [0,T]$ with the property $\max\{|t + c_i h - t^*| : 1 \leq i \leq s\} \leq \tau_0$, for $V_0 \in \mathcal{O}^s$ with $\|V_0 - \mathbb{1}\, v^*\|_{\mathcal{D}^s} \leq \delta_0$, and for $0 < h \leq \overline{h}_0$, a unique solution $V = (V_1, \ldots, V_s)^s \in \mathcal{O}^s$ of the equation*

$$V = V_0 + h\mathcal{Q}F(t + ch, V) \qquad \textit{where } F(t+ch, V) = \big(F(t + c_i h, V_i)\big)_{i=1}^s, \tag{4.6}$$

exists with $\|V - 1\!\!1 v^*\|_{\mathcal{D}^s} \le r_0$*. Moreover, if* $\|v_0 - v^*\|_{\mathcal{D}} \le \delta_0$*, the element* v *given by*

$$v = v_0 + hb^T F(t + ch, V)$$

satisfies the bound $\|v - v^*\|_{\mathcal{D}} \le \varrho_0$*. The constants* $\tau_0, \delta_0, r_0, \varrho_0$ *and* $\overline{h}_0$ *can be chosen uniformly for* (t^*, v^*) *lying in a relatively compact subset of* $[0, T] \times \mathcal{O}$*.*

Proof. As is the proof of Lemma 4.1 we set $A = D_v F(t^*, v^*)$ and $G(t, v) = F(t, v) - Av$ for $v \in \mathcal{O}$, and we suppose that Condition (1.2) from Assumption 1.1 holds with $R_0, K > 0$ and $L > 0$. Let

$$\mathcal{J} = (\mathcal{I} - hA\mathcal{Q})^{-1}.$$

Then, the fixed point iteration for (4.6) is defined through

$$\begin{aligned} \psi : \mathcal{W} = \{V \in \mathcal{O}^s : \|V - V^*\|_{\mathcal{D}^s} \le r_0\} &\longrightarrow \mathcal{W} \\ V &\longmapsto \mathcal{J}\big(V_0 + h\mathcal{Q}G(t + ch, V)\big), \end{aligned}$$

where $G(t + ch, V) = \big(G(t + c_i h, V_i)\big)_{i=1}^s$ and $V^* = 1\!\!1 v^*$. We choose $\overline{M}$ satisfying

$$\|h\mathcal{Q}\mathcal{J}\|_{\mathcal{D}^s \leftarrow \mathcal{X}^s} + \|\mathcal{J}\|_{\mathcal{D}^s \leftarrow \mathcal{D}^s} + \|\mathcal{Q}\mathcal{J}\|_{\mathcal{D}^s \leftarrow \mathcal{D}^s} \le \overline{M} \tag{4.7}$$

for $0 < h \le \overline{h}_0$ small enough, see Result C.1.

(i) Then, for τ_0 and r_0 small enough, ψ is contractive with contraction factor $\kappa < 1$. For $\max\{|t + c_i h - t^*| : 1 \le i \le s\} \le \tau_0$, $V, W \in \mathcal{W}$ and $0 < h \le \overline{h}_0$, this follows from (4.7) and Lemma E.1

$$\begin{aligned} \|\psi(V) - \psi(W)\|_{\mathcal{D}^s} &\le \|h\mathcal{Q}\mathcal{J}\|_{\mathcal{D}^s \leftarrow \mathcal{X}^s} \|G(t + ch, V) - G(t + ch, W)\|_{\mathcal{X}^s} \\ &\le \overline{M}(K\tau_0^\alpha + Lr_0)\|V - W\|_{\mathcal{D}^s}. \end{aligned}$$

(ii) Furthermore, we have the estimate

$$\|\psi(V) - V^*\|_{\mathcal{D}^s} \le r_0/2 + \|\psi(V^*) - V^*\|_{\mathcal{D}^s}.$$

Choosing $\widetilde{X} \in \mathcal{D}^s$ such that

$$\overline{M}\big\|AV^* + G(t + ch, V^*) - \widetilde{X}\big\|_{\mathcal{X}^s} \le r_0/4,$$

and applying again (4.7) yields the bound

$$\begin{aligned} \|\psi(V^*) - V^*\|_{\mathcal{D}^s} &\le \big\|\mathcal{J}\big\|_{\mathcal{D}^s \leftarrow \mathcal{D}^s}\|V_0 - V^*\|_{\mathcal{D}^s} + h\big\|\mathcal{Q}\mathcal{J}\big\|_{\mathcal{D}^s \leftarrow \mathcal{D}^s}\big\|\widetilde{X}\big\|_{\mathcal{D}^s} \\ &\quad + \big\|h\mathcal{Q}\mathcal{J}\big\|_{\mathcal{D}^s \leftarrow \mathcal{X}^s}\big\|AV^* + G(t + ch, V^*) - \widetilde{X}\big\|_{\mathcal{X}^s} \\ &\le \overline{M}\delta_0 + h\overline{M}\big\|\widetilde{X}\big\|_{\mathcal{D}} + r_0/4 \le r_0, \end{aligned}$$

whenever $0 < h \le \overline{h}_0$ and δ_0 are sufficiently small. Thus, the fixed point theorem of Banach is applicable. The same arguments as in the proof of Lemma 4.1 show that the constants can be chosen uniformly in a relatively compact subset around (t^*, v^*).

Remark further that the identity

$$F(t + ch, V) = h^{-1}\mathcal{A}^{-1}(V - V_0)$$

holds. We thus receive the estimate

$$\begin{aligned}\|v - v^*\|_{\mathcal{D}} &\le \|v_0 - v^*\|_{\mathcal{D}} + \left\|b^T\mathcal{A}^{-1}(V - V^*)\right\|_{\mathcal{D}} + \left\|b^T\mathcal{A}^{-1}(V^* - V_0)\right\|_{\mathcal{D}} \\ &\le (1 + C)\delta_0 + Cr_0 \le \varrho_0\end{aligned}$$

with $\left\|b^T\mathcal{A}^{-1}\right\|_{\boldsymbol{R}\leftarrow\boldsymbol{R}^n} \le C$. This concludes the proof of the lemma. □

4.2. Local existence results

In the present section, we show the existence and uniqueness of a local strict solution of equation (3.1) in the function space $\mathcal{C}^\alpha_\alpha((0, \delta], \mathcal{D})$ for some $\delta > 0$. Our result stated in Theorem 4.1 is Theorem 8.1.1 of Lunardi [25] and relies on a global fixed point iteration involving the modified variation-of-constants formula given in Section D.1. We then adopt the proof for the analytical solution to the backward Euler method, and we further give the generalization to strongly stable Runge-Kutta methods. This is done in Proposition 4.1 where we show that for all initial values $u_0 \in \mathcal{O}$ there exists some $t_N > 0$ such that (3.2) or (3.3), respectively, are uniquely solvable for all stepsizes $0 < h \le \overline{h}$ as long as $0 < t_{n+1} \le t_N$. The existence results of Theorem 4.1 and Proposition 4.1 are based on the fact that the nonlinear operators Γ, Φ and Ψ given below are contractions on suitably chosen sets $\mathcal{B}$, $\mathcal{V}$ and $\mathcal{W}$ of (α, α)-Hölder continuous functions and sequences. With the help of these fixed point iterations we further sketch the derivation of a convergence result for smooth solutions.

Due to the fact that the length of the time interval of existence might be very small, the results for the numerical solution are not satisfactory to practical applications, and, in particular, the convergence results are a first approach for further statements given later on. Nevertheless, the results presented here give a deeper insight in the analytical framework in use and serve as a guide line for Chapters 5 and 7.

Choose $R > 0$, and let $0 < \varrho \le R$. For constructing the analytical solution of equation (3.1), we set

$$A = D_vF(0, u(0)) \qquad \text{and} \qquad G(t, v) = F(t, v) - Av \quad \text{for } v \in \mathcal{O}, \tag{4.8a}$$

and we write problem (3.1) as a formally semilinear equation

$$u'(t) = Au(t) + G(t, u(t)), \quad t > 0, \qquad u(0) \text{ given}.$$

For some $\delta > 0$ the closed subset $\mathcal{B}$ of $\mathcal{C}^{\alpha}_{\alpha}((0,\delta],\mathcal{D})$, endowed with $\|\cdot\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{D})}$, is defined through

$$\mathcal{B} = \{v \in \mathcal{C}([0,\delta],\mathcal{D}) \cap \mathcal{C}^{\alpha}_{\alpha}((0,\delta],\mathcal{D}) : v(0) = u(0), \|v - u(0)\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{D})} \leq \varrho\}. \tag{4.8b}$$

The constants δ and ϱ are specified later. For functions w on $[0,\delta]$ taking values in $\mathcal{O}$, we define the convolution operator $\mathcal{K}(w)$ through

$$\big(\mathcal{K}(w)\big)(t) = \int_0^t e^{(t-\tau)A}\big(G(\tau,w(\tau)) - G(t,w(t))\big)\,\mathrm{d}\tau + A^{-1}\big(e^{tA} - I\big)G(t,w(t)), \tag{4.8c}$$

and we introduce the notation

$$E : [0,\infty) \longrightarrow \mathcal{L}(\mathcal{X}) : t \longmapsto e^{tA}. \tag{4.8d}$$

With these abbreviations, the fixed point iteration Γ on $\mathcal{B}$, mapping a function v to the solution of the nonhomogeneous problem

$$w'(t) = Aw(t) + G(t,v(t)), \quad t > 0, \qquad w(0) = u(0),$$

takes the compact form

$$\Gamma : \mathcal{B} \longrightarrow \mathcal{B} : v \longmapsto \Gamma(v) = Eu(0) + \mathcal{K}(G(\cdot,v)). \tag{4.8e}$$

For the construction of the numerical solution we choose $(0,u_0)$ as center of linearization, i.e. we set

$$G(t,v) = F(t,v) - Av \quad \text{for } v \in \mathcal{O} \qquad \text{with} \quad A = D_vF(0,u_0), \tag{4.9a}$$

and we apply then a Runge-Kutta method to the initial value problem

$$w'(t) = Aw(t) + G(t,v(t)), \quad t > 0, \qquad w(0) = u_0.$$

In case of the backward Euler method this yields the nonlinear operator

$$\Phi : \mathcal{V} \longrightarrow \mathcal{V} : \underline{v} \longmapsto \Phi(\underline{v}) = \underline{r}u_0 + \mathcal{K}_1(G(\underline{t},\underline{v})), \tag{4.9b}$$

see also (D.7), with $\underline{r}$ and $G(\underline{t},\underline{v})$ given by

$$\underline{r} = \big((I - hA)^{-n}\big)_{n=0}^{N} \qquad \text{and} \qquad G(\underline{t},\underline{v}) = \big(G(t_n,v_n)\big)_{n=0}^{N}. \tag{4.9c}$$

For a sequence $\underline{w} = (w_n)_{n=0}^N$ the convolution operator $\mathcal{K}_1$ is defined through

$$\begin{aligned}\mathcal{K}_1(\underline{w}) = \Bigg(h\sum_{\nu=0}^{n-1}(I-hA)^{-(n-\nu)}\big(w_{\nu+1} - w_n\big)\\ + A^{-1}\big((I-hA)^{-n} - I\big)w_n\Bigg)_{0\leq n\leq N}.\end{aligned} \tag{4.9d}$$

As domain of Φ we choose the closed subset

$$\mathcal{V} = \big\{ \underline{v} = (v_n)_{n=0}^N \in \mathcal{C}_\alpha^\alpha(\mathcal{N}, \mathcal{D}) : v_0 = u_0, \|\underline{v} - \underline{u}_0\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} \le \varrho \big\} \subset \mathcal{C}_\alpha^\alpha(\mathcal{N}, \mathcal{D}) \tag{4.9e}$$

where $\mathcal{N} = \{t_n = nh : n \in \boldsymbol{N}, 0 < n \le N\}$ for some $N \in \boldsymbol{N}$, $N > 0$, and $\underline{u}_0 = (u_0)_{n=0}^N$.

In order to write the fixed point iteration for a general Runge-Kutta method (3.2) more compactly, we henceforth employ the abbreviation

$$\mathcal{J} = (\mathcal{I} - hA\mathcal{Q})^{-1}.$$

By the discrete version of the modified variation-of-constants formula (D.9) we have

$$\begin{aligned} AU_n &= \mathcal{J}\mathbb{1} R(hA)^n A u_0 + hA\mathcal{Q}\mathcal{J} G(t_n + ch, V_n) \\ &\qquad + h \sum_{\nu=0}^{n-1} \mathcal{J}\mathbb{1} b^T \mathcal{J} A R(hA)^{n-\nu-1} \big(G(t_\nu + ch, V_\nu) - G(t_{n-1} + ch, V_{n-1}) \big) \\ &\qquad\quad + \mathcal{J}\mathbb{1} b^T \mathcal{J} \big(b^T \mathcal{J}\mathbb{1} \big)^{-1} \big(R(hA)^n - I \big) G(t_{n-1} + ch, V_{n-1}) \end{aligned}$$

where $G(t_n + ch, V_n) = \big(G(t_n + c_i h, V_{ni}) \big)_{i=1}^s$ for $V_n = (V_{n1}, \dots, V_{ns}) \in \mathcal{O}^s$. As before, we linearize F around $(0, u_0)$, and we set

$$G(t, v) = F(t, v) - Av \quad \text{for } v \in \mathcal{O} \qquad \text{where} \quad A = D_v F(0, u_0). \tag{4.10a}$$

We construct the internal stages U_n by using the fixed point iteration

$$\Psi : \mathcal{W} \longrightarrow \mathcal{W} : \underline{V} \longmapsto \Psi(\underline{V}) = \mathcal{J}\mathbb{1} \underline{R} u_0 + \mathcal{K}_2\big(G(\underline{t} + ch, \underline{V}) \big). \tag{4.10b}$$

Here, we employ the following sequence notation

$$\begin{gathered} \underline{R} = \big(R(hA)^n \big)_{n=0}^N \qquad \text{and} \qquad G(\underline{t} + ch, \underline{V}) = \big(G(t_n + ch, V_n) \big)_{n=0}^N \\ \text{with} \quad G(t_n + ch, V_n) = \big(G(t_n + c_i h, V_{ni}) \big)_{i=1}^s. \end{gathered} \tag{4.10c}$$

For $\underline{W} = (W_n)_{n=0}^N$ with $W_n \in \mathcal{X}^s$ the convolution operator $\mathcal{K}_2$ is defined through

$$\begin{aligned} \mathcal{K}_2(\underline{W}) = \Bigg(& h \sum_{\nu=0}^{n-1} \mathcal{J}\mathbb{1} b^T \mathcal{J} R(hA)^{n-\nu-1} \big(W_\nu - W_{n-1} \big) \\ & + A^{-1} \mathcal{J}\mathbb{1} b^T \mathcal{J} \big(b^T \mathcal{J}\mathbb{1} \big)^{-1} \big(R(hA)^n - I \big) W_{n-1} + h\mathcal{Q}\mathcal{J} W_n \Bigg)_{0 \le n \le N}. \end{aligned} \tag{4.10d}$$

Furthermore, the closed subset $\mathcal{W} \subset \mathcal{C}_\alpha^\alpha(\mathcal{N}, \mathcal{D}^s)$ is given by

$$\mathcal{W} = \{ \underline{V} = (V_n)_{n=0}^N \in \mathcal{C}_\alpha^\alpha(\mathcal{N}, \mathcal{D}^s) : V_0 = U_0, \|\underline{V} - \underline{U}_0\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}^s)} \le \varrho \} \tag{4.10e}$$

where $\mathcal{N} = \{t_n = nh : n \in \boldsymbol{N}, 0 < n \leq N\}$ for some $N \in \boldsymbol{N}$, $N > 0$. The abbreviation $\underline{U}_0 \in \mathcal{O}^s$ denotes the constant sequence with all components equal u_0.

In the sequel, we show that the fixed point iterations (4.8) to (4.10) are contractions for suitable choices of ϱ and δ or t_N, respectively, and for sufficiently small stepsizes $0 < h \leq \overline{h}$.

Unfortunately, in Theorem 4.1, due to the argumentation of the proof, the length δ of the time interval where the analytical solution of (3.1) exists, is limited by the fact that the quantity

$$\big\|(E - I)F(\cdot, u(0))\big\|_{\mathcal{C}_\alpha^\alpha((0,\delta],\mathcal{X})} \tag{4.11a}$$

has to be chosen sufficiently small, see Formula (4.14) and below. Correspondingly, for the backward Euler method and a general Runge-Kutta method, the quantities

$$\begin{aligned} &\big\|(\underline{r} - I)F(\underline{t}, \underline{u}_0)\big\|_{\mathcal{C}_\alpha^\alpha(\mathcal{N},\mathcal{X})} \qquad \text{and} \\ &\big\|\mathcal{J}\mathbb{1}\, b^T \mathcal{J}\big(b^T \mathcal{J}\mathbb{1}\big)^{-1}(\underline{R} - I)F(\underline{t} + ch, \underline{U}_0)\big\|_{\mathcal{C}_\alpha^\alpha(\mathcal{N},\mathcal{X}^s)} \end{aligned} \tag{4.11b}$$

have to be small enough. Due to Lemma F.1, nothing is known about the size of δ and t_N in general. Consequently, these existence results are valid possibly for very short times only, and thus they are not of use in practice. Nevertheless, for studying solutions in the neighbourhood of an equilibrium point, the ideas of Theorem 4.1 and Proposition 4.1 prove to be very useful. In this case, by linearizing around the equilibrium point, the quantities from (4.11) do not arise and therefore the existence of the exact and the numerical solution is ensured even on unbounded time intervals, see Chapter 7.

The local existence result for the analytical solution of (3.1), stated in the following, is Lunardi [25, Theorem 8.1.1]. In view of the analogous result for the numerical solution, given in Proposition 4.1, it is useful to adapt the proof of [25, Theorem 8.1.1] to our needs and to perform it in a more direct way.

Theorem 4.1. (Lunardi [25, Theorem 8.1.1]) *For every $u(0) \in \mathcal{O}$ there is some $\delta > 0$ such that there exists a strict solution $u \in \mathcal{C}([0,\delta],\mathcal{D}) \cap \mathcal{C}^1([0,\delta],\mathcal{X})$ of problem* (3.1)*, and further $u \in \mathcal{C}_\alpha^\alpha((0,\delta],\mathcal{D})$.*

Proof. Obviously, u is a mild local solution of (3.1) if and only if u is a fixed point of the nonlinear operator Γ defined by (4.8). To apply the fixed point theorem of Banach which ensures the existence of a unique fixed point of Γ in $\mathcal{B}$ we have to prove the following.

(i) Γ is a contraction on $\mathcal{B}$:

  (a) Γ is well-defined, i.e. $\Gamma(v) \in \mathcal{C}([0,\delta],\mathcal{D}) \cap \mathcal{C}_\alpha^\alpha((0,\delta],\mathcal{D})$ for all $v \in \mathcal{B}$.

  (b) Γ is contractive on $\mathcal{B}$, thus there exists some $\kappa < 1$ such that for all $v, w \in \mathcal{B}$

$$\|\Gamma(v) - \Gamma(w)\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} \leq \kappa \|v - w\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})}.$$

(c) Γ maps $\mathcal{B}$ into $\mathcal{B}$, and in particular the estimate $\|\Gamma(v) - u(0)\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} \leq \varrho$ holds whenever $\|v - u(0)\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} \leq \varrho$ for $v \in \mathcal{B}$.

In the sequel, we show that δ and ϱ can be chosen such that these requirements hold. Notice that, due to

$$G(\cdot, v) \in \mathcal{C}([0,\delta], \mathcal{X}) \qquad \text{and} \qquad u = \Gamma(u) \in \mathcal{C}([0,\delta], \mathcal{D}),$$

a mild solution $u \in \mathcal{B}$ of (3.1) on $[0,\delta]$ is already a strict solution, see Result B.1.

(i) In the first place, we have

$$G(\cdot, v) \in \mathcal{C}([0,\delta], \mathcal{X}) \cap \mathcal{C}_\alpha^\alpha((0,\delta], \mathcal{X}) \quad \text{for } v \in \mathcal{B}. \tag{4.12}$$

By assumption v as well as F and thus the composition $F(\cdot, v)$ are continuous maps. In addition Av is continuous since $v(t) \in \mathcal{D}$. This yields $G(\cdot, v) \in \mathcal{C}([0,\delta], \mathcal{X})$. The second statement of (4.12) is a direct consequence of Lemma E.2. Moreover, we receive

$$\Gamma(v) \in \mathcal{C}([0,\delta], \mathcal{D}) \cap \mathcal{C}_\alpha^\alpha((0,\delta], \mathcal{D}) \quad \text{for } v \in \mathcal{B}. \tag{4.13}$$

From the modified variation-of-constants formula (D.5) it can be seen that $\Gamma(v)$ is continuous in $\mathcal{D}$ for $t \in (0,\delta]$. Therefore it remains to study the case $t = 0$. For some constant $M_1 > 0$ it holds

$$\left\| A e^{tA} \right\|_{\mathcal{X} \leftarrow \mathcal{X}} \leq M_1 t^{-1}, \qquad 0 < t \leq T,$$

see Result A.1. The bound

$$\begin{aligned}
\big\| \big(\Gamma(v)\big)(t) - \big(\Gamma(v)\big)(0) \big\|_{\mathcal{D}} &\leq C_\gamma \big\| \big(e^{tA} - I\big)\big(Au(0) + G(t, v(t))\big) \big\|_{\mathcal{X}} \\
&\quad + C_\gamma \int_0^t \big\| A e^{(t-\tau)A} \big\|_{\mathcal{X} \leftarrow \mathcal{X}} \big\| G(\tau, v(\tau)) - G(\tau, w(\tau)) \big\|_{\mathcal{X}} \, \mathrm{d}\tau \\
&\leq C_\gamma \big\| \big(e^{tA} - I\big)\big(Au(0) + G(t, v(t))\big) \big\|_{\mathcal{X}} \\
&\quad + C_\gamma M_1 [G(\cdot, v)]_{\mathcal{C}_\alpha^\alpha(\mathcal{X})} \int_0^t (t - \tau)^{-1+\alpha} \tau^{-\alpha} \, \mathrm{d}\tau
\end{aligned}$$

implies $\|(\Gamma(v))(t) - (\Gamma(v))(0)\|_{\mathcal{D}} \to 0$ for $t \to 0$, i.e. $\Gamma(v) \in \mathcal{C}([0,\delta], \mathcal{D})$. From Lemma D.1 we know that $\Gamma(v) \in \mathcal{C}_\alpha^\alpha((0,\delta], \mathcal{D})$ since $G(v) \in \mathcal{C}_\alpha^\alpha((0,\delta], \mathcal{X})$ by (4.12). This yields (4.13).

(ii) Statement (4.12) implies $G(\cdot, v) - G(\cdot, w) \in \mathcal{C}_\alpha^\alpha((0,T], \mathcal{X})$. Thus, it follows from the Lemmas D.3 and E.3 that there exist constants $K > 0$ and $L > 0$ such that for elements $v, w \in \mathcal{B}$ the estimate

$$\begin{aligned}
\|\Gamma(v) - \Gamma(w)\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} = \big\| \mathcal{K}\big(G(\cdot, v) - G(\cdot, v)\big) \big\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} &\leq C_\gamma C \|G(\cdot, v) - G(\cdot, w)\|_{\mathcal{C}_\alpha^\alpha(\mathcal{X})} \\
&\leq 2 C_\gamma C (K\delta^\alpha + L\varrho) \|v - w\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})}
\end{aligned}$$

holds with constant $C > 0$ depending on α and on the bounds for the analytic semigroup. Since C depends continuously on δ, we can choose ϱ and δ such that Γ is contractive with contraction factor $2C_\gamma C(K\delta^\alpha + L\varrho) \le 1/2$.

(iii) From the above considerations we have $\Gamma(v) \in \mathcal{C}([0,\delta],\mathcal{D}) \cap \mathcal{C}_\alpha^\alpha((0,\delta],\mathcal{D})$, and the requirement $\Gamma(v)(0) = u(0)$ is obviously satisfied. On the other hand, the map $\Gamma(u(0)) - u(0)$ solves the nonhomogeneous problem

$$z'(t) = Az(t) + F(t,u(0)), \quad t > 0, \qquad z(0) = 0,$$

and thus can be represented by

$$A\big(\Gamma(u(0)) - u(0)\big) = AJ + (E - I)F(\cdot,u(0)). \tag{4.14}$$

Here the function J is defined through the integral

$$J(t) = \int_0^t e^{(t-\tau)A}\big(F(\tau,u(0)) - F(t,u(0))\big)\,\mathrm{d}\tau.$$

The norm of J in $\mathcal{C}_\alpha^\alpha((0,\delta],\mathcal{D})$ is bounded by a constant times δ^α, and $\|(E-I)x\|_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}$ tends to 0 as δ goes to 0 for all $x \in \mathcal{X}$, see Lemmas F.1 and F.2. Therefore the estimate

$$\|\Gamma(u(0)) - u(0)\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} \le \varrho/2$$

holds for δ sufficiently small. Combining this fact with the contractivity of Γ, yields

$$\begin{aligned}\|\Gamma(v) - u(0)\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} &\le \|\Gamma(v) - \Gamma(u(0))\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} + \|\Gamma(u(0)) - u(0)\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} \\ &\le 1/2\,\|v - u(0)\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} + \varrho/2 \le \varrho\end{aligned} \tag{4.15}$$

for $v \in \mathcal{B}$. Consequently the fixed point iteration Γ maps $\mathcal{B}$ into $\mathcal{B}$. This finally concludes the proof of the lemma. □

Slight modifications of the proceding proof show the corresponding result for the numerical solution of (3.1). For this, we assume that the resolvent bound

$$\big\|(\lambda I - D_vF(0,u_0))^{-1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \frac{M}{|\lambda - a|}, \qquad |\arg(\lambda - a)| < \varphi,$$

holds for constants $a \in \boldsymbol{R}$, $\pi/2 < \varphi < \pi$ and $M > 0$, see also Definition A.2, and that the Runge-Kutta method (3.2) is strongly $A(\vartheta)$-stable with $\vartheta > \pi - \varphi$.

Proposition 4.1. *In the above situation, for every initial value $u_0 \in \mathcal{O}$ there exists some $t_N > 0$ such that the Runge-Kutta scheme (3.2) is uniquely solvable in $\mathcal{D}$ as long as $0 < t_{n+1} \le t_N$ whenever the stepsize $0 < h \le \overline{h}$ is chosen sufficiently small.*

Proof. The present proof is very close to that of the previous result Theorem 4.1. At first, we consider the special case of the backward Euler method.

1. For sequences $\underline{u} \in \mathcal{O}^N$ the regularity property $\underline{u} \in \mathcal{C}_\alpha^\alpha(\mathcal{N},\mathcal{D})$ follows immediately. Therefore, it remains to show that the map Φ given by (4.9) is contractive and that Φ maps $\mathcal{V}$ onto $\mathcal{V}$.

(i) We apply again the Lemmas D.3 and E.3 and obtain

$$\|\Phi(\underline{v}) - \Phi(\underline{w})\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} = \big\|\mathcal{K}_1\big(G(\underline{t},\underline{v}) - G(\underline{t},\underline{w})\big)\big\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} \leq 2C_\gamma C(Kt_N^\alpha + L\varrho)\|\underline{v} - \underline{w}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})}$$

for $\underline{v},\underline{w} \in \mathcal{V}$ and $0 < h \leq \overline{h}$. The constant C depends on the bounds for the stability function and on α. Consequently, Φ is contractive with contraction factor $\kappa \leq 1/2$ for ϱ and t_N sufficiently small.

(ii) By using the abbreviation

$$J_1(\underline{t}) = \left(h\sum_{\nu=0}^{n-1}(I - hA)^{-(n-\nu)}\big(F(t_{\nu+1},u_0) - F(t_n,u_0)\big)\right)_{0\leq n\leq N}$$

we get the identity

$$A\big(\Phi(\underline{u}_0) - \underline{u}_0\big) = AJ_1(\underline{t}) + (\underline{r} - I)F(\underline{t},\underline{u}_0).$$

An application of the Lemmas F.1 and F.2 yields

$$\|\Phi(\underline{u}_0) - \underline{u}_0\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} \leq \varrho/2$$

for t_N chosen sufficiently small. Together with the contractivity of Φ for elements $\underline{v} \in \mathcal{V}$ the bound

$$\begin{aligned}\|\Phi(\underline{v}) - \underline{u}_0\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} &\leq \|\Phi(\underline{v}) - \Phi(\underline{u}_0)\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} + \|\Phi(\underline{u}_0) - \underline{u}_0\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})}\\ &\leq 1/2\,\|\underline{v} - \underline{u}_0\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} + \varrho/2 \leq \varrho\end{aligned}$$

provided that $0 < h \leq \overline{h}$. Thus, the relation $\Phi(\mathcal{V}) \subset \mathcal{V}$ follows.

2. For a general Runge-Kutta method of the form (3.2), we have to show that the map $\Psi : \mathcal{W} \to \mathcal{W}$ given by (4.10) is a contraction.

(i) Similarly to before, we have for $\underline{V},\underline{W} \in \mathcal{W}$

$$\begin{aligned}\|\Psi(\underline{V}) - \Psi(\underline{W})\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}^s)} &= \big\|\mathcal{K}_2\big(G(\underline{t}+ch,\underline{V})) - G(\underline{t}+ch,\underline{W})\big)\big\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}^s)}\\ &\leq 2C_\gamma C(Kt_N^\alpha + L\varrho)\|\underline{V} - \underline{W}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}^s)}\end{aligned}$$

from Lemmas D.3 and E.3. The constant C depends on the bounds for the stability function of the method and on α. Consequently, Ψ is contractive with contraction factor $\kappa \leq 1/2$ for constants ϱ and t_N chosen small enough and stepsizes $0 < h \leq \overline{h}$.

(ii) We define the sequence $J_2(\underline{t})$ through

$$J_2(\underline{t}) = \Big(h \sum_{\nu=0}^{n-1} \mathcal{J}\mathbb{1} b^T \mathcal{J} R(hA)^{n-\nu-1} \big(F(t_\nu + ch, U_0) - F(t_{n-1} + ch, U_0)\big) \\ + h\mathcal{A}\mathcal{J}F(t_n + ch, U_0)\Big)_{0 \le n \le N}.$$

Applying then Lemmas F.1 and F.2 to

$$A\big(\Psi(\underline{U}_0) - \underline{U}_0\big) = AJ_2(\underline{t}) + \mathcal{J}\mathbb{1} b^T \mathcal{J}\big(b^T \mathcal{J}\mathbb{1}\big)^{-1}(\underline{R} - I)F(\underline{t} + ch, \underline{U}_0)$$

shows that t_N can be chosen such that

$$\big\|\Psi(\underline{U}_0) - \underline{U}_0\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} \le \varrho/2$$

is fulfiled. Using the contractivity of Ψ gives

$$\begin{aligned}\big\|\Psi(\underline{V}) - \underline{U}_0\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} &\le \big\|\Psi(\underline{V}) - \Psi(\underline{U}_0)\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} + \big\|\Psi(\underline{U}_0) - \underline{U}_0\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} \\ &\le 1/2\big\|\underline{V} - \underline{U}_0\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} + \varrho/2 \le \varrho\end{aligned}$$

for elements $\underline{V} \in \mathcal{W}$ and proves that Ψ maps $\mathcal{W}$ onto $\mathcal{W}$ whenever $0 < h \le \overline{h}$ is sufficiently small.

Applying now the fixed point theorem of Banach yields the existence and the uniqueness of the internal stages U_n as long as $0 < t_n \le t_N$, and from this, the existence of a uniquely determined solution of (3.2) follows. This finally concludes the proof. $\square$

Provided that the analytical solution of equation (3.1) satisfies the following smoothness assumptions

$$u^{(r)} \in \mathcal{C}^\alpha_\alpha((0,\delta],\mathcal{D}) \quad \text{and} \quad u^{(r+1)} \in \mathcal{C}^\alpha_\alpha((0,\delta],\mathcal{X}), \qquad r = \min\{p, q+1\}, \tag{4.16}$$

where $t_N \le \delta$ denotes the quantity from Proposition 4.1, the order of convergence equals r. More precisely, we have the following error estimate for sufficiently small stepsizes $0 < h \le \overline{h}$ as long as $0 \le t_n \le t_N$

$$\|u_n - u(t_n)\|_{\mathcal{D}} \le C\Big(\|u_0 - u(0)\|_{\mathcal{D}} + \big\|u^{(r)}\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} \cdot h^r + \big\|u^{(r+1)}\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \cdot h^r\Big) \tag{4.17}$$

with constant C not depending on n or h. In the sequel we derive this convergence result for the backward Euler method and we sketch the proof for a general Runge-Kutta method.

Due to the fact that the nonlinear map Φ, defined by (4.9), is a contraction on $\mathcal{V}$ with contraction factor κ and that the backward Euler solution $\underline{u}$ is its fixed point, the triangular inequality yields the estimate

$$\begin{aligned}\|\underline{u}-\underline{\widehat{u}}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} &= \|\Phi(\underline{u})-\underline{\widehat{u}}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} \leq \|\Phi(\underline{u})-\Phi(\underline{\widehat{u}})\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} + \|\Phi(\underline{\widehat{u}})-\underline{\widehat{u}}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} \\ &\leq \kappa\|\underline{u}-\underline{\widehat{u}}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} + \|\Phi(\underline{\widehat{u}})-\underline{\widehat{u}}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})}\end{aligned}$$

which implies the error bound

$$\|u_n-\widehat{u}_n\|_{\mathcal{D}} \leq \frac{1}{1-\kappa}\|\Phi(\underline{\widehat{u}})-\underline{\widehat{u}}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})}. \tag{4.18}$$

The difference $\Phi(\underline{\widehat{u}})-\underline{\widehat{u}}$ fulfils the relation

$$\Phi(\underline{\widehat{u}})-\underline{\widehat{u}} = \underline{r}\big(u_0-u(0)\big)+\mathcal{K}_1\big(h^{-1}\underline{d}\big). \tag{4.19}$$

By definition, we namely have

$$\Phi(\underline{\widehat{u}}) = \underline{r}u_0+\mathcal{K}_1(G(\underline{t},\underline{\widehat{u}})). \tag{4.20a}$$

On the other hand we receive the following identity

$$\underline{\widehat{u}} = \underline{r}u(0)+\mathcal{K}_1(G(\underline{t},\underline{\widehat{u}}))+\mathcal{K}_1\big(h^{-1}\underline{d}\big) \tag{4.20b}$$

by inserting the exact solution in the backward Euler scheme and solving the resulting recursion. Taking the difference of (4.20a) and (4.20b) therefore gives (4.19). Note that especially for the backward Euler method the relation

$$\|\underline{d}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \leq h^2\big\|u''\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})}$$

holds, see Lemma 3.1. Together with the Lemmas D.2 and D.3 we thus get from (4.19)

$$\|\Phi(\underline{\widehat{u}})-\underline{\widehat{u}}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} \leq C\Big(\|u_0-u(0)\|_{\mathcal{D}}+\big\|u''\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})}\cdot h\Big)$$

with constant C depending on α and C_γ and on the bounds for the stability function. From (4.18) furthermore the error estimate

$$\|u_n-\widehat{u}_n\|_{\mathcal{D}} \leq C\big(\|u_0-u(0)\|_{\mathcal{D}}+\big\|u''\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})}\cdot h\big)$$

follows.

By using the same arguments as before, we receive the following relation for the stages of a general Runge-Kutta method

$$\big\|\underline{U}-\underline{\widehat{U}}\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} \leq \frac{1}{1-\kappa}\big\|\Psi\big(\underline{\widehat{U}}\big)-\underline{\widehat{U}}\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)}. \tag{4.21}$$

By definition, $\underline{W} = \Psi(\underline{\widehat{U}})$ is given by the scheme

$$W_n' = AW_n + G(t_n + ch, \widehat{U}_n), \qquad W_n = \mathbb{1}\, w_n + h\mathcal{O}W_n', \qquad w_{n+1} = w_n + hb^T W_n',$$

with initial value $w_0 = u_0$, and therefore the difference $\underline{Z} = \Psi(\underline{\widehat{U}}) - \underline{\widehat{U}}$ fulfils the scheme

$$Z_n' = AZ_n, \qquad Z_n = \mathbb{1}\, z_n + h\mathcal{O}Z_n' - D_n, \qquad z_{n+1} = z_n + hb^T Z_n' - d_{n+1}$$

with $z_0 = u_0 - u(0)$, see (3.4). A brief calculation yields the identity

$$Z_n = \mathcal{J}\mathbb{1}\, R(hA)^n(u_0 - u(0)) - h\sum_{\nu=0}^{n-1} \mathcal{J}\mathbb{1}\, R(hA)^{n-\nu-1}\big(b^T \mathcal{J} AD_\nu + h^{-1}d_{\nu+1}\big) - \mathcal{J}D_n.$$

Similar ideas to that arising in the proofs of the Lemmas D.2 and D.3, show the bound

$$\|\underline{Z}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} \leq C\big(\|u_0 - u(0)\|_{\mathcal{D}} + \|\underline{D}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} + \|h^{-1}\underline{d}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})}\big)$$

with constant C depending on α, C_γ, and on the bounds for the stability function of the Runge-Kutta method. From (4.21), due to Remark 3.1, we consequently have the following estimate for the internal stages of the method whenever u satisfies the smoothness requirements (4.16)

$$\big\|\underline{U} - \underline{\widehat{U}}\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} \leq C\Big(\|u_0 - u(0)\|_{\mathcal{D}} + \big\|u^{(r)}\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} \cdot h^r + \big\|u^{(r+1)}\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \cdot h^r\Big). \tag{4.22}$$

We show in Section 5.1.3 that the error of the Runge-Kutta solution, measured in $\|\cdot\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})}$, is essentially bounded by the quantity $\big\|\underline{U} - \underline{\widehat{U}}\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)}$ and by the (α,α)-Hölder norms of the defects in $\mathcal{D}$, see Estimate (5.16). From this result, the desired convergence estimate (4.17) follows.

5. Existence and convergence results for finite times

In the present chapter, we state the main results on existence and convergence. We show that for a Runge-Kutta method the order of convergence on finite time intervals is essentially the stage order plus one, under certain stability requirements on the method and smoothness assumptions on the analytical solution of the nonlinear problem (3.1). For our analysis, we employ two different approaches.

For the first approach we linearize the right-hand side F of the evolution equation (3.1) around the initial value, and we represent the numerical solution by the discrete modified variation-of-constants formula. This representation is then used in order to derive preliminary error bounds for the numerical solution. For the backward Euler method this is done in a direct way, whereas for a general Runge-Kutta method the proof is more involved. There, we first derive an error estimate for the internal stages and show then the error bound

$$\|u_n - u(t_n)\|_{\mathcal{D}} \le C\big(\|u_0 - u(0)\|_{\mathcal{D}} + h^r\big), \quad 0 \le t_n \le T, \qquad r = \min\{p, q+1\}, \tag{5.1}$$

for the solution of a strongly stable Runge-Kutta method with order p and stage order q, whenever the analytical solution satisfies

$$u^{(r)} \in \mathcal{C}_\alpha^\alpha((0,T],\mathcal{D}) \qquad \text{and} \qquad u^{(r+1)} \in \mathcal{C}_\alpha^\alpha((0,T],\mathcal{X})$$

and the stepsize $h > 0$ is chosen sufficiently small, see Section 5.1. For proving the existence and convergence result we then subdivide the considered time interval $[0,T]$ such that on each interval of the partition

$$0 = T_0 < T_1 < \cdots < T_j < T_{j+1} < \cdots < T_J = T$$

the quantity $\|u - u(T_k)\|_{\mathcal{C}_\alpha^\alpha((T_k,T_{k+1}],\mathcal{D})}$ is bounded from above by a certain quantity. On each interval $[T_j, T_{j+1}]$ we then combine a local approach and the preliminary convergence estimate (5.1). The desired result on $[0,T]$ finally follows by recursion. This program is carried out in Section 5.2.

An alternative existence and convergence result for finite times that avoids the consideration of the complicated (α,α)-Hölder norms is derived in Section 5.3 and relies on linearizing the right-hand side of (3.1) along the exact solution. We apply a Runge-Kutta method to the resulting problem which takes the form of a semilinear problem involving the time dependent sectorial operator $A(t) = D_v F(t, u(t))$, and we construct the numerical solution by a fixed point iteration based on a discrete variation-of-constants formula. Under the assumption that the analytical solution is Hölder continuous and under slightly stronger smoothness requirements on F, stability estimates for a strongly stable Runge-Kutta method are provided by GONZALEZ & PALENCIA [12, Theorem 1.1]. Whenever the analytical solution possesses the regularity properties

$$u^{(r)} \in \mathcal{B}([0,T], \mathcal{D}) \qquad \text{and} \qquad u^{(r+1)} \in \mathcal{B}([0,T], \mathcal{X}),$$

we obtain the following convergence estimate

$$\|u_n - u(t_n)\|_{\mathcal{D}} \le C\big(\|u_0 - u(0)\|_{\mathcal{D}} + h^r(1 + |\log(h)|)\big), \quad 0 \le t_n \le T, \tag{5.2}$$

where $r = \min\{p, q+1\}$, by using similar ideas to Section 4.2.

5.1. Preliminary convergence estimates

In the following, we derive preliminary error estimates for Runge-Kutta solutions of (3.1). At first, we consider in Section 5.1.1 the backward Euler method. We then generalize the error bound for the backward Euler solution to the stages of a general Runge-Kutta method of the form (3.2), see Section 5.1.2. Furthermore, it is shown in Section 5.1.3 that the error of the Runge-Kutta solution is bounded by the error of the stages and the defects of the method.

We henceforth denote by u the strict solution of equation (3.1) on some interval $[0, T]$ with the regularity property

$$u \in \mathcal{C}^\alpha_\alpha((0,T], \mathcal{D}),$$

see Chapter B, and we always assume that the Runge-Kutta solution

$$\underline{u} = (u_n)_{n=0}^N$$

of (3.1) with initial value $u_0 \in \mathcal{O}$ exists for some sequence of equidistant time levels

$$0 = t_0 < t_1 < \cdots < t_N \le T < t_{N+1}.$$

Clearly, $\underline{u}$ belongs to $\mathcal{C}^\alpha_\alpha(\mathcal{N}, \mathcal{D})$ for $\mathcal{N} = \{t_n = nh : n \in \boldsymbol{N}, 0 < t_n \le t_N\}$. Beyond, we suppose that the requirements of Section 1.1 on the evolution equation and the numerical method are valid, and we employ the notation introduced there. In particular, we denote by $K > 0$ and $L > 0$ the (α,α)-Hölder constant and the Lipschitz constant of the right-hand side F of (3.1), i.e. it holds

$$\|F(t,v)-F(s,v)\|_{\mathcal{X}} + \|D_vF(t,v)-D_vF(s,v)\|_{\mathcal{X}\leftarrow\mathcal{D}} \le K|t-s|^{\alpha}$$
$$\|D_vF(t,v)-D_vF(t,w)\|_{\mathcal{X}\leftarrow\mathcal{D}} \le L\|v-w\|_{\mathcal{D}}$$

for all $s,t \in [0,T]$ and $v,w \in \mathcal{O}$ with $\|v-u(0)\|_{\mathcal{D}} \le R$ and $\|w-u(0)\|_{\mathcal{D}} \le R$, see Assumption 1.1. As in the preceding chapter, we linearize the equation around $(0,u(0))$, and we set

$$A = D_vF(0,u(0)). \tag{5.3}$$

We denote by $a \in \boldsymbol{R}$, $\pi/2 < \varphi < \pi$ and $M > 0$ the constants arising in the resolvent estimate for the sectorial operator A

$$\big\|(\lambda I - A)^{-1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \frac{M}{|\lambda - a|}, \qquad |\arg(\lambda - a)| < \varphi,$$

see Definition A.2, and we assume that the Runge-Kutta method (3.2) of order $p \ge 1$ and stage order q is strongly $A(\vartheta)$-stable with $\vartheta > \pi - \varphi$. It is useful to introduce the abbreviation

$$\mathcal{J} = (\mathcal{I} - hA\mathcal{O}\!\iota)^{-1}$$

and to denote by $C_{\mathcal{J}}$ the bound of $\mathcal{J}$ on $\mathcal{X}^s$

$$\big\|(\mathcal{I} - hA\mathcal{O}\!\iota)^{-1}\big\|_{\mathcal{X}^s\leftarrow\mathcal{X}^s} \le C_{\mathcal{J}},$$

see Result C.1. Moreover, we make use of the notation introduced in Section 3.1, and we additionally denote by

$$\underline{u}(0) = \big(u(0)\big)_{n=0}^{N} \qquad \text{and} \qquad \underline{U}(0) = \big(U(0)\big)_{n=0}^{N} \quad \text{where} \quad U(0) = \big(u(0)\big)_{i=1}^{s}$$

the constant sequences with components equal $u(0)$.

5.1.1. Error estimate for the backward Euler solution

The error estimate for the backward Euler solution $\underline{u}$ of equation (3.1) is given in Lemma 5.1 below. For proving this auxiliary result, apart from the smoothness assumption $u'' \in \mathcal{C}^{\alpha}_{\alpha}((0,T],\mathcal{D})$, an essential requirement on the exact solution is that the quantity $u-u(0)$ does not exceed a certain size. In order to formulate this condition properly, in view of the discrete modified variation-of-constants formula (D.7), we introduce the following notation.

We denote by $\underline{r}$ the sequence containing the powers of the stability function of the backward Euler method, i.e.

$$\underline{r} = \big((I - hA)^{-n}\big)_{n=0}^{N},$$

and for $\underline{g} = (g_n)_{n=1}^{N} \in \mathcal{X}^N$ the convolution operator $\mathcal{K}$ is defined through

$$\mathcal{K}(\underline{g}) = \left(h \sum_{\nu=0}^{n-1} (I - hA)^{-(n-\nu)}(g_{\nu+1} - g_n) + A^{-1}\big((I - hA)^{-n} - I\big) g_n\right)_{0 \le n \le N}.$$

Then, due to the Lemmas D.2 and D.3, for all initial values $u_0 \in \mathcal{D}$ the relations

$$\|\underline{r} u_0\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{D})} \le C_0 \|u_0\|_{\mathcal{D}} \qquad \text{and} \qquad \|\mathcal{K}(\underline{g})\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{D})} \le C_1 \|\underline{g}\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{X})} \tag{5.4}$$

hold. The constants C_0 and C_1 depend on the bounds for the stability function of the backward Euler method reviewed in Result C.1, and further on α^{-1} and C_γ.

Now we are in the position to state the following result.

Lemma 5.1. (González et al. [11, Theorem 3.1]) *Suppose $u'' \in \mathcal{C}^{\alpha}_{\alpha}((0, T], \mathcal{X})$, and assume further that the analytical and the backward Euler solution of* (3.1) *fulfil the relations $\|\underline{\widehat{u}} - \underline{u}(0)\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{D})} \le \varrho$ and $\|\underline{u} - \underline{u}(0)\|_{\mathcal{B}(\mathcal{D})} \le \varrho$ such that*

$$2C_1(KT^{\alpha} + L\varrho) \le \kappa < 1$$

holds for some $0 < \varrho \le R$. Then the bound

$$\|\underline{u} - \underline{\widehat{u}}\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{D})} \le C\big(\|u_0 - u(0)\|_{\mathcal{D}} + \|u''\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{X})} \cdot h\big)$$

is valid with a constant C depending on the quantities C_0 and C_1 from (5.4) *and on the choice of κ.*

Proof. The backward Euler solution of equation (3.1) is given by the recursion

$$u_{n+1} = u_n + hF(t_{n+1}, u_{n+1}), \tag{5.5}$$

whereas the exact solution fulfils the numerical scheme with defects d_n

$$\widehat{u}_{n+1} = \widehat{u}_n + hF(t_{n+1}, \widehat{u}_{n+1}) + d_{n+1}. \tag{5.6}$$

As a first step, we linearize F around $(0, u(0))$, i.e. we set

$$A = D_v F(0, u(0)) \qquad \text{and} \qquad G(t, v) = F(t, v) - Av \quad \text{for } (t, v) \in [0, T] \times \mathcal{O}.$$

For the following, it is useful to introduce the sequence $\Delta G = (\Delta G_n)_{n=0}^{N}$ defined through

$$\Delta G_n = G(t_n, u_n) - G(t_n, \widehat{u}_n).$$

With this abbreviation, by taking the difference between (5.5) and (5.6) we then have

$$e_{n+1} = (I - hA)^{-1}\big(e_n + h\Delta G_{n+1} - d_{n+1}\big).$$

Solving this error recursion and employing the sequence notation introduced above, yields

$$\underline{e} = \underline{r}e_0 + \mathcal{K}(\Delta G) - \mathcal{K}\big(h^{-1}\underline{d}\big). \tag{5.7}$$

An application of the auxiliary result (5.4) and of Lemma E.3 with $(t^*, v^*) = (0, u(0))$ gives

$$\begin{aligned}\|\underline{e}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} &\le C_0\|e_0\|_{\mathcal{D}} + C_1\|\Delta G\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} + \big\|\mathcal{K}\big(h^{-1}\underline{d}\big)\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})}\\ &\le C_0\|e_0\|_{\mathcal{D}} + 2C_1(KT^\alpha + L\varrho)\|\underline{e}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} + \big\|\mathcal{K}\big(h^{-1}\underline{d}\big)\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})}.\end{aligned}$$

By assumption we have $2C_1(KT^\alpha + L\varrho) \le \kappa < 1$, and consequently

$$\|\underline{e}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} \le \frac{1}{1-\kappa}\Big(C_0\|e_0\|_{\mathcal{D}} + \big\|\mathcal{K}\big(h^{-1}\underline{d}\big)\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})}\Big)$$

follows. Especially for the backward Euler method the following relation holds

$$d_n = h^2\int_0^1 (-\tau)u''(t_{n-1} + \tau h)\,\mathrm{d}\tau,$$

see Lemma 3.1. Due to this, the requirement $u'' \in \mathcal{C}^\alpha_\alpha((0,T],\mathcal{X})$ together with (5.4) thus implies

$$\big\|\mathcal{K}\big(h^{-1}\underline{d}\big)\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} \le C_1\big\|h^{-1}\underline{d}\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \le C_1 h\|u''\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})}.$$

Therefore we finally have

$$\|\underline{e}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} \le \frac{1}{1-\kappa}\Big(C_0\|e_0\|_{\mathcal{D}} + C_1 h\|u''\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})}\Big)$$

which yields the desired result. □

5.1.2. Error estimate for the stages of a Runge-Kutta method

In the sequel, we extend the above ideas for the backward Euler method to the general case. In order to write the internal stages and the solution of a Runge-Kutta method more compactely, we define the sequence

$$\underline{R} = \big(R(hA)^n\big)_{n=0}^N$$

containing the powers of the stability function, and we further introduce the convolution operators $\mathcal{K}$ and $\mathcal{K}_1$. For $\underline{g} = \big(g(t_n + ch)\big)_{n=0}^N$ with $g(t_n + ch) \in \mathcal{X}^s$ the convolution operator $\mathcal{K}$ for the solution is defined by

$$\mathcal{K}(\underline{g}) = \Bigg(h\sum_{\nu=0}^{n-1} b^T\mathcal{J}R(hA)^{n-\nu-1}\big(g(t_\nu + ch) - g(t_{n-1} + ch)\big) \\ + A^{-1}b^T\mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1}\big(R(hA)^n - I\big)g(t_{n-1} + ch)\Bigg)_{0\le n\le N},$$

see also (D.9), and for $\underline{v} = (v_n)_{n=1}^N \in \mathcal{X}^N$ the operator $\mathcal{K}_1$ is given by

$$\mathcal{K}_1(\underline{v}) = \Bigg(h\sum_{\nu=0}^{n-1} R(hA)^{n-\nu-1}v_{\nu+1}\Bigg)_{0\le n\le N}.$$

From the Lemmas D.2 and D.3 we get the relations

$$\|\underline{R}u_0\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} \le C_0\|u_0\|_{\mathcal{D}} \qquad \text{and} \qquad \|\mathcal{K}(\underline{g})\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} \le C_1\|\underline{g}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X}^s)} \tag{5.8a}$$

for all $u_0 \in \mathcal{D}$ with constants C_0 and C_1 depending on the bounds for the stability function of the Runge-Kutta method, and further on α^{-1} and C_γ, see Result C.1. Note that for $\mathcal{J}\mathcal{K}_1(\underline{v})$ essentially the same bound holds as for $\mathcal{K}$. By following the proof of Lemma D.3, we receive

$$\|\mathcal{J}\mathcal{K}_1(\underline{v})\mathbb{1}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} \le C_1\|\underline{v}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \tag{5.8b}$$

after possibly changing the constant C_1.

Lemma 5.2. *Assume that the analytical solution u of* (3.1) *satisfies the regularity requirements $u^{(r)} \in \mathcal{C}^\alpha_\alpha((0,T],\mathcal{D})$ and $u^{(r+1)} \in \mathcal{C}^\alpha_\alpha((0,T],\mathcal{X})$ for $r = \min\{p, q+1\}$. Suppose further that $\big\|\widehat{\underline{U}} - \underline{U}(0)\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} \le \varrho$ and $\big\|\underline{U} - \underline{U}(0)\big\|_{\mathcal{B}(\mathcal{D}^s)} \le \varrho$ holds for $0 < \varrho \le R$ such that the relation*

$$2(KT^\alpha + L\varrho)C_{\mathcal{J}}\big(C_1 + C_\gamma(1 + C_{\mathcal{J}})\big) \le \kappa < 1$$

is valid. Then the internal stages $\underline{U}$ of the strongly stable Runge-Kutta method (3.2) *fulfil the estimate*

$$\big\|\underline{U} - \widehat{\underline{U}}\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} \le C\Big(\|u_0 - u(0)\|_{\mathcal{D}} + \big\|u^{(r)}\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})}\cdot h^r + \big\|u^{(r+1)}\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})}\cdot h^r\Big).$$

The constant C depends on C_0 and C_1 from (5.8), *on κ and C_γ and the bound for $(\mathcal{I} - hA\mathcal{Q})^{-1}$ and further on the method.*

Proof. As in the proof of Lemma 5.1 we linearize F around $(0,u(0))$ and set

$$A = D_v F(0,u(0)) \qquad \text{and} \qquad G(t,v) = F(t,v) - Av \quad \text{for } (t,v) \in [0,T] \times \mathcal{O}.$$

For a general Runge-Kutta method the sequence $\Delta G = (\Delta G(t_n + ch))_{n=0}^N$ is defined through

$$\Delta G(t_n + ch) = G(t_n + ch, U_n) - G\big(t_n + ch, \widehat{U}_n\big)$$

where $G(t_n + ch, U_n) = \big(G(t_n + c_i h, U_{ni})\big)_{i=1}^s$. The numerical solution is given by

$$U_n' = F(t_n + ch, U_n), \qquad U_n = \mathbb{1}\, u_n + h\mathcal{Q}U_n', \qquad u_{n+1} = u_n + hb^T U_n'.$$

Inserting the exact solution into this scheme yields

$$\widehat{U}_n' = F(t_n + ch, \widehat{U}_n), \qquad \widehat{U}_n = \mathbb{1}\, \widehat{u}_n + h\mathcal{Q}\widehat{U}_n' + D_n, \qquad \widehat{u}_{n+1} = \widehat{u}_n + hb^T \widehat{U}_n' + d_{n+1}.$$

By taking the difference and applying the linearization, we therefore get

$$\begin{aligned} E_n &= \mathbb{1}\, e_n + hA\mathcal{Q}E_n + h\mathcal{Q}\Delta G(t_n + ch) - D_n \\ e_{n+1} &= e_n + hAb^T E_n + hb^T \Delta G(t_n + ch) - d_{n+1}. \end{aligned}$$

An easy calculation yields

$$\begin{aligned} E_n &= \mathcal{J}\big(\mathbb{1}\, e_n + h\mathcal{Q}\Delta G(t_n + ch) - D_n\big) \\ e_{n+1} &= R(hA)e_n + hb^T \mathcal{J} G(t_n + ch) - hb^T \mathcal{J} A D_n - d_{n+1}. \end{aligned}$$

Solving the above error recursion for e_n thus yields

$$\begin{aligned} \underline{E} &= \mathcal{J}\mathbb{1}\, \underline{e} + h\mathcal{Q}\mathcal{J}\Delta G - \mathcal{J}\underline{D} \\ \underline{e} &= \underline{R}e_0 + \mathcal{K}(\Delta G) - \mathcal{K}(A\underline{D}) - \mathcal{K}_1\big(h^{-1}\underline{d}\big). \end{aligned} \tag{5.9}$$

In the following, we derive a bound for the norm of $\underline{E}$ in $\mathcal{C}_\alpha^\alpha(\mathcal{D}^s)$ involving e_0, $\underline{D}$ and $\underline{d}$. For this, we insert the second expression of (5.9) into the first relation which gives

$$\underline{E} = \mathcal{J}\mathbb{1}\, \underline{R}e_0 + \mathcal{J}\mathbb{1}\, \mathcal{K}(\Delta G) - \mathcal{J}\mathbb{1}\, \mathcal{K}(A\underline{D}) - \mathcal{J}\mathbb{1}\, \mathcal{K}_1\big(h^{-1}\underline{d}\big) + h\mathcal{Q}\mathcal{J}\Delta G - \mathcal{J}\underline{D}. \tag{5.10}$$

From Lemma E.3 we have

$$\|\Delta G\|_{\mathcal{C}_\alpha^\alpha(\mathcal{X}^s)} \le 2(KT^\alpha + L\varrho)\|\underline{E}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}^s)}.$$

Together with the bounds from (5.8) thus the estimate

$$\begin{aligned} \|\underline{E}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}^s)} \le{}& C_\mathcal{J} C_0 \|e_0\|_\mathcal{D} + C_\mathcal{J}\big(C_1 + C_\gamma \|hA\mathcal{Q}\mathcal{J}\|_{\mathcal{X}^s \leftarrow \mathcal{X}^s}\big)\|\Delta G\|_{\mathcal{C}_\alpha^\alpha(\mathcal{X}^s)} \\ &+ \|\mathcal{J}\mathbb{1}\, \mathcal{K}(A\underline{D})\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}^s)} + \|\mathcal{J}\mathbb{1}\, \mathcal{K}_1\big(h^{-1}\underline{d}\big)\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}^s)} + \|\mathcal{J}\underline{D}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}^s)} \\ \le{}& C_\mathcal{J} C_0 \|e_0\|_\mathcal{D} + 2(KT^\alpha + L\varrho) C_\mathcal{J}\big(C_1 + C_\gamma(1 + C_\mathcal{J})\big)\big\|\underline{E}\big\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}^s)} \\ &+ \|\mathcal{J}\mathbb{1}\, \mathcal{K}(A\underline{D})\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}^s)} + \big\|\mathcal{J}\mathbb{1}\, \mathcal{K}_1\big(h^{-1}\underline{d}\big)\big\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}^s)} + \|\mathcal{J}\underline{D}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}^s)} \end{aligned}$$

and further

$$\|\underline{E}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} \le \frac{1}{1-\kappa}\Big(C_{\mathcal{J}} C_0 \|e_0\|_{\mathcal{D}} + \|\mathcal{J}\mathbb{1}\mathcal{K}(A\underline{D})\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} + \big\|\mathcal{J}\mathbb{1}\mathcal{K}_1\big(h^{-1}\underline{d}\big)\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} + \|\mathcal{J}\underline{D}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)}\Big)$$

follows. If $u^{(r)} \in \mathcal{C}^\alpha_\alpha((0,T],\mathcal{D}))$ for $r = \min\{p, q+1\}$, we have

$$\|\underline{D}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} \le C\|u^{(r)}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} \cdot h^r$$

with constant C depending on the method, see Lemma 3.1. Consequently, we receive

$$\|\mathcal{J}\mathbb{1}\mathcal{K}(A\underline{D})\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} + \|\mathcal{J}\underline{D}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} \le (1 + C_1 C_\gamma) C_{\mathcal{J}} \|\underline{D}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X}^s)} \le C\|u^{(r)}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} \cdot h^r.$$

The constant C depends on C_1, C_γ, $C_{\mathcal{J}}$ and on the method. From (5.8) and with the help of Lemma 3.1, we further get

$$\big\|\mathcal{J}\mathbb{1}\mathcal{K}_1\big(h^{-1}\underline{d}\big)\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} \le C_1 \big\|h^{-1}\underline{d}\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \le C\|u^{(r+1)}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \cdot h^r$$

with C depending on C_1 and on the method. Combining these bounds finally shows

$$\|\underline{E}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} \le C\Big(\|e_0\|_{\mathcal{D}} + \|u^{(r)}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} \cdot h^r + \|u^{(r+1)}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \cdot h^r\Big).$$

The constant C depends on $\kappa, C_0, C_1, C_\gamma, C_{\mathcal{J}}$ and on the method. □

5.1.3. Error estimate for the Runge-Kutta solution

In the sequel we prove that the error of the Runge-Kutta solution is essentially bounded by the error of the internal stages and the defects of the method. For this, we consider the difference between the numerical solution and the values of the exact solution

$$\begin{aligned} &U_n' = F(t_n + ch, U_n), && U_n = \mathbb{1} u_n + h\mathcal{O}U_n', && u_{n+1} = u_n + hb^T U_n', \\ &\widehat{U}_n' = F(t_n + ch, \widehat{U}_n), && \widehat{U}_n = \mathbb{1}\widehat{u}_n + h\mathcal{O}\widehat{U}_n' + D_n, && \widehat{u}_{n+1} = \widehat{u}_n + hb^T\widehat{U}_n' + d_{n+1}. \end{aligned}$$

This yields the following recursion

$$E_n = \mathbb{1} e_n + h\mathcal{O}E_n' - D_n, \qquad e_{n+1} = e_n + hb^T E_n' - d_{n+1}. \tag{5.11}$$

From the first relation we receive the identity

$$E_n' = h^{-1}\mathcal{O}^{-1}\big(E_n - \mathbb{1} e_n + D_n\big).$$

By substituting this relation into the second formula of (5.11), we receive

$$e_{n+1} = R(\infty)e_n + b^T\mathcal{O}^{-1}(E_n + D_n) - d_{n+1} \tag{5.12}$$

with $R(\infty) = 1 - b^T \mathcal{O}\!\iota^{-1}\mathbb{1}$. Thus, for Runge-Kutta methods satisfying

$$R(\infty) = 0,$$

Relation (5.12) implies at once the estimate

$$\|\underline{e}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} \leq c_0\big(\|\underline{E}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)} + \|\underline{D}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}^s)}\big) + \|\underline{d}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})}$$

where $\left\|b^T \mathcal{O}\!\iota^{-1}\right\|_{\boldsymbol{R}\leftarrow\boldsymbol{R}^n} \leq c_0$.

In general, we solve the recursion in (5.12) yielding the following representation for the error

$$e_n = R(\infty)^n e_0 + \sum_{\nu=0}^{n-1} R(\infty)^{n-\nu-1}\big(b^T \mathcal{O}\!\iota^{-1}(E_\nu + D_\nu) - d_{\nu+1}\big), \tag{5.13}$$

and we derive a bound for the (α,α)-Hölder norm of $\underline{e}$ in $\mathcal{D}$. For this, it is useful to employ the abbreviation

$$\gamma = |R(\infty)|.$$

Notice that for a strongly stable Runge-Kutta methods it holds $\gamma < 1$. By using

$$\sum_{\nu=0}^{n-1} \gamma^{n-\nu-1} = \frac{1-\gamma^n}{1-\gamma},$$

we consequently receive from (5.13) the error estimate

$$\|\underline{e}\|_{\mathcal{B}(\mathcal{D})} \leq \|e_0\|_{\mathcal{D}} + \frac{1}{1-\gamma}\Big(c_0\big(\|\underline{E}\|_{\mathcal{B}(\mathcal{D}^s)} + \|\underline{D}\|_{\mathcal{B}(\mathcal{D}^s)}\big) + \|\underline{d}\|_{\mathcal{B}(\mathcal{D})}\Big). \tag{5.14}$$

From (5.13) a short calculation yields

$$\begin{aligned} e_n - e_m = {} & \big(R(\infty)^n - R(\infty)^m\big)e_0 \\ & + \sum_{\nu=0}^{m-1} \big(R(\infty)^{n-\nu-1} - R(\infty)^{m-\nu-1}\big)\big(b^T \mathcal{O}\!\iota^{-1}(E_\nu + D_\nu) - d_{\nu+1}\big) \\ & + \sum_{\nu=m}^{n-1} R(\infty)^{n-\nu-1}\big(b^T \mathcal{O}\!\iota^{-1}(E_\nu + D_\nu) - d_{\nu+1}\big). \end{aligned} \tag{5.15}$$

Note first that there is some $\beta > 0$ such that $\gamma = e^{-\beta h}$. By means of the bound $e^{-x} \leq x^{-1+\alpha}$ which is valid for all $x \geq 0$, we therefore receive

$$1 - \gamma^k = 1 - e^{-\beta t_k} = \beta \int_0^{t_k} e^{-\beta\tau}\,\mathrm{d}\tau \leq \beta^\alpha \int_0^{t_k} \tau^{-1+\alpha}\,\mathrm{d}\tau \leq \alpha^{-1}\beta^\alpha t_k^\alpha$$

for any $k \geq 0$. From this, we get

$$\begin{aligned}\big|R(\infty)^n - R(\infty)^m\big| &\leq \big|1 - R(\infty)\big| \sum_{\nu=m}^{n-1} \gamma^\nu = \gamma^m \big|1 - R(\infty)\big| \frac{1-\gamma^{n-m}}{1-\gamma} \\ &\leq \frac{2\beta^\alpha}{\alpha(1-\gamma)} (t_n - t_m)^\alpha\end{aligned}$$

and further the estimate

$$\begin{aligned}&\sum_{\nu=0}^{m-1} \big|R(\infty)^{n-\nu-1} - R(\infty)^{m-\nu-1}\big| \leq \big|R(\infty)^{n-m} - 1\big| \sum_{\nu=0}^{m-1} \gamma^{m-\nu-1} \\ &\quad\leq \big|1 - R(\infty)\big| \sum_{\nu=0}^{m-1} \gamma^{m-\nu-1} \cdot \sum_{\nu=0}^{n-m-1} \gamma^\nu = \big|1 - R(\infty)\big| \frac{1-\gamma^m}{(1-\gamma)^2} (1-\gamma^{n-m}) \\ &\quad\leq \frac{2\beta^\alpha}{\alpha(1-\gamma)^2} (t_n - t_m)^\alpha.\end{aligned}$$

Besides, the following relation is valid

$$\sum_{\nu=m}^{n-1} \gamma^{n-\nu-1} = \frac{1-\gamma^{n-m}}{1-\gamma} \leq \frac{1}{\alpha(1-\gamma)} \beta^\alpha (t_n - t_m)^\alpha.$$

Applying these auxiliary estimates in order to bound (5.13), in particular gives

$$\|e_n - e_m\|_{\mathcal{D}} \leq C\Big(\|e_0\|_{\mathcal{D}} + \big(c_0\big(\|\underline{E}\|_{\mathcal{B}(\mathcal{D}^s)} + \|\underline{D}\|_{\mathcal{B}(\mathcal{D}^s)}\big) + \|\underline{d}\|_{\mathcal{B}(\mathcal{D})}\big)\Big) \cdot t_{n_1}^\alpha (t_n - t_m)^\alpha$$

for all $0 < t_{n_1} \leq t_m < t_n$ with C depending on α^{-1} and γ. Together with the bound (5.14) we finally receive the desired error estimate

$$\|\underline{e}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} \leq C\Big(\|e_0\|_{\mathcal{D}} + \|\underline{E}\|_{\mathcal{B}(\mathcal{D}^s)} + \|\underline{D}\|_{\mathcal{B}(\mathcal{D}^s)} + \|\underline{d}\|_{\mathcal{B}(\mathcal{D})}\Big). \tag{5.16}$$

The constant C depends on α and on the method, especially on on γ.

5.2. Linearization around the initial value

In the present section, we derive existence and convergence results for Runge-Kutta methods applied to the nonlinear initial value problem (3.1). At first, we prove in Section 5.2.1 a convergence result of order one for the backward Euler method, and then the expected convergence order $r = \min\{p, q+1\}$ for a general Runge-Kutta method of order p and stage order q. However, as our proofs of Propositions 5.1 and 5.2 are in need of certain requirements on the growth of the exact solution which imply a severe restriction on the length of the considered time interval, these existence and convergence results are valid only locally. This restriction is overcome in Section 5.2.2 where we prove convergence in the large.

We further on suppose that the basic assumptions of Section 1.1 on the evolution equation (3.1) and on the numerical method (3.2) are fulfiled, and we henceforth employ the notation introduced at the beginning of Section 5.1. In particular, $K > 0$ and $L > 0$ are the (α, α)-Hölder constant and the Lipschitz constant of the function F arising on the right-hand side of (3.1). Moreover, the constant $C_{\mathcal{J}}$ denotes a bound for the linear operator $\mathcal{J} = (\mathcal{I} - hA\mathcal{Q})^{-1}$ on $\mathcal{X}^s$.

5.2.1. Local existence and convergence results

Our main tools for proving the following local existence and convergence result for the backward Euler method are the step-by-step approach from Lemma 4.1 and the preliminary error estimate stated in Lemma 5.1. Let C_1 denote the constant from (5.4) for $A = D_v F(0, u(0))$.

Proposition 5.1. (González et al. [11, Theorem 3.1]) *Assume that the analytical solution $u \in \mathcal{C}([0,T], \mathcal{D})$ of equation* (3.1) *has the property*

$$\|u - u(0)\|_{\mathcal{C}^\alpha_\alpha((0,T],\mathcal{D})} < \varrho \qquad \text{such that} \quad 2C_1(KT^\alpha + L\varrho) < 1, \tag{5.17}$$

and suppose further that $u'' \in \mathcal{C}^\alpha_\alpha((0,T], \mathcal{X})$. Then, there exist constants $\overline{h} > 0$, $\delta > 0$ and $C > 0$ such that, for $0 < h \leq \overline{h}$ and initial values u_0 with $\|u_0 - u(0)\|_{\mathcal{D}} \leq \delta$, the backward Euler solution u_n exists as long as $t_n \leq T$. Furthermore, the error estimate

$$\|\underline{u} - \underline{\widehat{u}}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} \leq C \left(\|u_0 - u(0)\|_{\mathcal{D}} + \|u''\|_{\mathcal{C}^\alpha_\alpha((0,T],\mathcal{X})} \cdot h\right)$$

is valid. The constant C in general depends on T but not on h.

Proof. We set $R_0 = \varrho - \|u - u(0)\|_{\mathcal{B}([0,T],\mathcal{D})} > 0$ with $\varrho \leq R$ denoting the quantity from (5.17). Then, in particular, we have

$$\|\widehat{u}_n - u(0)\|_{\mathcal{D}} \leq \varrho - R_0 \qquad \text{for } t_n \leq T. \tag{5.18}$$

Under the assumption that the numerical solution u_n is defined as long as $t_n \leq T$, and that the bound

$$\|u_n - u(0)\|_{\mathcal{D}} \leq \varrho, \qquad t_n \leq T,$$

holds, Lemma 5.1 yields the validity of the desired error bound

$$\|\underline{u} - \underline{\widehat{u}}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} \leq C\big(\|u_0 - u(0)\|_{\mathcal{D}} + \|u''\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \cdot h\big). \tag{5.19}$$

Thus it remains to justify this supposition.

For this, we apply Lemma 4.1 to R_0 defined above and elements (t^*, v^*) varying in the compact set $[0,T] \times \{u(t) : 0 \leq t \leq T\}$ and denote by $\tau_0 > 0$, $0 < \delta_0 < \varrho_0 \leq R_0$ and $\overline{h}_0 > 0$ the quantities there. After a possible reduction of $0 < \overline{h} \leq \overline{h}_0$, we can choose some $\delta > 0$ satisfying

$$C\big(\delta + \|u''\|_{\mathcal{C}_\alpha^\alpha((0,T],\mathcal{X})} \cdot \overline{h}\big) \leq \delta_0/2 \tag{5.20}$$

where C is the constant from (5.19). Since the exact solution is uniformly continuous, we can also assume

$$\|\widehat{u}_n - \widehat{u}_{n-1}\|_{\mathcal{D}} \leq \delta_0/2, \qquad \text{for} \quad t_n \leq T \quad \text{if} \quad h \leq \overline{h}. \tag{5.21}$$

Besides, we let the initial value u_0 fulfil $\|u_0 - u(0)\|_{\mathcal{D}} \leq \delta$.

We suppose by induction that u_n exists and that $\|u_n - u(0)\|_{\mathcal{D}} \leq \varrho$ is satisfied for all integers $n \leq k$. Thus it remains to show that then the following holds

$$u_{k+1} \text{ exists} \qquad \text{and} \qquad \|u_{k+1} - u(0)\|_{\mathcal{D}} \leq \varrho.$$

Note first, that, due to our choices of δ, $\overline{h}$ and u_0, the relations (5.19), (5.20) and (5.21) imply

$$\|u_k - \widehat{u}_{k+1}\|_{\mathcal{D}} \leq \|u_k - \widehat{u}_k\|_{\mathcal{D}} + \|\widehat{u}_k - \widehat{u}_{k+1}\|_{\mathcal{D}} \leq \delta_0 \qquad \text{for } h \leq \overline{h}.$$

We apply Lemma 4.1 with $t^* = t_k$, $v^* = \widehat{u}_{k+1}$ and $v_0 = u_k$ which yields the existence of u_{k+1} and the validity of

$$\|u_{k+1} - \widehat{u}_{k+1}\|_{\mathcal{D}} \leq \varrho_0 \leq R_0.$$

Consequently, from this relation together with (5.18) the estimate

$$\|u_{k+1} - u(0)\|_{\mathcal{D}} \leq \|u_{k+1} - \widehat{u}_{k+1}\|_{\mathcal{D}} + \|\widehat{u}_{k+1} - u(0)\|_{\mathcal{D}} \leq R_0 + (\varrho - R_0) = \varrho$$

follows. This concludes the proof. □

With the help of the local existence result Lemma 4.2, the preliminary error estimate from Lemma 5.2 and the considerations from Section 5.1.3, we are in the position to generalize the proceeding result to a Runge-Kutta method of the form (3.2). For this, we suppose that the numerical method has order $p \geq 1$ and stage order q and that it is strongly $A(\vartheta)$-stable with $\vartheta > \pi - \varphi$. Here, the angle $\pi/2 < \varphi < \pi$ is chosen such that the resolvent condition

$$\big\|\big(\lambda I - D_v F(0, u(0))\big)^{-1}\big\|_{\mathcal{X} \leftarrow \mathcal{X}} \leq \frac{M}{|\lambda - a|}, \qquad |\arg(\lambda - a)| < \varphi,$$

is satisfied for some $a \in \boldsymbol{R}$ and $M > 0$. Let C_1 denote the constant arising in Estimate (5.8). Then, the following result holds.

Proposition 5.2. *In the above situation suppose that the exact solution $u \in \mathcal{C}([0,T],\mathcal{D})$ of equation* (3.1) *fulfils the requirement*

$$\|u - u(0)\|_{\mathcal{C}_\alpha^\alpha((0,T],\mathcal{D})} < \varrho \quad \text{such that} \;\; 2(KT^\alpha + L\varrho)C_{\mathcal{J}}\big(C_1 + C_\gamma(1+C_{\mathcal{J}})\big) < 1 \tag{5.22}$$

and assume further that it has the smoothness properties

$$u^{(r)} \in \mathcal{C}_\alpha^\alpha((0,T],\mathcal{D}) \qquad \text{and} \qquad u^{(r+1)} \in \mathcal{C}_\alpha^\alpha((0,T],\mathcal{X}), \qquad r = \min\{p, q+1\}.$$

Then there exist constants $\overline{h} > 0$, $\delta > 0$ and $C > 0$ such that, for stepsizes $0 < h \le \overline{h}$ and for initial values u_0 with $\|u_0 - u(0)\|_{\mathcal{D}} \le \delta$, the Runge-Kutta solution u_n exists as long as $t_n \le T$, and the following error estimate holds

$$\|\underline{u} - \underline{\widehat{u}}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} + \big\|\underline{U} - \underline{\widehat{U}}\big\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}^s)} \le C\|u_0 - u(0)\|_{\mathcal{D}} + Ch^r\Big(\big\|u^{(r)}\big\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} + \big\|u^{(r+1)}\big\|_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}\Big)$$

with a constant C depending in general on T but not on h.

Proof. As before, we set $R_0 = \varrho - \|u - u(0)\|_{\mathcal{B}([0,T],\mathcal{D})} > 0$ for the constant $\varrho \le R$ from (5.22), and therefore it holds

$$\|\widehat{u}_n - u(0)\|_{\mathcal{D}} \le \varrho - R_0 \qquad \text{for } t_n \le T. \tag{5.23}$$

Provided that the internal stages U_n and thus the Runge-Kutta solution u_{n+1} are defined for all $t_{n+1} \le T$ and that the bound

$$\|U_n - U(0)\|_{\mathcal{D}^s} \le \varrho, \qquad t_n \le T,$$

is valid, Lemma 5.2 together with the considerations from Section 5.1.3, yields the desired error estimate

$$\begin{aligned}\|\underline{u} - \underline{\widehat{u}}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} + \big\|\underline{U} - \underline{\widehat{U}}\big\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}^s)} \le{}& C\|u_0 - u(0)\|_{\mathcal{D}} \\ &+ C\Big(\big\|u^{(r)}\big\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} + \big\|u^{(r+1)}\big\|_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}\Big)\cdot h^r.\end{aligned} \tag{5.24}$$

Therefore, it remains to justify this assumption. Again, this is done recursively.

We apply Lemma 4.2 to R_0 and (t^*, v^*) belonging to $[0,T] \times \{u(t) : 0 \le t \le T\}$, and we denote by $\tau_0 > 0$, $0 < \delta_0 < \varrho_0 \le r_0 \le R_0$ and $\overline{h}_0 > 0$ the constants arising there. After a possible reduction of $0 < \overline{h} \le \overline{h}_0$, we choose $\delta > 0$ such that the relation

$$C\bigg(\delta + \Big(\big\|u^{(r)}\big\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} + \big\|u^{(r+1)}\big\|_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}\Big)\cdot \overline{h}^r\bigg) \le \delta_0/2 \tag{5.25}$$

is fulfiled with the constant C from (5.24). Due to the fact that the analytical solution of (3.1) is uniformly continuous, we may assume that

$$\|\widehat{u}_n - \widehat{u}_{n-1}\|_{\mathcal{D}} \le \delta_0/2, \qquad \text{for} \quad t_n \le T \quad \text{if} \quad 0 < h \le \overline{h}, \tag{5.26}$$

holds as well. Moreover, we suppose that the initial value is such that $\|u_0 - u(0)\|_{\mathcal{D}} \leq \delta$.

Assume that U_{n-1} and u_n exist and that the bound $\|U_{n-1} - U(0)\|_{\mathcal{D}^s} \leq \varrho$ is valid for $n \leq k$. In the following we prove that then U_k and u_{k+1} exist as well and that $\|U_k - U(0)\|_{\mathcal{D}^s} \leq \varrho$ holds. Our choice of the parameter δ, the stepsize $\overline{h}$ and the initial value u_0, implies

$$\|u_k - \widehat{u}_{k+1}\|_{\mathcal{D}} \leq \delta_0 \qquad \text{for } h \leq \overline{h}.$$

By adding $\widehat{u}_k$ and applying the triangular inequality, this follows from the relations (5.24), (5.25) and (5.26). Now, Lemma 4.2 with $t^* = t_k$, $v^* = \widehat{u}_{k+1}$, $v_0 = u_k$ and $V_0 = \mathbb{1} u_k$ ensures the existence of U_k and u_{k+1} and yields the estimates

$$\|U_k - \mathbb{1}\widehat{u}_{k+1}\|_{\mathcal{D}} \leq r_0 \leq \varrho_0 \leq R_0 \qquad \text{and} \qquad \|u_{k+1} - \widehat{u}_{k+1}\|_{\mathcal{D}} \leq \varrho_0 \leq R_0.$$

By using (5.23), we finally receive the bound

$$\|U_k - U(0)\|_{\mathcal{D}^s} \leq \|U_{k+1} - \mathbb{1}\widehat{u}_{k+1}\|_{\mathcal{D}} + \|\widehat{u}_{k+1} - u(0)\|_{\mathcal{D}} \leq R_0 + (\varrho - R_0) = \varrho$$

which concludes the proof. □

5.2.2. Convergence in the large

Our main assumption for proving the local existence and convergence result Proposition 5.2 of the previous section is that the quantity $\|u - u(0)\|_{\mathcal{C}^{\alpha}_{\alpha}((0,T],\mathcal{D})}$ remains sufficiently small on the interval $[0, T]$. In general, this implies a severe restriction on the size of T. In order to overcome this, we construct a partition

$$0 = T_0 < T_1 < \cdots < T_J = T$$

such that the (α, α)-Hölder norm of $u - u(T_j)$ is small enough on each interval $[T_j, T_{j+1}]$ for $0 \leq j \leq J - 1$, see Lemma B.1, and we apply Proposition 5.2 piecewisely. This procedure is sketched below.

Let $N_j \in \boldsymbol{N}$ be defined through $t_{N_j} \leq T_j < t_{N_j+1}$ for $1 \leq j \leq J$. For simplicity, we assume $t_{N_j} = T_j$ for all $1 \leq j \leq J$. If $\|u_0 - u(0)\|_{\mathcal{D}} \leq \delta$ and $0 < h \leq \overline{h}$, then the following estimate holds for the Runge-Kutta solution

$$\|u_n - u(t_n)\|_{\mathcal{D}} \leq C\big(\|u_0 - u(0)\|_{\mathcal{D}} + h^r\big), \qquad 0 \leq t_n \leq t_{N_1}.$$

Here, $\overline{h}$ and δ denote the constants from Proposition 5.2. After a possible reduction of $\|u_0 - u(0)\|_{\mathcal{D}} \leq \delta^*$ and $0 < \overline{h}^* \leq \overline{h}$ such that the bound

$$\|u_{N_1} - u(t_{N_1})\|_{\mathcal{D}} \leq C\left(\delta^* + \overline{h}^r\right) \leq \delta$$

is still valid, a further application of Proposition 5.2 yields

$$\begin{aligned}\|u_n - u(t_n)\|_{\mathcal{D}} &\le C\big(\|u_{N_1} - u(t_{N_1})\|_{\mathcal{D}} + h^r\big)\\ &\le C^2\|u_0 - u(0)\|_{\mathcal{D}} + \widetilde{C}h^r, \qquad t_{N_1} \le t_n \le t_{N_2},\end{aligned}$$

with some constant $\widetilde{C} > 0$. Reiterating this process shows the desired error estimate for the Runge-Kutta solution on the interval $[0,T]$. Especially for the backward Euler method we have the following result. Notice that Theorem 5.1 provides only estimates for $\|u_n - u(t_n)\|_{\mathcal{D}}$.

Theorem 5.1. (González et al. [11, Theorem 3.3]) *Let $u : [0,T] \to \mathcal{D}$ be the analytical solution of problem* (3.1) *such that*

$$u'' \in \mathcal{C}^{\alpha}_{\alpha}((0,T],\mathcal{X}).$$

Then, there exist $\overline{h} > 0$, $\delta > 0$ and $C > 0$ such that, for stepsizes $0 < h \le \overline{h}$ and for initial values u_0 with $\|u_0 - u(0)\|_D \le \delta$, the backward Euler solution u_n exists for $t_n \le T$ and the error estimate

$$\|u_n - u(t_n)\|_{\mathcal{D}} \le C\big(\|u_0 - u(0)\|_{\mathcal{D}} + h\big), \qquad 0 \le t_n \le T,$$

holds with a constant C depending in general on T but not on h or n.

Proof. In view of the generalization to Runge-Kutta methods we introduce the parameter $r = \min\{p, q+1\}$ and set $r = 1$.

Due to the continuity of the analytical solution, there are constants $R, K > 0$ and $L > 0$ such that condition (1.2) of Assumption 1.1 is uniformly satisfied for all elements $v^* \in \{u(t)\}_{0\le t\le T}$. Furthermore, by Lemma B.1, there exists a partition

$$0 = T_0 < T_1 < \cdots < T_J = T$$

of the interval $[0,T]$ such that for all $0 \le j \le J-1$ the requirement (5.17) of Proposition 5.1

$$\|u - u(T_j)\|_{\mathcal{C}^{\alpha}_{\alpha}((T_j,T_{j+1}],\mathcal{D})} < \varrho \qquad \text{such that} \quad 2C_1\big(K(T_{j+1}-T_j)^{\alpha} + L\varrho\big) < 1 \tag{5.27}$$

holds. As outlined above, we deduce from Proposition 5.1, applied piece-by-piece, that there exist constants $\overline{h}, C > 0$ and $\widetilde{C} > 0$ such that for stepsizes $0 < h \le \overline{h}$ the bound

$$\|u_n - \widehat{u}_n\|_{\mathcal{D}} \le C^j\|u_0 - u(0)\|_{\mathcal{D}} + \widetilde{C}h^r, \qquad 0 \le t_n \le T_j,$$

is valid. This implies the following estimate

$$\|u_n - u(t_n)\|_{\mathcal{D}} \le \delta,$$

after a possible reduction of the constants δ and $\overline{h}$ from Proposition 5.1. Remark that J does not depend on the stepsize h. The desired result thus follows by induction. ☐

The analogous result for a general Runge-Kutta method of order p and stage order q follows literally from the proof of Theorem 5.1 by replacing (5.27) with the condition

$$\|u - u(T_j)\|_{\mathcal{C}_\alpha^\alpha((T_j,T_{j+1}],\mathcal{D})} < \varrho, \quad 2(K(T_{j+1}-T_j)^\alpha + L\varrho)C_{\mathcal{J}}\big(C_1 + C_\gamma(1+C_{\mathcal{J}})\big) < 1,$$

and Proposition 5.1 with the corresponding auxiliary result Proposition 5.2. Notice that there are constants $a \in \boldsymbol{R}$, $\pi/2 < \varphi < \pi$ and $M > 0$ such that the resolvent estimate

$$\big\|\big(\lambda I - D_v F(T_j, u(T_j))\big)^{-1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \leq \frac{M}{|\lambda - a|}, \qquad |\arg(\lambda - a)| < \varphi,$$

remains valid for an arbitrary partition

$$0 = T_0 < T_1 < \cdots < T_J = T$$

of the interval $[0,T]$. We henceforth suppose that (3.2) is strongly $A(\vartheta)$-stable with $\vartheta > \pi - \varphi$ for a suitable choice of φ.

Theorem 5.2. *In the above situation, assume that the analytical solution $u : [0,T] \to \mathcal{D}$ of equation* (3.1) *satisfies*

$$u^{(r)} \in \mathcal{C}_\alpha^\alpha((0,T],\mathcal{D}) \quad \text{and} \quad u^{(r+1)} \in \mathcal{C}_\alpha^\alpha((0,T],\mathcal{X}) \qquad \text{for} \quad r = \min\{p, q+1\}.$$

Then, there exist constants $\overline{h} > 0$, $\delta > 0$ and $C > 0$ such that, for stepsizes $0 < h \leq \overline{h}$ and for initial values u_0 with $\|u_0 - u(0)\|_D \leq \delta$, the Runge-Kutta solution u_n exists as long as $t_n \leq T$. Further, the following error estimate holds

$$\|u_n - u(t_n)\|_{\mathcal{D}} + \|U_{ni} - u(t_n + c_i h)\|_{\mathcal{D}} \leq C\big(\|u_0 - u(0)\|_{\mathcal{D}} + h^r\big), \qquad 0 \leq t_n \leq T,$$

for $1 \leq i \leq s$. The constant C depends on T but not on h or n.

5.3. Linearization around the exact solution

For the following analysis, it is convenient to use a simplified version of the analytical framework of Lunardi [25] reviewed in Chapter 1, and to assume that the right-hand side of equation (3.1) is twice differentiable. The precise requirements on F for the remainder of this section are the following, see also Assumption 1.1.

Assumption 5.1. *Suppose that $F : [0,T] \times \mathcal{O} \to \mathcal{X}$ is twice continuously Fréchet differentiable, and assume further that for every element $(t^*, v^*) \in [0,T] \times \mathcal{O}$, the derivative $D_v F(t^*, v^*)$ is a sectorial operator in $\mathcal{X}$ and that its graph norm is equivalent to $\|\cdot\|_{\mathcal{D}}$.*

Our basic concept for proving existence and convergence results for Runge-Kutta time discretizations of (3.1) which avoid the consideration of the complicated (α,α)-Hölder norms is the following. We linearize the right-hand side of (3.1) along the analytical solution and apply the numerical method to the resulting formally semilinear equation by using the fact that Runge-Kutta methods are invariant under such transformations. For this, we define the time dependent linear operator A and the map G through

$$A(t) = D_v F\big(t, u(t)\big) \quad \text{and} \quad G(t,v) = F(t,v) - A(t)v \quad \text{for } (t,v) \in [0,T] \times \mathcal{O}. \tag{5.28a}$$

Then the original problem (3.1) writes as

$$u'(t) = A(t)u(t) + G(t, u(t)), \qquad u(0) \text{ given}. \tag{5.28b}$$

For the construction of the numerical solution we use a fixed point iteration relying on the discrete variation-of-constants formula. Provided that the analytical solution is α-Hölder continuous, A has the regularity property

$$A \in \mathcal{C}^\alpha\big((0,T], \mathcal{L}(\mathcal{D}, \mathcal{X})\big),$$

and therefore stability estimates for Runge-Kutta time discretizations of (5.28b) are available by González & Palencia [12], see also Result C.2. The Hölder continuity of u implies that $A(t)$ is uniformly sectorial on $[0,T]$. Thus there are constants $a \in \boldsymbol{R}$, $\pi/2 < \varphi < \pi$ and $M > 0$ such that the resolvent estimate

$$\big\|(\lambda I - A(t))^{-1}\big\|_{\mathcal{X} \leftarrow \mathcal{X}} \leq \frac{M}{|\lambda - a|}, \qquad |\arg(\lambda - a)| < \varphi, \tag{5.29}$$

uniformly holds for $0 \leq t \leq T$, see Definition A.2. We henceforth assume that the Runge-Kutta method (3.2) is strongly $A(\vartheta)$-stable with $\vartheta > \pi - \varphi$. For simplicity, we set

$$u(0) = u_0 \in \mathcal{D},$$

i.e. the initial value of the numerical method equals the initial value of the evolution equation.

In Section 5.3.1 we derive an existence and convergence result for the backward Euler method. The generalization to a Runge-Kutta method of the form (3.2) is stated in Section 5.3.2.

5.3.1. Existence and convergence of the backward Euler method

For proving the existence result for the backward Euler method, we use a fixed point iteration in a strip around the exact solution of (3.1) which relies on a global representation of the backward Euler solution by the discrete variation-of-constants formula. With the help of this fixed point iteration the convergence order one follows easily.

Theorem 5.3. (Ostermann & Thalhammer [27, Theorem 6]) *Assume that the analytical solution of equation* (3.1) *has the regularity properties*

$$u \in \mathcal{C}^\alpha([0,T], \mathcal{D}) \qquad \textit{and} \qquad u'' \in \mathcal{B}([0,T], \mathcal{X}).$$

Then, there exist constants $C > 0$ and $\overline{h} > 0$ such that for $0 < h \le \overline{h}$ the backward Euler solution u_n exists for $0 \le t_n \le T$ and satisfies

$$\|u_n - u(t_n)\|_{\mathcal{D}} \le Ch\big(1 + |\log(h)|\big), \qquad 0 \le t_n \le T.$$

The quantity C depends on T but not on n or h.

Proof. As a first step, we derive a global representation for the backward Euler solution of the formally semilinear equation (5.28) which is given by

$$u_{n+1} = \big(I - hA(t_{n+1})\big)^{-1}\big(u_n + hG(t_{n+1}, u_{n+1})\big).$$

Solving this recursion yields the discrete variation-of-constants formula

$$\begin{aligned} u_n &= \big(I - hA(t_n)\big)^{-1} \cdots \big(I - hA(t_1)\big)^{-1} u_0 \\ &\quad + h \sum_{\nu=0}^{n-1} \big(I - hA(t_n)\big)^{-1} \cdots \big(I - hA(t_{\nu+1})\big)^{-1} G(t_{\nu+1}, u_{\nu+1}). \end{aligned} \tag{5.30}$$

For the discrete transition operator we employ the useful abbreviation

$$r(t_n, t_k) = \big(I - hA(t_n)\big)^{-1} \cdots \big(I - hA(t_{k+1})\big)^{-1}, \qquad 0 \le k < n \le N,$$

and we set $r(t_n, t_n) = I$. With this notation Relation (5.30) writes as

$$u_n = r(t_n, 0)u_0 + h \sum_{\nu=0}^{n-1} r(t_n, t_\nu) G(t_{\nu+1}, u_{\nu+1}). \tag{5.31}$$

By inserting the exact solution into the numerical scheme and performing a similar calculation to before, we obtain the following representation

$$\widehat{u}_n = r(t_n, 0)u_0 + h \sum_{\nu=0}^{n-1} r(t_n, t_\nu) G(t_{\nu+1}, \widehat{u}_{\nu+1}) + \sum_{\nu=0}^{n-1} r(t_n, t_\nu) d_{\nu+1}, \tag{5.32}$$

involving the defects d_n which are given by

$$d_{n+1} = h^2 \int_0^1 (-\tau) u''(t_n + \tau h)\, \mathrm{d}\tau, \tag{5.33}$$

see Lemma 3.1.

1. For constructing the backward Euler solution we use the fixed point iteration Φ mapping a sequence $\underline{v} = (v_n)_{n=0}^N \in \mathcal{O}^{N+1}$ where $t_N \le T < t_{N+1}$ to

$$\Phi(\underline{v}) = \Big(r(t_n, 0)u_0 + h \sum_{\nu=0}^{n-1} r(t_n, t_\nu) G(t_{\nu+1}, v_{\nu+1}) \Big)_{n=0}^{N}. \tag{5.34}$$

As domain of Φ we choose the closed subset

$$\mathcal{V} = \Big\{ \underline{v} = (v_n)_{n=0}^N \in \mathcal{O}^{N+1} : v_0 = u_0, \|\underline{v} - \underline{\widehat{u}}\|_{\mathcal{B}(\mathcal{D})} \leq c_0 h^\eta \Big\},$$

endowed with the norm $\|\cdot\|_{\mathcal{B}(\mathcal{D})}$, for some constants $c_0 > 0$ and $0 < \eta < 1$. In order to apply the fixed point theorem of Banach, we have to show the following.

Φ is a contraction on $\mathcal{V}$.

(i) Φ is contractive on $\mathcal{V}$, i.e. for all elements $\underline{v}, \underline{w} \in \mathcal{V}$ the relation

$$\big\|\Phi(\underline{v}) - \Phi(\underline{w})\big\|_{\mathcal{B}(\mathcal{D})} \leq \kappa \|\underline{v} - \underline{w}\|_{\mathcal{B}(\mathcal{D})}$$

holds for some $\kappa < 1$.

(ii) Φ maps $\mathcal{V}$ into $\mathcal{V}$, and in particular the estimate $\|\underline{v} - \underline{\widehat{u}}\|_{\mathcal{B}(\mathcal{D})} \leq c_0 h^\eta$ implies

$$\big\|\Phi(\underline{v}) - \underline{\widehat{u}}\big\|_{\mathcal{B}(\mathcal{D})} \leq c_0 h^\eta.$$

In the sequel we show that these relations hold for $0 < h \leq \overline{h}$ sufficiently small.

(i) From the definition of Φ we get

$$\big(\Phi(\underline{v}) - \Phi(\underline{w})\big)_n = h \sum_{\nu=0}^{n-1} r(t_n, t_\nu) \big(G(t_{\nu+1}, v_{\nu+1}) - G(t_{\nu+1}, w_{\nu+1}) \big). \tag{5.35}$$

For the backward Euler method the following stability bounds are valid for some constants $\hat{M}_0 > 0$ and $\hat{M}_1 > 0$ whenever $0 < h \leq \overline{h}$ is sufficiently small

$$\|r(t_n, t_k)\|_{\mathcal{X} \leftarrow \mathcal{X}} \leq \hat{M}_0 \quad \text{and} \quad \|r(t_n, t_k)\|_{\mathcal{D} \leftarrow \mathcal{X}} \leq \hat{M}_1 t_{n-k}^{-1}, \qquad 0 \leq k < n \leq N, \tag{5.36}$$

see Result C.2. Notice that for all $t \in [0, T]$ and $v, w \in \mathcal{O}$ the identity

$$G(t, v) - G(t, w) = \int_0^1 \big(D_v F(t, \tau v + (1 - \tau) w) - D_v F(t, u(t)) \big) (v - w) \,\mathrm{d}\tau$$

holds. Consequently, we receive for elements $\underline{v}, \underline{w} \in \mathcal{V}$ the estimate

$$\|G(t_k, v_k) - G(t_k, w_k)\|_{\mathcal{X}} \leq c_0 L h^\eta \|\underline{v} - \underline{w}\|_{\mathcal{B}(\mathcal{D})}. \tag{5.37}$$

Estimating the following Riemann sum by the corresponding integral implies

$$\begin{aligned} h \sum_{\nu=0}^{n-1} t_{n-\nu}^{-1} &= h \sum_{\nu=0}^{n-1} t_{n-\nu}^{-1+\beta} t_{n-\nu}^{-\beta} \leq h^{-\beta} \cdot h \sum_{\nu=0}^{n-1} t_{n-\nu}^{-1+\beta} \\ &\leq h^{-\beta} \int_0^{t_n} (t_n - \tau)^{-1+\beta} \,\mathrm{d}\tau = \beta^{-1} h^{-\beta} t_n^\beta \end{aligned}$$

for arbitrary $0 < \beta < 1$. The optimal choice $\beta = |\log(n)|^{-1}$ thus yields the bound

$$h \sum_{\nu=0}^{n-1} t_{n-\nu}^{-1} \leq c_1 \big(1 + |\log(h)|\big) \tag{5.38}$$

with constant c_1 depending on t_n. Applying the above auxiliary results to (5.35) yields

$$\begin{aligned} \big\| \big(\Phi(\underline{v}) - \Phi(\underline{w})\big)_n \big\|_{\mathcal{D}} &\leq h \sum_{\nu=0}^{n-1} \| r(t_n, t_\nu) \|_{\mathcal{D} \leftarrow \mathcal{X}} \big\| G(t_{\nu+1}, v_{\nu+1}) - G(t_{\nu+1}, w_{\nu+1}) \big\|_{\mathcal{X}} \\ &\leq c_0 c_1 L \hat{M}_1 h^\eta \big(1 + |\log(h)|\big) \| \underline{v} - \underline{w} \|_{\mathcal{B}(\mathcal{D})}. \end{aligned}$$

For stepsizes $0 < h \leq \overline{h}$ small enough the contractivity of Φ follows.

(ii) Since Φ is contractive with contraction factor κ, we get the following estimate for elements $\underline{v} \in \mathcal{V}$

$$\begin{aligned} \| \Phi(\underline{v}) - \underline{\widehat{u}} \|_{\mathcal{B}(\mathcal{D})} &\leq \| \Phi(\underline{v}) - \Phi(\underline{\widehat{u}}) \|_{\mathcal{B}(\mathcal{D})} + \| \Phi(\underline{\widehat{u}}) - \underline{\widehat{u}} \|_{\mathcal{B}(\mathcal{D})} \\ &\leq \kappa \| \underline{v} - \underline{\widehat{u}} \|_{\mathcal{B}(\mathcal{D})} + \| \Phi(\underline{\widehat{u}}) - \underline{\widehat{u}} \|_{\mathcal{B}(\mathcal{D})} \\ &\leq \kappa c_0 h^\eta + \| \Phi(\underline{\widehat{u}}) - \underline{\widehat{u}} \|_{\mathcal{B}(\mathcal{D})}. \end{aligned} \tag{5.39}$$

In the sequel, we derive a relation for the difference $\Phi(\underline{\widehat{u}}) - \underline{\widehat{u}}$. From the definition of Φ and the identity (5.32), it follows

$$\big(\Phi(\underline{\widehat{u}})\big)_n - \widehat{u}_n = - \sum_{\nu=0}^{n-1} r(t_n, t_\nu) d_{\nu+1}.$$

Provided that the exact solution has the regularity property $u'' \in \mathcal{B}([0,T], \mathcal{X})$, the defects are bounded by

$$\| d_n \|_{\mathcal{X}} \leq h^2 \big\| u'' \big\|_{\mathcal{B}([0,T],\mathcal{X})},$$

see (5.33). Together with the stability estimate (5.36) and Relation (5.38) we thus have

$$\begin{aligned} \big\| \big(\Phi(\underline{\widehat{u}})\big)_n - \widehat{u}_n \big\|_{\mathcal{D}} &\leq \sum_{\nu=0}^{n-1} \| r(t_n, t_\nu) \|_{\mathcal{D} \leftarrow \mathcal{X}} \| d_{\nu+1} \|_{\mathcal{X}} \\ &\leq c_1 \hat{M}_1 \big\| u'' \big\|_{\mathcal{B}([0,T],\mathcal{X})} \cdot h \big(1 + |\log(h)|\big). \end{aligned}$$

Therefore, we receive the following bound with constant C depending on t_N, the bound for the stability function and on $\big\| u'' \big\|_{\mathcal{B}([0,T],\mathcal{X})}$

$$\big\| \Phi(\underline{\widehat{u}}) - \underline{\widehat{u}} \big\|_{\mathcal{B}(\mathcal{D})} \leq C h \big(1 + |\log(h)|\big). \tag{5.40}$$

For sufficiently small stepsizes $0 < h \le \overline{h}$ this expression is bounded by $(1-\kappa)c_0 h^\eta$ which proves

$$\left\|\Phi(\underline{v}) - \underline{\widehat{u}}\right\|_{\mathcal{B}(\mathcal{D})} \le c_0 h^\eta,$$

see (5.39).

2. Due to the fact that the backward Euler solution $\underline{u}$ is a fixed point of the nonlinear map Φ, it holds

$$\begin{aligned}\|\underline{u} - \underline{\widehat{u}}\|_{\mathcal{B}(\mathcal{D})} = \left\|\Phi(\underline{u}) - \underline{\widehat{u}}\right\|_{\mathcal{B}(\mathcal{D})} &\le \left\|\Phi(\underline{u}) - \Phi(\underline{\widehat{u}})\right\|_{\mathcal{B}(\mathcal{D})} + \left\|\Phi(\underline{\widehat{u}}) - \underline{\widehat{u}}\right\|_{\mathcal{B}(\mathcal{D})} \\ &\le \kappa\|\underline{u} - \underline{\widehat{u}}\|_{\mathcal{B}(\mathcal{D})} + \left\|\Phi(\underline{\widehat{u}}) - \underline{\widehat{u}}\right\|_{\mathcal{B}(\mathcal{D})}.\end{aligned}$$

From this, we further receive the error estimate

$$\|\underline{u} - \underline{\widehat{u}}\|_{\mathcal{B}(\mathcal{D})} \le \frac{1}{1-\kappa}\left\|\Phi(\underline{\widehat{u}}) - \underline{\widehat{u}}\right\|_{\mathcal{B}(\mathcal{D})}. \tag{5.41}$$

By means of (5.40), now the desired convergence estimate

$$\|u_n - u(t_n)\|_{\mathcal{D}} \le Ch\big(1 + |\log(h)|\big)$$

follows. The constant C depends on t_N and on the bounds for the stability function and u''.

Combining the above local existence and convergence result which is valid for times $t_n \ge 0$ as long as

$$\|u_n - u(t_n)\|_{\mathcal{D}} \le c_0 h^\eta,$$

yields the desired result on existence and convergence for the whole interval $[0,T]$. This concludes the proof. $\square$

5.3.2. Existence and convergence of a general Runge-Kutta method

For deriving the existence and convergence result for a general Runge-Kutta method, we mainly follow the proof of Theorem 5.3 and consider a fixed point iteration for the internal stages. The error estimate for the stages is proven in a similar way to Section 4.2, see Relation (4.21) and below, and the convergence result for the Runge-Kutta solution then follows by means of the auxiliary estimate stated in Section 5.1.3. Recall that p and q denote the order and stage order of the Runge-Kutta method. Let the method (3.2) be strongly $A(\vartheta)$-stable for $\vartheta > \pi - \varphi$ where φ denotes the angle from (5.29).

Theorem 5.4. (Ostermann & Thalhammer [27, Theorem 8]) *In the above situation, suppose that the analytical solution of equation* (3.1) *fulfils the smoothness requirements*

$$u^{(r)} \in \mathcal{B}([0,T],\mathcal{D}) \qquad \textit{and} \qquad u^{(r+1)} \in \mathcal{B}([0,T],\mathcal{X}) \qquad \textit{for } r = \min\{p, q+1\}.$$

Then, there exist constants $C > 0$ and $\overline{h} > 0$ such that for $0 < h \le \overline{h}$ the Runge-Kutta solution u_n exists as long as $t_n \le T$, and the error estimate

$$\|u_n - u(t_n)\|_{\mathcal{D}} + \|U_{ni} - u(t_n + c_i h)\|_{\mathcal{D}} \le Ch^r\big(1 + |\log(h)|\big), \qquad 0 \le t_n \le T,$$

holds for $1 \le i \le s$. The quantity C depends on T but not on n or h.

Proof. As in the proof of Theorem 5.3, we consider the formally semilinear problem (5.28) and represent the numerical solution by a discrete variation-of-constants formula. For this, we define the linear operator $\mathcal{A}(t_n + ch) : \mathcal{X}^s \to \mathcal{X}^s$ through

$$\mathcal{A}(t_n + ch) = \operatorname{diag}\big(A(t_n + c_1 h), \dots, A(t_n + c_s h)\big),$$

and we introduce the useful abbreviations

$$\mathcal{J}(t_n + ch) = \big(\mathcal{I} - h\mathcal{O}\!\mathcal{A}(t_n + ch)\big)^{-1} \quad \text{and} \quad \mathcal{K}(t_n + ch) = \big(\mathcal{I} - h\mathcal{A}(t_n + ch)\mathcal{O}\big)^{-1}.$$

Due to the resolvent condition (5.29) and the strong stability of the Runge-Kutta method, these operators are well-defined and bounded on $\mathcal{X}^s$ for stepsizes $h > 0$ small enough, see Result C.2. We denote by $C_{\mathcal{J}}$ a constant such that the following estimate is valid

$$\big\|\mathcal{J}(t_n + ch)\big\|_{\mathcal{D}^s \leftarrow \mathcal{D}^s} + h\big\|\mathcal{J}(t_n + ch)\big\|_{\mathcal{D}^s \leftarrow \mathcal{X}^s} \le C_{\mathcal{J}}.$$

Furthermore, we employ the sequence notation

$$G(t_n + ch, U_n) = \big(G(t_n + c_i h, U_{ni})\big)_{i=1}^{s} \qquad \text{for} \quad U_n = (U_{ni})_{i=1}^{s}.$$

With this abbreviations, the Runge-Kutta scheme (3.2) takes the form

$$U_n' = \mathcal{A}(t_n + ch)U_n + G(t_n + ch, U_n), \quad U_n = \mathbb{1}\, u_n + h\mathcal{O}U_n', \quad u_{n+1} = u_n + hb^T U_n',$$

and a short calculation gives

$$\begin{aligned} U_n &= \mathcal{J}(t_n + ch)\big(\mathbb{1}\, u_n + h\mathcal{O}G(t_n + ch, U_n)\big) \\ u_{n+1} &= R\big(h\mathcal{A}(t_n + ch)\big)u_n + hb^T\mathcal{K}(t_n + ch)G(t_n + ch, U_n) \end{aligned}$$

with the stability function

$$R\big(h\mathcal{A}(t_n + ch)\big) = 1 + hb^T\mathcal{A}(t_n + ch)\mathcal{J}(t_n + ch)\mathbb{1}.$$

Solving this recursion for the numerical solution yields

$$\begin{aligned} u_n = {} & R\big(h\mathcal{A}(t_{n-1} + ch)\big) \cdots R\big(h\mathcal{A}(t_0 + ch)\big)u_0 \\ & + h\sum_{\nu=0}^{n-1} R\big(h\mathcal{A}(t_{n-1} + ch)\big) \cdots R\big(h\mathcal{A}(t_{\nu+1} + ch)\big)b^T\mathcal{K}(t_\nu + ch)G(t_\nu + ch, U_\nu). \end{aligned}$$

We denote by $R(t_n, t_k)$ the discrete transition operator

$$R(t_n, t_k) = R\big(h\mathcal{A}(t_{n-1} + ch)\big) \cdots R\big(h\mathcal{A}(t_k + ch)\big), \qquad 0 \le k < n \le N,$$

and set $R(t_n, t_n) = \mathcal{I}$.

By González & Palencia [12, Theorem 1.1] the stability estimate

$$\big\| R(t_n, t_k) - R(\infty)^{n-k} \big\|_{\mathcal{D}^s \leftarrow \mathcal{X}^s} \le C t_{n-k}^{-1}, \qquad 0 \le k < n \le N,$$

is available. The constant C depends on T but is independent of n and h. In view of estimations performed later on, we choose a constant $\hat{M}_1$ such that following relation holds for $0 \le k < n \le N$, see also Result C.2,

$$\begin{aligned} &\big\| \mathcal{J}(t_n + ch) \mathbb{1} R(t_n, t_{k+1}) \big\|_{\mathcal{D}^s \leftarrow \mathcal{X}^s} + \big\| \mathcal{J}(t_n + ch) \mathbb{1} R(t_n, t_{k+1}) b^T \mathcal{K}(t_k + ch) \big\|_{\mathcal{D}^s \leftarrow \mathcal{X}^s} \\ &\quad + \big\| \mathcal{J}(t_n + ch) \mathbb{1} R(t_n, t_{k+1}) b^T \mathcal{K}(t_k + ch) \mathcal{A}(t_k + ch) \big\|_{\mathcal{D}^s \leftarrow \mathcal{D}^s} \le \hat{M}_1 t_{n-k}^{-1}. \end{aligned} \tag{5.42}$$

In accordance with the discrete variation-of-constants formula (5.31) for the backward Euler method, we consequently receive from above

$$\begin{aligned} U_n &= \mathcal{J}(t_n + ch)\big(\mathbb{1} u_n + h\mathcal{Q}G(t_n + ch, U_n)\big) \\ u_n &= R(t_n, 0)u_0 + h \sum_{\nu=0}^{n-1} R(t_n, t_{\nu+1}) b^T \mathcal{K}(t_\nu + ch) G(t_\nu + ch, U_\nu). \end{aligned} \tag{5.43}$$

For the values of the exact solution the analogous representation is the following

$$\begin{aligned} \widehat{U}_n &= \mathcal{J}(t_n + ch)\Big(\mathbb{1} \widehat{u}_n + h\mathcal{Q}G(t_n + ch, \widehat{U}_n) + D_n\Big) \\ \widehat{u}_n &= R(t_n, 0)u_0 + \sum_{\nu=0}^{n-1} R(t_n, t_{\nu+1}) d_{\nu+1} \\ &\quad + h \sum_{\nu=0}^{n-1} R(t_n, t_{\nu+1}) b^T \mathcal{K}(t_\nu + ch)\big(G(t_\nu + ch, \widehat{U}_\nu) + \mathcal{A}(t_\nu + ch) D_\nu\big). \end{aligned}$$

The defects d_n and D_n are given by Lemma 3.1.

1. For the construction of the internal stages we use the fixed point iteration based on the discrete variation-of-constants formula (5.43). The nonlinear operator Ψ maps a sequence $\underline{V} = (V_n)_{n=0}^N$ where $t_N \le T < t_{N+1}$ to another sequence $\Psi(\underline{V})$ with components

$$\begin{aligned} \Psi(\underline{V})_n &= \mathcal{J}(t_n + ch) \mathbb{1} R(t_n, 0)u_0 + h\mathcal{J}(t_n + ch)\mathcal{Q}G(t_n + ch, V_n) \\ &\quad + h \sum_{\nu=0}^{n-1} \mathcal{J}(t_n + ch) \mathbb{1} R(t_n, t_{\nu+1}) b^T \mathcal{K}(t_\nu + ch) G(t_\nu + ch, V_\nu). \end{aligned}$$

As domain of Φ we choose the closed subset

$$\mathcal{W} = \Big\{ \underline{V} = (V_n)_{n=0}^N \in \mathcal{O}^{(N+1)s} : \big\|\underline{V} - \underline{\widehat{U}}\big\|_{\mathcal{B}(\mathcal{D}^s)} \le c_0 h^\eta \Big\},$$

endowed with $\|\cdot\|_{\mathcal{B}(\mathcal{D}^s)}$, for some constants $c_0 > 0$ and $0 < \eta < 1$. In the sequel, we show that the following holds for $0 < h \le \overline{h}$ chosen sufficiently small.

Ψ is a contraction on $\mathcal{W}$.

(i) Ψ is contractive on $\mathcal{W}$, i.e. for all elements $\underline{V}, \underline{W} \in \mathcal{W}$ the bound

$$\big\|\Psi(\underline{V}) - \Psi(\underline{W})\big\|_{\mathcal{B}(\mathcal{D}^s)} \le \kappa \|\underline{V} - \underline{W}\|_{\mathcal{B}(\mathcal{D}^s)}$$

is valid for some $\kappa < 1$.

(ii) Ψ maps $\mathcal{W}$ into $\mathcal{W}$. Thus it holds

$$\big\|\Psi(\underline{V}) - \underline{\widehat{U}}\big\|_{\mathcal{B}(\mathcal{D}^s)} \le c_0 h^\eta$$

whenever $\big\|\underline{V} - \underline{\widehat{U}}\big\|_{\mathcal{B}(\mathcal{D}^s)} \le c_0 h^\eta$.

Then, the fixed point theorem of Banach ensures the existence of a unique fixed point $\underline{U}$ belonging to $\mathcal{W}$.

(i) By definition, we have for elements $\underline{V}, \underline{W} \in \mathcal{W}$

$$\begin{aligned}\big(\Psi(\underline{V}) - \Psi(\underline{W})\big)_n &= h\mathcal{J}(t_n + ch)\mathcal{Q}\big(G(t_n + ch, V_n) - G(t_n + ch, W_n)\big)\\ &\quad + h\sum_{\nu=0}^{n-1} \mathcal{J}(t_n + ch)\mathbb{1}\, R(t_n, t_{\nu+1}) b^T \mathcal{K}(t_\nu + ch)\big(G(t_\nu + ch, V_\nu) - G(t_\nu + ch, W_\nu)\big).\end{aligned}$$

Estimating this expression in $\mathcal{D}^s$ yields

$$\begin{aligned}\big\|\big(\Psi(\underline{V}) - \Psi(\underline{W})\big)_n\big\|_{\mathcal{D}^s} &\le h\big\|\mathcal{J}(t_n + ch)\mathcal{Q}\big\|_{\mathcal{D}^s\leftarrow\mathcal{X}^s}\|G(t_n + ch, V_n) - G(t_n + ch, W_n)\|_{\mathcal{X}^s}\\ &\quad + h\sum_{\nu=0}^{n-1} \big\|\mathcal{J}(t_n + ch)\mathbb{1}\, R(t_n, t_{\nu+1}) b^T \mathcal{K}(t_\nu + ch)\big\|_{\mathcal{D}^s\leftarrow\mathcal{X}^s}\\ &\qquad \cdot \|G(t_\nu + ch, V_\nu) - G(t_\nu + ch, W_\nu)\|_{\mathcal{X}^s}.\end{aligned}$$

Remark that, analogously to (5.37), the relation

$$\big\|G(t_k + ch, V_k) - G(t_k + ch, W_k)\big\|_{\mathcal{X}^s} \le c_0 L h^\eta \big\|\underline{V} - \underline{W}\big\|_{\mathcal{B}(\mathcal{D}^s)}$$

holds. Together with the stability bounds from (5.42) and the auxiliary estimate (5.38), this implies

$$\big\|\Psi(\underline{V}) - \Psi(\underline{W})\big\|_{\mathcal{B}(\mathcal{D}^s)} \le c_0 L\Big(c_2 C_{\mathcal{J}} h^\eta + c_1 \hat{M}_1 h^\eta \big(1 + |\log(h)|\big)\Big)\big\|\underline{V} - \underline{W}\big\|_{\mathcal{B}(\mathcal{D}^s)}$$

where $\|\mathcal{Q}^{-1}\|_{\boldsymbol{R}^n \leftarrow \boldsymbol{R}^n} \leq c_2$. Thus, Ψ is contractive for $0 < h \leq \overline{h}$ small enough.

(ii) Arguments similar to that of the proof of Theorem 5.3 yield the following estimate for every $\underline{V} \in \mathcal{W}$

$$\|\Psi(\underline{V}) - \underline{\widehat{U}}\|_{\mathcal{B}(\mathcal{D}^s)} \leq \kappa c_0 h + \|\Psi(\underline{\widehat{U}}) - \underline{\widehat{U}}\|_{\mathcal{B}(\mathcal{D}^s)}.$$

Consequently, it remains to estimate the difference

$$\begin{aligned}\Psi(\underline{\widehat{U}}) - \widehat{U} = &-\sum_{\nu=0}^{n-1} \mathcal{J}(t_n + ch)\mathbb{1}\, R(t_n, t_{\nu+1}) d_{\nu+1} - \mathcal{J}(t_n + ch) D_n \\ &- h \sum_{\nu=0}^{n-1} \mathcal{J}(t_n + ch)\mathbb{1}\, R(t_n, t_{\nu+1}) b^T \mathcal{K}(t_\nu + ch)\mathcal{A}(t_\nu + ch) D_\nu.\end{aligned}$$

By means of (5.38) and (5.42) we get

$$\begin{aligned}\big\|\big(\Psi(\underline{\widehat{U}}) - \widehat{U}\big)_n\big\|_{\mathcal{D}^s} \leq &\sum_{\nu=0}^{n-1} \big\|\mathcal{J}(t_n + ch)\mathbb{1}\, R(t_n, t_{\nu+1})\big\|_{\mathcal{D}^s \leftarrow \mathcal{X}} \big\|d_{\nu+1}\big\|_{\mathcal{X}} \\ &+ h \sum_{\nu=0}^{n-1} \big\|\mathcal{J}(t_n + ch)\mathbb{1}\, R(t_n, t_{\nu+1}) b^T \mathcal{K}(t_\nu + ch)\mathcal{A}(t_\nu + ch)\big\|_{\mathcal{D}^s \leftarrow \mathcal{D}^s} \big\|D_\nu\big\|_{\mathcal{D}^s} \\ &+ \big\|\mathcal{J}(t_n + ch)\big\|_{\mathcal{D}^s \leftarrow \mathcal{D}^s} \|D_n\|_{\mathcal{D}^s} \\ \leq{}& c_1\big(1 + |\log(h)|\big)\hat{M}_1\Big(\big\|h^{-1}\underline{d}\big\|_{\mathcal{B}(\mathcal{X})} + \|\underline{D}\|_{\mathcal{B}(\mathcal{D}^s)}\Big) + C_{\mathcal{J}}\|\underline{D}\|_{\mathcal{B}(\mathcal{D}^s)}.\end{aligned}$$

An application of Lemma 3.1 and Remark 3.1 below, further shows

$$\|\Psi(\underline{\widehat{U}}) - \underline{\widehat{U}}\|_{\mathcal{B}(\mathcal{D}^s)} \leq C\Big(\big\|u^{(r)}\big\|_{\mathcal{B}(\mathcal{D})} + \big\|u^{(r+1)}\big\|_{\mathcal{B}(\mathcal{X})}\Big) \cdot h^r\big(1 + |\log(h)|\big) \tag{5.44}$$

with constant C depending on t_N, the stability bound $\hat{M}_1$, and on the method. Now the estimate

$$\|\Psi(\underline{V}) - \underline{\widehat{U}}\|_{\mathcal{B}(\mathcal{D}^s)} \leq c_0 h^\eta$$

follows for $0 < h \leq \overline{h}$ small enough.

2. In order to derive the error estimate for the internal stages, we use essentially the same arguments as for the backward Euler solution and obtain

$$\|U_n - \widehat{U}_n\|_{\mathcal{B}(\mathcal{D}^s)} \leq \frac{1}{1-\kappa}\,\|\Psi(\underline{\widehat{U}}) - \underline{\widehat{U}}\|_{\mathcal{B}(\mathcal{D}^s)} \leq Ch^r\big(1 + |\log(h)|\big),$$

see (5.41) and (5.44). The constant C depends on t_N, the bound $\hat{M}_1$ for the stability function, on the method and further on the bounds for the derivatives $u^{(r)}$ in $\mathcal{D}$ and $u^{(r+1)}$ in $\mathcal{X}$, respectively. Now the desired error bound for the Runge-Kutta solution

$$\|u_n - u(t_n)\|_{\mathcal{D}} \leq Ch^r\big(1 + |\log(h)|\big)$$

follows from Relation (5.14) of Section 5.1.3 together with Remark 3.1. This concludes the proof. □

Part III.

Dynamical behaviour near steady state solutions

6. Basic concepts of geometric theory

In the present part, we study the long-term behaviour of Runge-Kutta time discretizations of nonlinear parabolic evolution equations

$$u'(t) = F(u(t)), \qquad t > 0, \qquad u(0) \text{ given},$$

near a stationary solution. In order to keep this part in a reasonable length, we elaborate the saddle point case only. Sections 6.2 and 6.3 contain the preliminary remarks for Chapter 7. There, we state the assumptions on the equation and the method, and we further introduce the basic notion of stability in the sense of Lyapunov. For the sake of completeness, we recall in Section 6.1 basics from the theory of dynamical systems. Due to the fact that these definitions are specialized to our present needs later on, Sections 6.2 and 6.3 are selfcontained.

In Chapter 7 we then demonstrate the principle of linearized stability which says that the stationary solution of the nonlinear equation has the same stability properties as the null solution of the linearized equation, see LUNARDI [25, Chapter 9], and we prove the corresponding results for the numerical discretization. In the first place, we consider the asymptotically stable case where analytical and numerical solutions lying nearby the equilibrium point are attracted exponentially fast. For this, we expand the concepts of Section 4.2. Secondly, for a stationary solution of saddle point type, we construct the local unstable and stable manifold, respectively, and their discrete counterparts, by using the concepts of LUBICH & OSTERMANN [24]. The auxiliary results for Chapter 7 are mainly provided by Chapters D and E.

The precise contents of Sections 6.1 to 6.3 are as follows. In Section 6.1 we show that every nonlinear evolution equation of parabolic type defines a local dynamical system, and we further introduce fundamental notions from the theory of dynamical systems such as stable orbits and invariant sets. For convenience, we recall in Section 6.2 the precise assumptions on the evolution equation, and we remark that a stationary solution of the equation is also an equilibrium point of the Runge-Kutta scheme. We further specialize the important notion of stability introduced in Section 6.2 for dynamical systems to evolution equations. Besides, we define the unstable manifold of an equilibrium point which is an important tool for characterizing instability. In view of Chapter 7, we linearize in Section 6.3 the nonlinear equation around a stationary solution. As a first step, we then study the stability properties of the associated homogeneous problem and the asymptotic behaviour of the Runge-Kutta time discretization under certain assumptions on the spectrum of the involved sectorial operator.

For a thorough representation of the theory of dynamical systems and the effect of numerical discretizations, we refer to the monographs Stuart & Humphries [32] and Wiggins [35]. The geometric properties of evolution equations in Banach spaces or Hilbert spaces, respectively, are studied in Henry [15] and Stuart [31].

6.1. Nonlinear evolution equations as local dynamical systems

In the present section, we consider the following nonlinear evolution equation

$$u'(t) = F(u(t)), \qquad t > 0,$$

on a Banach space $\mathcal{X}$, and we require that the right-hand side $F : \mathcal{O} \subset \mathcal{D} \to \mathcal{X}$ satisfies the assumptions from Chapter 1. Assume for a moment that for each element $u_0 \in \mathcal{Z}$ of a closed subset $\mathcal{Z} \subset \mathcal{O}$ a strict solution u satisfying the initial condition $u(0) = u_0$ exists for all positive times and that $u(t)$ belongs to $\mathcal{Z}$ for every $t \geq 0$. In order to indicate the dependence of u on the initial value u_0, we employ the notation $u(\cdot, u_0)$. For some $t \geq 0$ we define the map $S(t)$ on $\mathcal{Z}$ through

$$S(t)u_0 = u(t, u_0), \qquad u_0 \in \mathcal{Z}.$$

Then, it is obvious that $S(t)$ possesses the semigroup properties

$$S(0) = I \qquad \text{and} \qquad S(s)S(t) = S(s+t) \quad \text{for all } s, t \geq 0.$$

Besides, $u(\cdot, u_0)$ depends continuously on the initial value u_0, i.e. $S(t)$ is a continuous map on $\mathcal{Z}$, see Lunardi [25, Proposition 8.2.3]. These considerations imply that the the *flow* of the equation

$$T(t, u_0) = S(t)u_0 = u(t, u_0), \qquad t \geq 0, \quad u_0 \in \mathcal{Z},$$

defines a dynamical system on $\mathcal{Z}$.

Definition 6.1. (Henry [15, Def. 4.1.1]) Let $\mathcal{Z}$ denote a Banach space. Then, a continuous map T from $\mathcal{M} = [0, \infty) \times \mathcal{Z}$ to $\mathcal{Z}$ is called a *dynamical system* or a *semiflow* on $\mathcal{Z}$ if the identity $T(0, z) = z$ is valid for every $z \in \mathcal{Z}$ and if further for all $s, t \geq 0$ and $z \in \mathcal{Z}$ the relation $T(s, T(t, z)) = T(s + t, z)$ holds.

For usual, the solution $u(\cdot, u_0)$ of a nonlinear problem exists only locally, and, besides, the length of the time interval of existence, denoted by

$$\tau(u_0),$$

depends strongly on the actual initial value u_0. Therefore, it is useful to introduce the notion of a *local dynamical system*. The precise definition is given below. It is shown in LUNARDI [25], see Proposition 8.2.3 and remark below, that the above nonlinear evolution equation of parabolic type defines a local dynamical system on the domain $\mathcal{O}$ of the right-hand side F through $T(t,u_0) = u(t,u_0)$.

Definition 6.2. (LUNARDI [25, Section 8.2]) Let $\mathcal{Z}$ denote a closed subset of a Banach space, and assume that the set $\mathcal{M} = \{(t,z) : z \in \mathcal{Z}, 0 \le t < \tau(z)\} \subset [0,\infty) \times \mathcal{Z}$ is open. A continuous map $T : \mathcal{M} \to \mathcal{Z}$ is called a *local dynamical system* or a *local semiflow* on $\mathcal{Z}$ if $T(0,z) = z$ holds for every $z \in \mathcal{Z}$ and if further the following requirement is satisfied. For all $z \in \mathcal{Z}$ and times $0 \le t < \tau(z)$, $0 \le s \le \tau(T(t,z))$ it holds $s+t < \tau(z)$, and moreover the relation $T(s,T(t,z)) = T(s+t,z)$ is valid.

In the following we define the fundamental notions of a stable orbit in the sense of Lyapunov and an invariant set for a dynamical systems.

Definition 6.3. (HENRY [15, Chapters 4 and 5], STUART [31, Section 2.9]) Let T be a dynamical system on the Banach space $(\mathcal{Z}, \|\cdot\|_{\mathcal{Z}})$.

(i) The *(positive) orbit* through $z_0 \in \mathcal{Z}$ is the set $\gamma(z_0) = \{T(t,z_0) : t \ge 0\}$.

Assume that there exists a map $z : (-\infty, 0] \to \mathcal{Z}$ such that

$$z(0) = z_0 \qquad \text{and} \qquad T(s,z(t)) = z(s+t)$$

for all $0 \le s \le -t$ and $t \le 0$. Then, the set $\{z(t) : t \le 0\}$ is called a *negative orbit* through z_0.

(ii) An orbit $\gamma(z_0)$ or the point z_0, respectively, is called *(Lyapunov) stable* if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $t \ge 0$ and for elements $z \in \mathcal{Z}$ with $\|z - z_0\|_{\mathcal{Z}} < \delta$ also $\|T(t,z) - T(t,z_0)\|_{\mathcal{Z}} < \varepsilon$ holds.

A stable orbit is *asymptotically stable* whenever $\|T(t,z) - T(t,z_0)\|_{\mathcal{Z}}$ converges to 0 for $t \to \infty$.

An orbit is called *unstable* if it is not stable.

(iii) A subset $Z \subset \mathcal{Z}$ is called *invariant* if $\{T(t,z) : t \ge 0, z \in Z\} = Z$.

The simplest case of an invariant set is an *equilibrium point* $\overline{z}$ which is defined by the following property

$$\gamma(\overline{z}) = \{\overline{z}\},$$

see HENRY [15, Definition 4.1.2]. Another example for an invariant set is the *unstable set* $\mathcal{U}$ of an equilibrium point $\overline{z}$

$$\mathcal{U} = \big\{z_0 \in \mathcal{Z} : \text{there exists a negative orbit } \{z(t) : t \le 0\} \text{ through } z_0 \\ \text{and } z(t) \text{ converges to } \overline{z} \text{ as } t \text{ goes to } -\infty\big\}.$$

For some $\varrho > 0$ the *local unstable manifold* $\mathcal{U}_\varrho$ of an equilibrium point $\overline{z}$ is defined through

$$\mathcal{U}_\varrho = \big\{ z_0 \in \mathcal{U} : \|z(t) - \overline{z}\|_{\mathcal{Z}} \leq \varrho \text{ for all } t \leq 0 \big\}.$$

Analogously, the *stable set* $\mathcal{S}$ and the *local stable manifold* $\mathcal{S}_\varrho$ of the equilibrium point $\overline{z}$ are given by

$$\mathcal{S} = \big\{ z_0 \in \mathcal{Z} : \text{ there exists a orbit } \{z(t) : t \geq 0\} \text{ through } z_0 \text{ and } z(t) \to \overline{z} \text{ as } t \to \infty \big\}$$
$$\mathcal{S}_\varrho = \big\{ z_0 \in \mathcal{S} : \|z(t) - \overline{z}\|_{\mathcal{Z}} \leq \varrho \text{ for all } t \geq 0 \big\}.$$

The above definitions of unstable sets and local unstable manifolds for dynamical systems are in the lines of Stuart [31, Chapter 5]). The adequate definitions for evolution equations are found in Section 6.2 below.

6.2. Stationary solutions and local manifolds

In the sequel, we consider solely nonlinear evolution equations where the right-hand side does not depend explicitely on the time variable

$$u'(t) = F(u(t)), \qquad t > 0. \tag{6.1}$$

Our requirements on the nonlinearity F, stated in Section 1.1.1, then are reducible to the following assumptions.

Assumption 6.1. *We suppose that $F : \mathcal{O} \subset \mathcal{D} \to \mathcal{X}$ is Fréchet differentiable and that its derivative F' is locally Lipschitz continuous, i.e. for every $v^* \in \mathcal{O}$ there exists a constant $L > 0$ such that the following relation holds for all elements $v, w \in \mathcal{O}$ with $\|v - v^*\|_{\mathcal{D}} \leq R$ and $\|w - v^*\|_{\mathcal{D}} \leq R$*

$$\|F'(v) - F'(w)\|_{\mathcal{X} \leftarrow \mathcal{D}} \leq L \|v - w\|_{\mathcal{D}}.$$

Moreover, we assume that $F'(v^*)$ is sectorial in $\mathcal{X}$ and that its graph norm is equivalent to the norm in $\mathcal{D}$ for all $v^* \in \mathcal{O}$.

All constants which arise from interchanging equivalent norms such as $\|\cdot\|_{\mathcal{D}}$ and the graph norm $\|\cdot\|_{\mathcal{X}} + \|F'(\overline{u}) \cdot\|_{\mathcal{X}}$ are henceforth denoted by C_γ.

In accordance with the notion of the theory of dynamical systems, a solution u of equation (6.1) is called an *equilibrium point* or a *stationary solution* of (6.1) if it does not dependent on the time

$$u'(t) = 0 \qquad \text{for all} \quad t > 0,$$

see Henry [15, Section 5.1] and Lunardi [25, Chapter 9].

We henceforth assume that the considered equation (6.1) possesses a stationary solution denoted by $\overline{u}$. Clearly, the element $\overline{u} \in \mathcal{O}$ fulfils $F(\overline{u}) = 0$. Notice that $\overline{u}$ is likewise an equilibrium point of the Runge-Kutta method

$$U_n' = F(U_n), \qquad U_n = \mathbb{1}\, u_n + h\mathcal{Q}U_n', \qquad u_{n+1} = u_n + hb^T U_n', \tag{6.2}$$

where $F(U_n) = \big(F(U_{n1}), \ldots, F(U_{ns})\big)^T$ for $U_n = (U_{n1}, \ldots, U_{ns})^T$, and, in particular, this is also true for the backward Euler method

$$\frac{u_{n+1} - u_n}{h} = F(u_{n+1}). \tag{6.3}$$

Within the framework of geometric theory, it is of interest whether solutions with initial values that lie close to a stationary solution remain close to it for all positive times. In this case, the equilibrium point is called *stable*. If furthermore all nearby solutions converge to the equilibrium point it is said to be *asymptotically stable*. The precise definition of stability and asymptotic stability for stationary solutions is given below.

For indicating the dependence of the solution u of the initial value problem

$$u'(t) = F(u(t)), \quad t > 0, \qquad u(0) = u_0,$$

on the actual initial value u_0, we henceforth employ the notation $u(\cdot, u_0)$. For the definition of strict solutions we refer to Chapter B.

Definition 6.4. (Henry [15, Section 5.1], Lunardi [25, Chapter 9])

(i) A stationary solution $\overline{u}$ of (6.1) is *stable in $\mathcal{D}$ (in the sense of Lyapunov)* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for every $u_0 \in \mathcal{D}$ with $\|u_0 - \overline{u}\|_{\mathcal{D}} \leq \delta$ the following conditions hold. The strict solution u of (6.1) with initial value $u(0) = u_0$ exists for all positive times and satisfies $\|u(t) - \overline{u}\|_{\mathcal{D}} \leq \varepsilon$, uniformly for $t \geq 0$.

(ii) A stable solution $\overline{u}$ is *asymptotically stable* if in addition the relation

$$\lim_{t \to \infty} \|u(t, u_0) - \overline{u}\|_{\mathcal{D}} = 0$$

is valid, uniformly for all $u_0 \in \mathcal{O}$ belonging to a neighbourhood of $\overline{u}$.

(iii) A stationary solution $\overline{u}$ is called *unstable* if it is not stable.

Our concern is the characterization of stability and instability for stationary solutions. For this, it is useful to introduce the notion of backward solutions and positive or negative orbits, respectively.

Definition 6.5. (i) For some $u_0 \in \mathcal{O}$ let $u(\cdot, u_0)$ be the strict solution of (6.1) with initial value u_0. Then, the *(positive) orbit* through $u_0 \in \mathcal{O}$ is the set $\{u(t, u_0) : t \geq 0\}$.

(ii) Assume that a map $\varphi : (-\infty, 0] \to \mathcal{O}$ with $\varphi(0) = u_0$ satisfies the following requirements for every $t \leq 0$. For the solution ψ_t of (6.1) with initial value $\psi_t(0) = \varphi(t)$ the identity

$$\psi_t(s) = \varphi(s+t)$$

holds for all $0 \leq s \leq -t$. Then, φ is said to be a *backward solution* of (6.1), and the set $\{\varphi(t) : t \geq 0\}$ is a *negative orbit* through u_0.

In the following we define local unstable and stable manifolds for equilibrium points. Roughly spoken, the local unstable manifold of an equilibrium point is the set of points from which nearby solutions defined backward in time converge to the equilibrium point. The local stable manifold of an equilibrium point $\overline{u}$ contains the initial values of all solutions lying close to the equilibrium point which converge to $\overline{u}$ for $t \to \infty$.

Definition 6.6. Let $\overline{u}$ be an equilibrium point of (6.1) and choose $\varrho > 0$. The *local unstable manifold* $\mathcal{U}_\varrho$ of $\overline{u}$ is defined through

$$\begin{aligned} \mathcal{U}_\varrho = \big\{ u_0 \in \mathcal{O} :\ & \text{there exists a negative orbit } \{\varphi(t) : t \leq 0\} \text{ through } u_0 \\ & \text{with } \|\varphi(t) - \overline{u}\|_{\mathcal{D}} \leq \varrho \text{ for all } t \leq 0 \text{ and } \varphi(t) \to \overline{u} \text{ as } t \to -\infty \big\}, \end{aligned}$$

and the set $\mathcal{S}_\varrho$ given by

$$\begin{aligned} \mathcal{S}_\varrho = \big\{ u_0 \in \mathcal{O} :\ & \text{the solution } u(\cdot, u_0) \text{ exists on } [0,\infty) \text{ and satisfies} \\ & \|u(t,u_0) - \overline{u}\|_{\mathcal{D}} \leq \varrho \text{ for } t \geq 0 \text{ and } u(t,u_0) \to \overline{u} \text{ as } t \to \infty \big\} \end{aligned}$$

is called the *local stable manifold* $\mathcal{S}_\varrho$ of $\overline{u}$.

It is shown in Section 7.2 that $\mathcal{U}_\varrho$ and $\mathcal{S}_\varrho$ can essentially be represented as graphs and thus have the structure of a manifold.

6.3. Principle of linearized stability

For our analysis, it is useful to linearize the right-hand side F around the equilibrium point and to interpret equation (6.1) as a semilinear problem

$$u'(t) = Au(t) + G(u(t)), \qquad t > 0, \tag{6.4a}$$

where the linear operator A and the map $G : \mathcal{O} \to \mathcal{X}$ are defined through

$$A = F'(\overline{u}) \qquad \text{and} \qquad G(v) = F(v) - Av \quad \text{for } v \in \mathcal{O}. \tag{6.4b}$$

Notice that there are constants $\varphi \in (\pi/2, \pi)$ and $a \in \boldsymbol{R}$ such that the spectrum of A is contained in the set

$$\sigma(A) \subset S_\varphi(a) = \big\{ \lambda \in \boldsymbol{C} : |\arg(\lambda - a)| \geq \varphi \big\}, \tag{6.5}$$

see Assumption 6.1 and Definition A.2. We henceforth assume that the Runge-Kutta method (6.2) is strongly $A(\vartheta)$-stable with $\vartheta > \pi - \varphi$.

In the following chapter we study Runge-Kutta time discretizations of the nonlinear parabolic evolution equation (6.1), and we investigate the stability properties of the equilibrium point $\overline{u}$. As a first step, we consider now the homogeneous equation

$$v'(t) = Av(t), \qquad t > 0, \tag{6.6}$$

and we determine the asymptotic behaviour of the exact and the numerical solution under certain assumptions on the spectrum of the sectorial operator $A = F'(\overline{u}) : \mathcal{D} \to \mathcal{X}$. Obviously, the origin

$$\overline{v} = 0$$

is an equilibrium point of (6.6). For arbitrary initial values $v_0 \in \mathcal{D}$, the analytical solution of (6.6) exists for all positive times and is given by

$$v(t) = e^{tA} v_0, \qquad t \geq 0,$$

whereas the solution $\underline{v} = (v_n)_{n=0}^{\infty}$ of the Runge-Kutta scheme (6.2) has a similar representation involving the stability function of the method

$$v_n = R(hA)^n v_0, \qquad n \geq 0.$$

We henceforth suppose that the spectrum of A does not intersect the imaginary axis, i.e. $\sigma(A) \cap i\boldsymbol{R} = \emptyset$. Thus, it is convenient to introduce the following abbreviations for denoting the part of the spectrum contained in the positive and negative complex plane

$$\sigma_1(A) = \big\{\lambda \in \sigma(A) : \Re(\lambda) > 0\big\} \qquad \text{and} \qquad \sigma_2(A) = \big\{\lambda \in \sigma(A) : \Re(\lambda) < 0\big\}.$$

In accordance with Lubich & Ostermann [24], our precise requirements on the spectrum of the sectorial operator $A = F'(\overline{u})$ are the following, see also Figure 6.1.

Assumption 6.2. *Suppose that the positive part $\sigma_1(A)$ of the spectrum of A is contained in a rectangle $\mathcal{R}$ such that $\Re(\lambda) \geq a_1 > 0$ for all elements $\lambda \in \mathcal{R}$ and that the negative part $\sigma_2(A)$ lies in a sector $\mathcal{S}$ with top $-a_2 < 0$ and opening angle smaller than $\pi/2$. Assume further that $\sigma_1(A)$ consists of a finite number of eigenvalues with finite multiplicity.*

The projections P and Q associated with the negative and the positive part of the spectrum of A are defined through

$$P = \frac{1}{2\pi i} \int_{\partial\mathcal{R}} (\lambda I - A)^{-1} \, \mathrm{d}\lambda \qquad \text{and} \qquad Q = I - P. \tag{6.7}$$

For the following considerations, bounds for the analytical semigroup and the corresponding estimates for the stability function of the Runge-Kutta method (6.2) are all

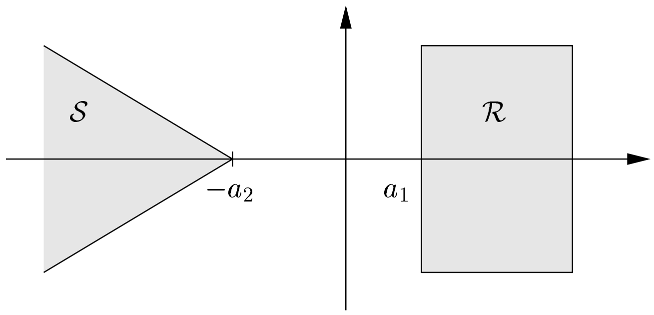

Figure 6.1.: The spectrum of A is contained in the shaded area

important. We collect these auxiliary results below, see also Results A.1, A.3 and C.1. In order to simplify the presentation, we suppose

$$R(\infty) = 0,$$

i.e. it holds $b^T \mathcal{Q}^{-1} \mathbb{1} = 1$.

We consider first the general situation where A is a sectorial operator satisfying the resolvent condition on a sector with top a and opening angle φ, see (6.5). Then, it is well known from Henry [15, Theorem 1.4.3] or Lunardi [25, Propositions 2.1.1 and 2.3.1] that for $\omega > a$ and every integer $k \geq 0$ there exists a constant $M_k > 0$ such that

$$\left\| A^k e^{tA} \right\|_{\mathcal{X} \leftarrow \mathcal{X}} \leq M_k e^{\omega t} t^{-k}, \qquad t > 0, \quad k \geq 0, \tag{6.8a}$$

holds. For a strongly stable Runge-Kutta method the corresponding stability result is valid. For every $\omega > a$ there exist $\overline{h} > 0$ and $\hat{M}_k > 0$ such that for stepsizes $0 < h \leq \overline{h}$ the following bound holds

$$\left\| A^k R(hA)^n \right\|_{\mathcal{X} \leftarrow \mathcal{X}} \leq \hat{M}_k e^{\omega t_n} t_n^{-k}, \qquad t_n \geq t_k, \quad 0 \leq k \leq 2. \tag{6.8b}$$

This result follows from Lubich & Nevanlinna [20] by modifying slightly the proof of Theorem 3.5 there, see also Lubich & Ostermann [24, Lemma 3.1]. Due to the fact that the sectorial operator AQ generates an analytic semigroup in $Q(\mathcal{X})$ and that $\sigma(AQ) = \sigma_2(A)$, for the composition $e^{tA}Q$ the analogous estimate

$$\left\| A^k e^{tA} Q \right\|_{\mathcal{X} \leftarrow \mathcal{X}} \leq M_k^Q e^{-\omega t} t^{-k}, \qquad t > 0, \quad k \geq 0, \tag{6.8c}$$

is valid for $0 \leq \omega < a_2$, see Lunardi [25, Propositions 2.3.3] . Similarly, we have

$$\left\| A^k R(hA)^n Q \right\|_{\mathcal{X} \leftarrow \mathcal{X}} \leq \hat{M}_k^Q e^{-\omega t_n} t_n^{-k}, \qquad t_n \geq t_k, \quad 0 \leq k \leq 2, \tag{6.8d}$$

whenever $0 < h \leq \overline{h}$ is small enough. From LUNARDI [25, Proposition 2.3.3] follows that $t \mapsto e^{tA}P$ as a map from $[0,\infty)$ to $\mathcal{L}(\mathcal{X})$ has an analytic extension to the whole real line satisfying

$$\left\|A^k e^{tA} P\right\|_{\mathcal{X}\leftarrow\mathcal{X}} \leq M_k^P e^{-\omega t}, \qquad t \leq 0, \quad k \geq 0, \tag{6.8e}$$

where $-a_1 < \omega \leq 0$. Using similar ideas as in LUBICH & OSTERMANN [24, Lemma 3.2] shows

$$\left\|A^k R(hA)^n P\right\|_{\mathcal{X}\leftarrow\mathcal{X}} \leq \hat{M}_k^P e^{-\omega t_n}, \qquad t_n \leq 0, \quad 0 \leq k \leq 2, \tag{6.8f}$$

for stepsizes $0 < h \leq \overline{h}$.

The stability properties of the null solution of the linear problem (6.6) follow readily by means of the above auxiliary results. At first, we consider the case where the spectrum of A is contained in the negative complex plane.

Provided that $\sigma_1(A) = \emptyset$ holds then $\overline{v} = 0$ is asymptotically stable.

In this situation, we apply (6.8a) with $-\omega < a < 0$. For arbitrary $\varepsilon > 0$ we set $\delta = \varepsilon / M_0$. Then, for every element $v_0 \in \mathcal{D}$ with $\|v_0\|_{\mathcal{D}} \leq \delta$ the analytical solution $v(t, v_0) = e^{tA} v_0$ of the linear equation (6.6) exists for all positive times and fulfils

$$\|v(t,v_0)\|_{\mathcal{D}} \leq M_0 e^{-\omega t} \|v_0\|_{\mathcal{D}} \leq M_0 \delta \leq \varepsilon \qquad \text{for} \quad t \geq 0.$$

Moreover, the solution $v(\cdot, v_0)$ converges to $\overline{v}$ uniformly for $\|v_0\|_{\mathcal{D}} \leq R$, since

$$\lim_{t\to\infty} \|v(t,v_0) - \overline{v}\|_{\mathcal{D}} = \lim_{t\to\infty} \left\|e^{tA} v_0\right\|_{\mathcal{D}} \leq M_0 R \lim_{t\to\infty} e^{-\omega t} = 0.$$

This proves the asymptotic stability of the null solution, see Definition 6.4. Note further that any solution $v(\cdot, v_0)$ approximates the equilibrium point $\overline{v} = 0$ with exponential speed

$$\|v(t,v_0)\|_{\mathcal{D}} \leq M_0 e^{-\omega t} \|v_0\|_{\mathcal{D}}, \qquad t \geq 0.$$

The corresponding result for the Runge-Kutta solution $\underline{v} = (v_n)_{n=0}^{\infty}$ with initial value v_0

$$\|v_n\|_{\mathcal{D}} \leq \hat{M}_0 e^{-\omega t_n} \|v_0\|_{\mathcal{D}}, \qquad t_n \geq 0.$$

follows in a similar manner by using (6.8d) instead of (6.8c). In case of the original nonlinear problem, the existence of the analytical and the numerical solution is proven by means of a fixed point iteration relying on the modified variation-of-constants formula in the space of (α,α)-Hölder continuous functions and sequences, respectively. In a ball around the equilibrium point $\overline{u}$ of sufficiently small radius for initial values close to $\overline{u}$ the iteration is a contraction and the fixed point theorem of Banach applies.

If the part of the spectrum of A contained in the positive complex plane is not empty, we have the following result.

Whenever $\sigma_1(A) \neq \emptyset$ then $\overline{v} = 0$ is unstable. The unstable manifold $\mathcal{U}$ and the stable manifold $\mathcal{S}$ of the origin are given by $\mathcal{U} = P(\mathcal{X})$ and $\mathcal{S} = Q(\mathcal{D})$.

Due to the linearity of the equation, for arbitrary $p \in P(\mathcal{X})$ the existence of a uniquely determined backward solution v_p, given by

$$v_p(t) = e^{tA}p, \qquad t \leq 0,$$

follows at once and implies $\mathcal{U} = P(\mathcal{X})$. Recall that AP is a bounded operator on $\mathcal{X}$, see (6.8e). Thus, for negative times $t_n \leq 0$, the Runge-Kutta solution $\underline{v} = (v_n)_{n\leq 0}$ is given by

$$v_n = R(hAP)^n p \qquad \text{for} \quad n \leq 0,$$

provided that the stepsize $h > 0$ is sufficiently small. Consequently, the discrete local unstable manifold equals $\mathcal{U}^h = P(\mathcal{X})$. Notice that the analytical solution v_p fulfils the estimate

$$\|v_p(t) - \overline{v}\|_{\mathcal{D}} \leq C_\gamma \left\|Ae^{tA}p\right\|_{\mathcal{X}} \leq C_\gamma M_1^P e^{-\omega t}\|p\|_{\mathcal{X}}, \qquad t \leq 0,$$

for any $-a_1 < \omega \leq 0$, see (6.8e), and that $v_p(t)$ tends to 0 as $t \to -\infty$. The numerical solution has the same asymptotic behaviour. Similarly, for any $q \in Q(\mathcal{D})$, the analytical and numerical solution are given by

$$v_q(t) = e^{tA}q, \quad t \geq 0, \qquad \text{and} \qquad v_n = R(hA)^n q, \quad n \geq 0,$$

wherefore we obtain $\mathcal{S} = Q(\mathcal{D}) = \mathcal{S}^h$. From the relations (6.8c) and (6.8d) it further follows that $v_p(t)$ and v_n approach $\overline{v} = 0$ as $t \to \infty$ and $n \to \infty$, respectively.

In the sequel we summarize briefly the previously obtained results for the linear problem (6.6) under the requirements of Assumption 6.2 on the spectrum of $A = F'(\overline{u})$.

(i) Provided that the spectrum of A is contained in the negative complex plane, then the stationary solution $\overline{v} = 0$ is asymptotically stable, and it attracts all nearby solutions exponentially fast. Besides, the Runge-Kutta solution exists for all positive times and shows alike the exponentially decaying behaviour towards the origin.

(ii) Whenever the positive part of $\sigma(A)$ is not empty, the null solution is unstable. The unstable set $\mathcal{U} = P(\mathcal{X})$ of $\overline{v} = 0$ has a manifold structure whereas the stable manifold is given by $\mathcal{S} = Q(\mathcal{D})$. For sufficiently small stepsizes $0 < h \leq \overline{h}$ the corresponding discrete objects exist and equal $\mathcal{U}^h = P(\mathcal{X})$ and $\mathcal{S}^h = Q(\mathcal{D})$.

It is shown in Lunardi [25, Section 9.1] that the stationary solution $\overline{u}$ of the originally considered nonlinear equation (6.1) has the same stability properties as the null solution $\overline{v} = 0$ of the linearized equation (6.6), i.e. $\overline{u}$ is asymptotically stable if $\sigma(F'(\overline{u}))$ is contained in the negative complex plane and unstable otherwise. In other words, under Assumption 6.2 the *Principle of linearized stability* holds. Moreover, the local unstable and stable manifolds of $\overline{u}$ are constructed. Our further concern is it, to adopt the statements of LUNARDI [25, Section 9.1] for the continuous nonlinear problem in such a way that the corresponding results for Runge-Kutta time discretizations follow easily. This program is carried out in Chapter 7.

7. Long-term behaviour of continuous and discrete dynamical system

In the present chapter, we study Runge-Kutta time discretizations of the nonlinear parabolic equation

$$u'(t) = F(u(t)), \qquad t > 0, \tag{7.1}$$

and their long-term behaviour in the neighbourhood of a stationary solution. We show in Section 7.1 that numerical solutions lying nearby an equilibrium point $\overline{u}$ of (7.1) decay exponentially fast towards $\overline{u}$ whenever the spectrum of the derivative $F'(\overline{u})$ lies in the negative complex plane. Under the assumption that the stationary solution $\overline{u}$ is of saddle point type, we construct in Section 7.2 the discrete local unstable manifold and the discrete local unstable manifold of $\overline{u}$.

For proving the results of Section 7.1 we use similar concepts as the proof of Proposition 4.1. There, basic tools are the modified variation-of-constants formulas (D.7) and (D.9) obtained by linearization and further the space of Hölder continuous sequences. In the present situation, we consider (7.1) on unbounded time intervals, and we introduce additional weights in the Hölder norm in order to capture the exponentially decaying behaviour of the numerical solution. More involved techniques are needed for showing the results of Section 7.2 for the case of a saddle point $\overline{u}$ of (7.1). The main idea for constructing the discrete local unstable manifold of $\overline{u}$ is to consider a boundary value problem involving the projections on the positive and negative part of the spectrum of $F'(\overline{u})$ and to represent the numerical solution of (7.1) by means of the discrete modified variation-of-constants formula applied forward and backward in time. The discrete stable manifold is constructed in an analogous way.

The statements for the continuous problem are proven in Lunardi [25, Chapter 9]. Nevertheless, we derive the Theorems 7.1, 7.5 and 7.7 here as well, since then the proofs of the corresponding results for Runge-Kutta time discretizations can be adapted rather easily. For variable stepsize backward Euler discretizations the Theorems 7.2 and 7.3 are given in González et. al. [11]. Auxiliary results needed below are mainly appropriated by Sections D.2.2, D.2.3 and E.1.3.

We further on employ the assumptions and notations introduced in the previous chapter and in Section 3.1. In particular, we assume that the right-hand side of (7.1) satisfies Assumption 6.1. Beyond, we suppose that the nonlinear equation (7.1) possesses an equilibrium point $\overline{u}$ such that the spectrum of the sectorial operator

$$A = F'(\overline{u})$$

fulfils Assumption 6.2. We consider the strongly $A(\vartheta)$-stable Runge-Kutta method

$$U_n' = F(U_n), \qquad U_n = \mathbb{1} u_n + h\mathcal{O}U_n', \qquad u_{n+1} = u_n + hb^T U_n', \tag{7.2}$$

where $\vartheta > \pi - \varphi$ and $F(U_n) = \big(F(U_{n1}), \dots, F(U_{ns})\big)^T$. The angle φ is specified in the resolvent condition (6.5). Then, for stepsizes $0 < h \leq \overline{h}$ sufficiently small

$$\mathcal{J} = (\mathcal{I} - hA\mathcal{O})^{-1}$$

is bounded on $\mathcal{X}^s$ by a constant $C_{\mathcal{J}} > 0$, see Result C.1. We denote by

$$\underline{u} = (u_n)_{n=0}^{\infty} \qquad \text{and} \qquad \underline{\widehat{u}} = (\widehat{u}_n)_{n=0}^{\infty}$$

the numerical solution and the values of the exact solution of (7.1). Furthermore, we set $\underline{\overline{u}} = (\overline{u})_{n=0}^{\infty}$ and we denote by $\underline{\overline{U}}$ the constant sequence with components $\mathbb{1}\overline{u}$.

7.1. Asymptotically stable stationary solutions

The fundamental assumption for the subsequent considerations is that the spectrum of the sectorial operator $A = F'(\overline{u})$ lies in the negative complex plane, i.e. $\sigma(A)$ is contained in a sector with top $a < 0$ and opening angle less that $\pi/2$, see also (6.5). This situation is illustrated in Figure 7.1. For the sequel, we set

$$a_0 = -a > 0.$$

Then, the following result is valid.

Theorem 7.1. (Lunardi [25, Theorem 9.1.2]) *Let $\overline{u}$ denote an equilibrium point of equation* (7.1) *and suppose that the spectrum of $F'(\overline{u})$ is contained in the negative complex plane. Then, $\overline{u}$ is* exponentially asymptotically stable, *i.e. for every $0 \leq \omega < a_0$ there exist constants $\delta > 0$ and $M > 0$ such that for all elements $u_0 \in \mathcal{O}$ with $\|u_0 - \overline{u}\|_{\mathcal{D}} \leq \delta$ the following holds. The solution u of* (7.1) *with initial value u_0 exists for all positive times $t \geq 0$ and satisfies*

$$\|u(t) - \overline{u}\|_{\mathcal{D}} \leq Me^{-\omega t}\|u_0 - \overline{u}\|_{\mathcal{D}}, \qquad t \geq 0,$$

with a constant M depending on ω.

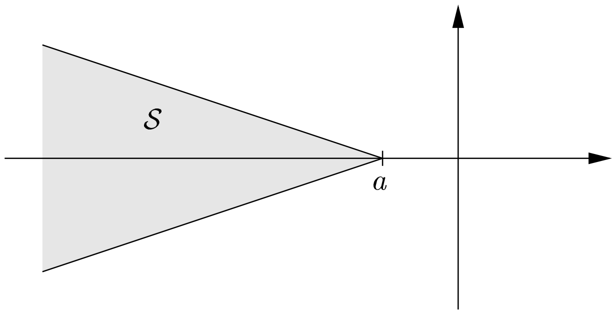

Figure 7.1.: The spectrum of A is contained in the sector $\mathcal{S}$

Proof. Fix $0 < \alpha < 1$ and $0 \leq \omega < a_0$. We denote by $R > 0$ and $L > 0$ the constants from Assumption 6.1. For constructing the analytical solution of equation (7.1) with initial value u_0, we use a fixed point iteration involving the nonlinear operator

$$\Gamma : \mathcal{B} \longrightarrow \mathcal{B} : v \longmapsto \Gamma(v) = Eu_0 + \mathcal{K}(G(v)),$$

where $E(t) = e^{tA}$ for $t \geq 0$. We allow initial values $u_0 \in \mathcal{O}$ satisfying $\|u_0 - \overline{u}\|_{\mathcal{D}} \leq \delta$. The convolution operator $\mathcal{K}$ is given by

$$\big(\mathcal{K}(w)\big)(t) = \int_0^t e^{(t-\tau)A}\big(G(w(\tau)) - G(w(t))\big)\,\mathrm{d}\tau + A^{-1}\big(e^{tA} - I\big)G(w(t))$$

for functions w on $[0,\infty)$ taking values in $\mathcal{O}$. The closed subset $\mathcal{B}$ of $\mathcal{C}_\alpha^{\alpha,\omega}((0,\infty),\mathcal{D})$ is defined through

$$\mathcal{B} = \Big\{v \in \mathcal{C}([0,\infty),\mathcal{D}) \cap \mathcal{C}_\alpha^{\alpha,\omega}((0,\infty),\mathcal{D}) : v(0) = u_0, \|v - \overline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} \leq \varrho\Big\}$$

and endowed with the norm of $\mathcal{C}_\alpha^{\alpha}((0,\infty),\mathcal{D})$. From the Lemmas E.4 and D.5 follows that $G(v) \in \mathcal{C}_\alpha^{\alpha,\omega}((0,\infty),\mathcal{X})$ and $\Gamma(v) \in \mathcal{C}_\alpha^{\alpha,\omega}((0,\infty),\mathcal{D})$. Therefore, Γ is well-defined on $\mathcal{B}$.

In the sequel we show that $0 < \varrho \leq R$ and $\delta > 0$ can be chosen such that the following conditions hold.

Γ is a contraction on $\mathcal{B}$:

(i) Γ is contractive on $\mathcal{B}$. Thus there exists some $\kappa < 1$ such that the relation

$$\|\Gamma(v) - \Gamma(w)\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} \leq \kappa\|v - w\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})}$$

holds for all $v, w \in \mathcal{B}$.

(ii) Γ maps $\mathcal{B}$ into $\mathcal{B}$, i.e. for elements $v \in \mathcal{B}$ the bound $\|v - \overline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} \leq \varrho$ implies $\|\Gamma(v) - \overline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} \leq \varrho$.

Then, the fixed point theorem of Banach can be applied and ensures the existence of a unique fixed point u of Γ in $\mathcal{B}$. This fixed point is just the mild solution of (7.1) and approximates the equilibrium with exponential decay. Note that u is also a strict solution, see Result B.1.

1.(i) By using the Lemmas D.5 and E.5, for $v, w \in \mathcal{B}$ the estimate

$$\begin{aligned}\|\Gamma(v) - \Gamma(w)\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} &\leq C\big\|G(v) - G(w)\big\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{X})} \\ &\leq \big(2 + \alpha^{-1}\omega^\alpha\big) LC\varrho\,\|v - w\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})}\end{aligned}$$

follows. The constant C depends on α, ω, C_γ and on the bounds for the analytical semigroup. Thus, Γ is contractive with contraction factor $\kappa = \varrho\big(2 + \alpha^{-1}\omega^\alpha\big)LC_1 < 1$ for ϱ sufficiently small.

(ii) From the definition of Γ and the identity $G(\overline{u}) = F(\overline{u}) - A\overline{u} = -A\overline{u}$, we obtain the relation

$$\Gamma(\overline{u}) - \overline{u} = E(u_0 - \overline{u}).$$

For $\|u_0 - \overline{u}\|_{\mathcal{D}} \leq \delta$ sufficiently small, this implies the estimate

$$\|\Gamma(\overline{u}) - \overline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} \leq C\|u_0 - \overline{u}\|_{\mathcal{D}} \leq (1-\kappa)\varrho, \tag{7.3}$$

see Lemma D.5. The constant C depends on α, ω and C_γ and on the bounds for the analytic semigroup. Combining this result with the contractivity of Γ, yields

$$\begin{aligned}\|\Gamma(v) - \overline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} &\leq \|\Gamma(v) - \Gamma(\overline{u})\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} + \|\Gamma(\overline{u}) - \overline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} \\ &\leq \kappa\,\|v - \overline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} + (1-\kappa)\varrho \leq \varrho\end{aligned}$$

for $v \in \mathcal{B}$. This proves $\Gamma(\mathcal{B}) \subset \mathcal{B}$.

2. The above considerations show that Γ is a contraction on $\mathcal{B}$ with contraction factor κ and that u is its fixed point. By adding $\Gamma(\overline{u})$ and using the first relation in (7.3), we receive the bound

$$\|u - \overline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} = \|\Gamma(u) - \overline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} \leq \kappa\|u - \overline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} + C\|u_0 - \overline{u}\|_{\mathcal{D}}.$$

From the definition of the weighted (α, α)-Hölder norm we further have

$$e^{\omega t}\|u(t) - \overline{u}\|_{\mathcal{D}} \leq \|u - \overline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} \leq \frac{C}{1-\kappa}\,\|u_0 - \overline{u}\|_{\mathcal{D}}.$$

This yields the desired result and concludes the proof. □

The corresponding result for the backward Euler method follows in a similar way.

Theorem 7.2. (González et al. [11, Theorem 4.1]) *Assume that for the equilibrium $\overline{u}$ of (7.1) the spectrum of $F'(\overline{u})$ lies in the negative complex plane. Then, for any non-negative number $0 \le \omega < a_0$ there are constants $\overline{h} > 0$, $\delta > 0$ and $M > 0$ such that for all initial values $u_0 \in \mathcal{D}$ with $\|u_0 - \overline{u}\|_{\mathcal{D}} \le \delta$ the following statements holds. The backward Euler solution $(u_n)_{n=0}^{\infty}$ of (7.1) with initial value u_0 exists for all stepsizes $0 < h < \overline{h}$ and satisfies*

$$\|u_n - \overline{u}\|_{\mathcal{D}} \le M e^{-\omega t_n} \|u_0 - \overline{u}\|_{\mathcal{D}} \qquad \text{for all} \quad t_n \ge 0.$$

The constant M depends on ω and $\overline{h}$ but not on h.

Proof. The above result for the backward Euler solution mainly follows from the proof of Theorem 7.1 by replacing the continuous quantities with the discrete ones.

For constructing the backward Euler solution of (7.1), we define the fixed point operator Φ through

$$\Phi : \mathcal{V} \longrightarrow \mathcal{V} : \underline{v} \longmapsto \Phi(\underline{v}) = \underline{r} u_0 + \mathcal{K}(G(\underline{v})).$$

The sequence $\underline{r}$ contains the powers of the stability function

$$\underline{r} = \big((I - hA)^{-n} \big)_{n=0}^{\infty},$$

and for a sequence $\underline{w} = (w_n)_{n=0}^{\infty}$ the convolution operator $\mathcal{K}$ is given by

$$\mathcal{K}(\underline{w}) = \left(h \sum_{\nu=0}^{n-1} (I - hA)^{-(n-\nu)} \big(w_{\nu+1} - w_n \big) + A^{-1} \big((I - hA)^{-n} - I \big) w_n \right)_{n \ge 0}.$$

We set $G(\underline{v}) = \big(G(v_n) \big)_{n=0}^{\infty}$ for $\underline{v} = (v_n)_{n=0}^{\infty}$ and $\mathcal{N} = \{ t_n = nh : n \in \boldsymbol{N} \}$ for some $h > 0$. Further, we denote by

$$\mathcal{V} = \Big\{ \underline{v} = (v_n)_{n=0}^{\infty} \in \mathcal{C}_{\alpha}^{\alpha,\omega}(\mathcal{N}, \mathcal{D}) : v_0 = u_0, \|\underline{v} - \underline{\overline{u}}\|_{\mathcal{C}_{\alpha}^{\alpha,\omega}(\mathcal{D})} \le \varrho \Big\}$$

the closed subset $\mathcal{V} \subset \mathcal{C}_{\alpha}^{\alpha,\omega}(\mathcal{N}, \mathcal{D})$, endowed with the associated norm $\|\cdot\|_{\mathcal{C}_{\alpha}^{\alpha,\omega}(\mathcal{D})}$.

For sequences $\underline{v} \in \mathcal{V}$ the regularity property $\Gamma(\underline{v}) \in \mathcal{C}_{\alpha}^{\alpha,\omega}(\mathcal{N}, \mathcal{D})$ follows from the Lemmas D.5 and E.4. In order to apply the fixed point theorem of Banach, it therefore remains to show that Φ is contractive and that $\Phi(\mathcal{V}) \subset \mathcal{V}$ holds.

1.(i) We apply the Lemmas D.5 and E.5 and obtain for $\underline{v}, \underline{w} \in \mathcal{V}$ the relation

$$\|\Phi(\underline{v}) - \Phi(\underline{w})\|_{\mathcal{C}_{\alpha}^{\alpha,\omega}(\mathcal{D})} \le \big(2 + \alpha^{-1} \omega^{\alpha} \big) L C \varrho \, \|\underline{v} - \underline{w}\|_{\mathcal{C}_{\alpha}^{\alpha,\omega}(\mathcal{D})}$$

with constant C depending on α, ω and C_γ and on the bounds for the stability function. Therefore, Φ is contractive with factor $\kappa < 1$ for ϱ small enough.

(ii) In order to show that Φ maps $\mathcal{V}$ onto $\mathcal{V}$, similarly to the proof of Theorem 7.1, we use the estimate

$$\|\Phi(\underline{v}) - \overline{\underline{u}}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} \leq \kappa \|\underline{v} - \overline{\underline{u}}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} + \|\Phi(\overline{\underline{u}}) - \overline{\underline{u}}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})}$$

and furthermore the identity

$$\Phi(\overline{\underline{u}}) - \overline{\underline{u}} = \underline{r}(u_0 - \overline{u}).$$

Together with Lemma D.5 this implies

$$\|\Phi(\overline{\underline{u}}) - \overline{\underline{u}}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} \leq C\|u_0 - \overline{u}\|_{\mathcal{D}}.$$

The constant C depends on α, ω and C_γ and further on the bounds for the stability function. For $\|u_0 - \overline{u}\|_{\mathcal{D}} \leq \delta$ chosen sufficiently small this yields $\Phi(\mathcal{V}) \subset \mathcal{V}$.

2. Let C denote the constant from (ii). Essentially the same arguments as in the proof of Theorem 7.1 show that then the bound

$$e^{\omega t_n}\|u_n - \overline{u}\|_{\mathcal{D}} \leq \|\underline{u} - \overline{\underline{u}}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} \leq \frac{C}{1-\kappa}\|u_0 - \overline{u}\|_{\mathcal{D}}$$

holds. This yields the exponentially decaying behaviour of the backward Euler solution and concludes the proof. □

In the previous theorem, the supposition that ω lies close to the top $-a_0$ of the sector containing $\sigma(A)$ has a strong restrictive effect on the size of the maximal stepsize $\overline{h} > 0$, as it requires that the numerical solution decays nearly as fast towards the equilibrium point as the analytical solution. The following result shows that an exponentially decaying behaviour is obtained as long as the stepsize remains bounded, and it uses the fact that the bounds for the stability function of the backward Euler method remain valid for arbitrary stepsizes $0 < h \leq \overline{h}$ whenever the real number $\omega \geq 0$ is chosen sufficiently small, see Result C.1.

Theorem 7.3. (González et al. [11, Theorem 4.1]) *Assume that for the equilibrium $\overline{u}$ of (7.1) the spectrum of $F'(\overline{u})$ is contained in the negative complex plane. Then, for every $\overline{h} > 0$ there exist constants $0 \leq \omega < a_0$, $\delta > 0$ and $M > 0$ such that the backward Euler solution $(u_n)_{n=0}^\infty$ of (7.1) exists for stepsizes $0 < h \leq \overline{h}$ and initial values $u_0 \in \mathcal{D}$ with $\|u_0 - \overline{u}\|_{\mathcal{D}} \leq \delta$. Further, the following relation holds*

$$\|u_n - \overline{u}\|_{\mathcal{D}} \leq Me^{-\omega t_n}\|u_0 - \overline{u}\|_{\mathcal{D}}, \qquad t_n \geq 0,$$

with constant M depending on $\overline{h}$ and ω but not on h.

Proof. The proof of Theorem 7.3 is very close to that of the previous result. It makes use of the fact that the stability bounds the backward Euler method and thus the auxiliary result Lemma D.5 holds for all stepsizes provided that the real number $\omega \geq 0$ is small enough. We omit the details. □

We next generalize the Theorems 7.2 and 7.3 to a strongly stable Runge-Kutta method. The existence of the internal stages and their exponentially decaying behaviour towards the equilibrium point is proven in a similar way as for the backward Euler method. The result for the solution then follows by means of the discrete modified variation-of-constants formula. The stability estimates are provided by Result C.1, and, as already discussed before, there is an interplay between the choice of $\omega \geq 0$ and the maximal stepsize $\overline{h} > 0$, see also Bakaev & Ostermann [4, Theorem 4.1 and Remark 1]. For non-optimal choices of ω larger stepsizes are admitted and the Runge-Kutta solution still shows exponential decay. However, choosing ω close to a_0 implies a severe restriction on the size of $\overline{h}$.

Theorem 7.4. *Let $\overline{u}$ denote an equilibrium point of* (7.1)*. Suppose that the spectrum of $F'(\overline{u})$ belongs to the negative complex plane and that the Runge-Kutta method* (7.2) *is strongly $A(\vartheta)$-stable for some $\vartheta > \pi - \varphi$ where φ denotes the quantity from* (6.5)*.*

(i) Then, for every $0 \leq \omega < a_0$ there are constants $\overline{h} > 0$, $\delta > 0$ and $M > 0$ such that for all initial values $u_0 \in \mathcal{D}$ with $\|u_0 - \overline{u}\|_{\mathcal{D}} \leq \delta$ the following holds. The internal stages $(U_n)_{n=0}^{\infty}$ and the Runge-Kutta solution $(u_n)_{n=0}^{\infty}$ with initial value u_0 exist for all $0 < h < \overline{h}$. Moreover, the following relation holds

$$\|u_n - \overline{u}\|_{\mathcal{D}} + \|U_{ni} - \overline{u}\|_{\mathcal{D}} \leq M e^{-\omega t_n} \|u_0 - \overline{u}\|_{\mathcal{D}}, \qquad t_n \geq 0, \tag{7.4}$$

for $1 \leq i \leq s$.

(ii) For every $\overline{h} > 0$ there are $0 \leq \omega < a_0$, $\delta > 0$ and $M > 0$ such that the Runge-Kutta solution $(u_n)_{n=0}^{\infty}$ exists for $0 < h \leq \overline{h}$ and $u_0 \in \mathcal{D}$ with $\|u_0 - \overline{u}\|_{\mathcal{D}} \leq \delta$ and satisfies (7.4)*.*

The constants M depend on ω and $\overline{h}$ and further on the method, but not on the stepsize h.

Proof. As the second statement of the theorem follows in analogous way, we only derive the first result.

For a general Runge-Kutta method of the form (7.2) the fixed point iteration for the internal stages $\underline{U} = (U_n)_{n=0}^{\infty}$ is defined through

$$\Psi : \mathcal{W} \longrightarrow \mathcal{W} : \underline{V} \longmapsto \Psi(\underline{V}) = \mathcal{J}\mathbb{1}\underline{R}u_0 + \mathcal{K}(G(\underline{V}))$$

Here, we employ the following sequence notation

$$\underline{R} = \big(R(hA)^n\big)_{n=0}^{\infty} \qquad \text{and} \qquad G(\underline{V}) = \big(G(V_n)\big)_{n=0}^{\infty} \quad \text{for} \quad \underline{V} = (V_n)_{n=0}^{\infty}.$$

The convolution operator $\mathcal{K}$ for the internal stages is given by

$$\begin{aligned}\mathcal{K}(\underline{W}) = \bigg(& h \sum_{\nu=0}^{n-1} \mathcal{J}\mathbb{1} b^T \mathcal{J} R(hA)^{n-\nu-1} \big(W_\nu - W_{n-1}\big) \\ & + A^{-1} \mathcal{J}\mathbb{1} b^T \mathcal{J} \big(b^T \mathcal{J}\mathbb{1}\big)^{-1} \big(R(hA)^n - I\big) W_{n-1} + h\mathcal{Q}\mathcal{J} W_n \bigg)_{n \geq 0}\end{aligned}$$

for $\underline{W} = (W_n)_{n\geq 0}$ with $W_n \in \mathcal{X}^s$. The closed subset $\mathcal{W}$

$$\mathcal{W} = \Big\{ \underline{V} = (V_n)_{n=0}^{\infty} \in \mathcal{C}_{\alpha}^{\alpha,\omega}(\mathcal{N}, \mathcal{D}^s) : V_0 = \mathbb{1}\, u_0, \big\| \underline{V} - \underline{\overline{U}} \big\|_{\mathcal{C}_{\alpha}^{\alpha,\omega}(\mathcal{D}^s)} \leq \varrho \Big\}$$

is endowed with the norm $\|\cdot\|_{\mathcal{C}_{\alpha}^{\alpha,\omega}(\mathcal{D}^s)}$, where $\mathcal{N} = \{t_n = nh : n \in \boldsymbol{N}\}$ for fixed $h > 0$.

The present proof is close to that of the previous result for the backward Euler method. An application of the auxiliary results Lemmas D.5 and E.5 proves the contractivity of Ψ for $\varrho > 0$ small enough and sufficiently small stepsizes $0 < h \leq \overline{h}$. From Lemma D.5, due to the identity

$$\Psi(\underline{\overline{U}}) - \underline{\overline{U}} = \mathcal{J}\mathbb{1}\, \underline{R}(u_0 - \overline{u})$$

we further receive the following bound

$$\big\|\Psi(\underline{\overline{U}}) - \underline{\overline{U}}\big\|_{\mathcal{C}_{\alpha}^{\alpha,\omega}(\mathcal{D}^s)} \leq C\|u_0 - \overline{u}\|_{\mathcal{D}}.$$

Consequently, for $\underline{V} \in \mathcal{W}$ this yields

$$\big\|\Psi(\underline{V}) - \underline{\overline{U}}\big\|_{\mathcal{C}_{\alpha}^{\alpha,\omega}(\mathcal{D}^s)} \leq \kappa \big\|\underline{V} - \underline{\overline{U}}\big\|_{\mathcal{C}_{\alpha}^{\alpha,\omega}(\mathcal{D}^s)} + \big\|\Psi\big(\underline{\overline{U}}\big) - \underline{\overline{U}}\big\|_{\mathcal{C}_{\alpha}^{\alpha,\omega}(\mathcal{D}^s)} \leq \varrho$$

and proves the existence of the internal stages $\underline{U}$ as the unique fixed point of Ψ in $\mathcal{W}$, whenever $\|u_0 - \overline{u}\|_{\mathcal{D}} \leq \delta$ is chosen small enough. As in the second part of the proof of Theorem 7.2, we get

$$\big\|\underline{U} - \underline{\overline{U}}\big\|_{\mathcal{C}_{\alpha}^{\alpha,\omega}(\mathcal{D}^s)} \leq C\|u_0 - \overline{u}\|_{\mathcal{D}}, \tag{7.5}$$

and now the estimate

$$\|U_n - \mathbb{1}\,\overline{u}\|_{\mathcal{D}^s} \leq M e^{-\omega t_n}\|u_0 - \overline{u}\|_{\mathcal{D}}$$

follows at once. We omit the details.

In order to derive the corresponding relation for the Runge-Kutta solution, we consider the discrete modified variation-of-constants formula (D.9). By using the identity

$$G(\overline{u}) = F(\overline{u}) - A\overline{u} = -A\overline{u},$$

we obtain the following representation

$$\begin{aligned} u_n - \overline{u} = R(hA)^n(u_0 - \overline{u}) + h\sum_{\nu=0}^{n-1} b^T \mathcal{J} R(hA)^{n-\nu-1}\big(G(U_\nu) - G(U_{n-1})\big) \\ + A^{-1} b^T \mathcal{J}\big(b^T \mathcal{J}\mathbb{1}\big)^{-1}\big(R(hA)^n - I\big)\big(G(U_{n-1}) - G(\overline{U})\big). \end{aligned} \tag{7.6}$$

For estimating the difference $u_n - \overline{u}$, several auxiliary bounds are needed which we state in the following.

For the following we choose a constant c_0 such that the estimate

$$\left\|b^T \mathcal{J}\left(b^T \mathcal{J} \mathbb{1}\right)^{-1}\right\|_{\mathcal{X}\leftarrow\mathcal{X}^s} \le c_0$$

holds, and we denote by $\hat{M}_0 > 0$ and $\hat{M}_1 > 0$ the bounds of the stability function

$$\begin{aligned} \left\|R(hA)^n\right\|_{\mathcal{X}\leftarrow\mathcal{X}} &\le \hat{M}_0 e^{-\omega t_n}, \quad t_n \ge 0, \\ \left\|b^T \mathcal{J} A R(hA)^n\right\|_{\mathcal{X}\leftarrow\mathcal{X}^s} &\le \hat{M}_1 e^{-\omega t_n} t_n^{-1}, \quad t_n > 0, \end{aligned}$$

for $0 < h \le \overline{h}$ sufficiently small, see Result C.1. Besides, from the definition of the map G for all elements $v, w \in \mathcal{O}$ the relation

$$\begin{aligned} \|G(v) - G(w)\|_{\mathcal{X}} &= \left\|F(v) - F(w) - F'(\overline{u})(v - w)\right\|_{\mathcal{X}} \\ &\le \int_0^1 \left\|F'(\tau v + (1-\tau) w) - F'(\overline{u})\right\|_{\mathcal{X}\leftarrow\mathcal{D}} \|v - w\|_{\mathcal{D}} \,\mathrm{d}\tau \\ &\le \frac{L}{2}\Big(\|v - \overline{u}\|_{\mathcal{D}} + \|w - \overline{u}\|_{\mathcal{D}}\Big)\|v - w\|_{\mathcal{D}} \end{aligned}$$

follows From the previous considerations we know that $\underline{U} - \underline{\overline{U}}$ belongs to $\mathcal{C}_\alpha^{\alpha,\omega}((0,\infty),\mathcal{D}^s)$. Due to Remark E.1 and the bound (7.5) for the internal stages, for all $0 < t_m < t_n < \infty$ we receive the bound

$$\begin{aligned} \|U_n - U_m\|_{\mathcal{D}^s} &\le [\underline{U}]_{\mathcal{C}_\alpha^\alpha(\mathcal{D}^s)} \cdot (t_n - t_m)^\alpha t_m^{-\alpha} \\ &\le \left(1 + \alpha^{-1}\omega^\alpha\right)\left\|\underline{U} - \underline{\overline{U}}\right\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} \cdot (t_n - t_m)^\alpha t_m^{-\alpha} \\ &\le C\|u_0 - \overline{u}\|_{\mathcal{D}} \cdot (t_n - t_m)^\alpha t_m^{-\alpha} \end{aligned}$$

with some constant C depending on α and ω. Moreover, the stages decay exponentially fast towards the equilibrium, i.e. it holds

$$\left\|U_n - \overline{U}\right\|_{\mathcal{D}^s} \le M e^{-\omega t_n}\|u_0 - \overline{u}\|_{\mathcal{D}}.$$

Combining the above bounds shows

$$\|G(U_n) - G(U_m)\|_{\mathcal{X}^s} \le C\|u_0 - \overline{u}\|_{\mathcal{D}}\, e^{-\omega t_m}(t_n - t_m)^\alpha t_m^{-\alpha}$$

where the constant C depends on α, ω, L and further on $\|u_0 - \overline{u}\|_{\mathcal{D}}$. We use the representation (7.6) in order to estimate $u_n - \overline{u}$ in $\mathcal{D}$ and apply the above auxiliary results. For stepsizes $0 < h \le \overline{h}$ small enough, this yields

$$\begin{aligned}
\|u_n - \overline{u}\|_{\mathcal{D}} &\le \big\|R(hA)^n\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \|u_0 - \overline{u}\|_{\mathcal{D}} \\
&\quad + C_\gamma h \big\|b^T \mathcal{J} A R(hA)^{n-1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}^s} \big\|G(U_0) - G(U_{n-1})\big\|_{\mathcal{X}^s} \\
&\qquad + C_\gamma h \sum_{\nu=1}^{n-1} \big\|b^T \mathcal{J} A R(hA)^{n-\nu-1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}^s} \big\|G(U_\nu) - G(U_{n-1})\big\|_{\mathcal{X}^s} \\
&\qquad\quad + C_\gamma \big\|b^T \mathcal{J} \big(b^T \mathcal{J} \mathbb{1}\big)^{-1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}^s} \big\|R(hA)^n - I\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \big\|G(U_{n-1}) - G(\overline{U})\big\|_{\mathcal{X}^s} \\
&\le C e^{-\omega t_n} \|u_0 - \overline{u}\|_{\mathcal{D}} + C e^{-\omega t_n} \Big(\big\|U_0 - \overline{U}\big\|_{\mathcal{D}^s} + \big\|U_{n-1} - \overline{U}\big\|_{\mathcal{D}^s}\Big)^2 \\
&\quad + C \|u_0 - \overline{u}\|_{\mathcal{D}}\, e^{-\omega t_n}\, h \sum_{\nu=1}^{n-1} t_{n-\nu}^{-1+\alpha} t_\nu^{-\alpha} + C \|U_{n-1} - \overline{U}\|_{\mathcal{D}^s}^2 .
\end{aligned}$$

From this, we finally obtain the following estimate

$$\|u_n - \overline{u}\|_{\mathcal{D}} \le C e^{-\omega t_n} \|u_0 - \overline{u}\|_{\mathcal{D}} \qquad \text{for} \quad t_n \ge 0.$$

The constant C depends on α, ω, L, C_γ, on the bounds for the stability function of the method and further on $\|u_0 - \overline{u}\|_{\mathcal{D}} \le \delta$ and $e^{\omega \overline{h}}$. This concludes the proof. □

7.2. Exponentially decaying backward solutions and local manifolds

For the following, we assume that the equilibrium point $\overline{u}$ of (7.1) is of saddle point type. In this situation, the spectrum of the sectorial operator

$$A = F'(\overline{u})$$

does not intersect the imaginary axis and the positive part of the spectrum of A is not empty, i.e. it holds

$$a_1 > 0 \qquad \text{and} \qquad a_2 > 0.$$

Here, we denote by a_1 the left verge of the rectangle $\mathcal{R}$ where the positive part $\sigma_1(A)$ of the spectrum lies and by $-a_2$ the top of the sector $\mathcal{S}$ where the negative part $\sigma_2(A)$ is contained in, see also Assumption 6.2. This situation in displayed in Figure 7.2.

In the first place, we construct in Section 7.2.1 the local unstable manifold of the equilibrium point $\overline{u}$ and then the discrete local unstable manifold for the Runge-Kutta method (7.2). In Section 7.2.2 we briefly sketch the proofs of the corresponding results for the local stable manifold and its discrete counterpart. For the definition of the spectral projections P and Q associated with $\sigma_1(A)$ and $\sigma_2(A)$ we refer to Relation (6.7) of Section 6.3.

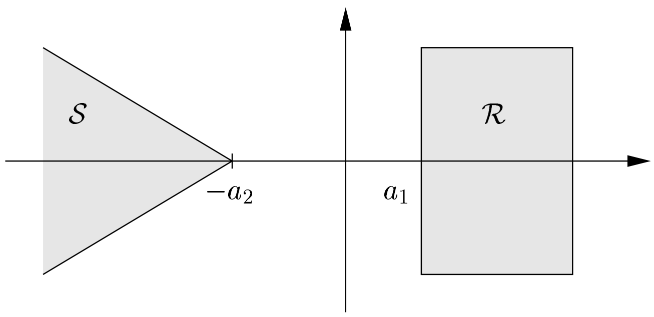

Figure 7.2.: The spectrum of A is contained in the shaded area

7.2.1. The local unstable manifold and its discrete counterpart

Recall that for some $\varrho > 0$ the local unstable manifold $\mathcal{U}_\varrho$ of $\overline{u}$ is defined through

$$\begin{aligned} \mathcal{U}_\varrho = \big\{u_0 \in \mathcal{O} :\ & \text{there exists a negative orbit } \{u(t) : t \leq 0\} \text{ through } u_0 \\ & \text{with } \|u(t) - \overline{u}\|_{\mathcal{D}} \leq \varrho \text{ for all } t \leq 0 \text{ and } u(t) \to \overline{u} \text{ as } t \to -\infty\big\}. \end{aligned}$$

The following result shows that $\mathcal{U}_\varrho$ is essentially given as the graph of a Lipschitz continuous function.

Theorem 7.5. (Lunardi [25, Theorem 9.1.4]) *Let $\overline{u}$ denote an equilibrium point of equation (7.1), and suppose that the spectrum of $F'(\overline{u})$ satisfies Assumption 6.2. Choose $0 < \alpha < 1$. Then, there is some $\varrho > 0$ such that for some $\delta > 0$ and a Lipschitz continuous map*

$$\varphi : \mathcal{B}_\delta(\overline{u}) = \{p \in P(\mathcal{X}) : \|p - \overline{u}\|_{\mathcal{X}} \leq \delta\} \longrightarrow Q(\mathcal{D})$$

satisfying $\varphi(\overline{u}) = 0$, the local unstable manifold $\mathcal{U}_\varrho$ of $\overline{u}$ is given by

$$\mathcal{U}_\varrho = \big\{u_0 \in \mathcal{O} : u_0 = p + \varphi(p), p \in \mathcal{B}_\delta(\overline{u})\big\}.$$

For every element $u_0 \in \mathcal{U}_\varrho$ a unique backward solution u of (7.1) through u_0 exists which belongs to $\mathcal{C}^\alpha((-\infty, 0], \mathcal{D})$ and fulfils

$$\|u - \overline{u}\|_{\mathcal{C}^\alpha(\mathcal{D})} \leq \varrho \qquad \text{and} \qquad \lim_{t\to-\infty} u(t) = \overline{u}.$$

Furthermore, for all $0 < \beta < 1$ and $\omega \leq 0$ such that $-a_2 < \omega \leq -\omega < a_1$ it holds $u \in \mathcal{C}^{\beta,\omega}((-\infty, 0], \mathcal{D})$.

Proof. For constructing the set $\mathcal{U}_\varrho$ we combine the concepts arising in the proofs of Lunardi [24, Theorem 9.1.4] and Lubich & Ostermann [24, Theorem 2.1].

Fix $0 < \alpha < 1$ and $-a_1 < \omega \leq 0$. We denote by $R > 0$ and $L > 0$ the constants from Assumption 6.1. In the first part of the proof we show that a unique solution u of the following boundary value problem exists

$$u'(t) = F(u(t)), \quad t < 0, \qquad Pu(0) = p, \qquad \limsup_{t \to -\infty} \|Qu(t)\|_{\mathcal{D}} < \infty, \tag{7.7}$$

for each element $p \in P(\mathcal{X})$ with $\|p - \overline{u}\|_{\mathcal{X}} \leq \delta$ small enough. For this, we linearize F around the equilibrium point $\overline{u}$ and set $A = F'(\overline{u})$ and $G(v) = F(v) - Av$ for $v \in \mathcal{O}$. Then, the analytical solution of

$$u'(t) = Au(t) + G(u(t)), \quad t < 0, \qquad Pu(0) = p, \qquad \limsup_{t \to -\infty} \|Qu(t)\|_{\mathcal{D}} < \infty,$$

has the following formal representation

$$\begin{aligned} u(t) = e^{tA}p + \int_0^t e^{(t-\tau)A} P\big(G(u(\tau)) - G(u(t))\big)\, \mathrm{d}\tau + A^{-1}\big(e^{tA}P - P\big)G(u(t)) \\ + \int_{-\infty}^t e^{(t-\tau)A} Q\big(G(u(\tau)) - G(u(t))\big)\, \mathrm{d}\tau - A^{-1}QG(u(t)), \qquad t \leq 0. \end{aligned} \tag{7.8}$$

This follows from the following formal considerations. We split u into $u = Pu + Qu$ and apply the modified variation-of-constants formula (D.5) twice in order to represent the solutions of the initial value problems

$$\begin{aligned} Pu'(t) &= APu(t) + PG(u(t)), \quad t < 0, \qquad Pu(0) = p, \\ Qu'(t) &= AQu(t) + QG(u(t)), \quad t < 0, \qquad Qu(T) \text{ given}, \end{aligned}$$

for some $T < 0$. Therefore, we receive

$$\begin{aligned} Pu(t) &= e^{tA}p + \int_0^t e^{(t-\tau)A} P\big(G(u(\tau)) - G(u(t))\big)\, \mathrm{d}\tau + A^{-1}\big(e^{tA}P - P\big)G(u(t)), \\ Qu(t) &= e^{(t-T)A}Qu(T) + \int_T^t e^{(t-\tau)A} Q\big(G(u(\tau)) - G(u(t))\big)\, \mathrm{d}\tau \\ &\quad + A^{-1}\big(e^{(t-T)A}Q - Q\big)G(u(t)), \qquad t \leq 0. \end{aligned}$$

By taking the limit $T \to -\infty$, due to the relations

$$\lim_{t \to \infty} \big\|e^{tA}Q\big\|_{\mathcal{X} \leftarrow \mathcal{X}} = 0 \qquad \text{and} \qquad \limsup_{t \to -\infty} \|Qu(t)\|_{\mathcal{D}} < \infty,$$

see (6.8c), we obtain (7.8).

This formula is used to construct the analytical solution of (7.7) by a fixed point iteration, see below. We look for such a backward solution u in a ball $\mathcal{B} \subset \mathcal{C}^{\alpha,\omega}((-\infty, 0], \mathcal{D})$

around $\overline{u}$ with radius ϱ sufficiently small. From LUNARDI [25, Theorem 4.4.6] follows that the fixed point u is just the searched strict solution of (7.7) with representation (7.8). In particular, we then have

$$e^{\omega t}\|u(t)-\overline{u}\|_{\mathcal{D}} \le \|u-\overline{u}\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{D})} \le \varrho, \qquad t\le 0, \quad \omega \le 0,$$

which implies $u(t)\to\overline{u}$ as $t\to-\infty$.

In order to indicate the dependence of the unique solution of (7.7) on p we henceforth employ the notation u_p. We define the map φ through

$$\varphi : \mathcal{B}_\delta(\overline{u}) = \{p\in P(\mathcal{X}) : \|p-\overline{u}\|_{\mathcal{X}} \le \delta\} \longrightarrow Q(\mathcal{D}) : p \longmapsto \varphi(p) = Qu_p(0), \tag{7.9}$$

and we show in the second part of the proof that φ possesses the desired smoothness properties. The local unstable manifold $\mathcal{U}_\varrho$ is then given by

$$\mathcal{U}_\varrho = \big\{u_0\in\mathcal{O} : u_0 = p+\varphi(p), p\in\mathcal{B}_\delta(\overline{u})\big\}.$$

1. For constructing the analytical solution u_p of (7.7) we use a fixed point iteration involving the nonlinear operator

$$\Gamma_p : \mathcal{B}\longrightarrow\mathcal{B} : v\longmapsto \Gamma_p(v) = Ep+\mathcal{K}(G(v))$$

where $E(t)=e^{tA}$. For functions $w:(-\infty,0]\to\mathcal{O}$ the operator $\mathcal{K}$ is given by

$$\begin{aligned}\big(\mathcal{K}(w)\big)(t) = &\int_0^t e^{(t-\tau)A}P\big(G(w(\tau))-G(w(t))\big)\,\mathrm{d}\tau + A^{-1}\big(e^{tA}P-P\big)G(w(t))\\ &+\int_{-\infty}^t e^{(t-\tau)A}Q\big(G(w(\tau))-G(w(t))\big)\,\mathrm{d}\tau - A^{-1}QG(w(t)), \quad t\le 0.\end{aligned}$$

The closed subset $\mathcal{B}$ of $\mathcal{C}^{\alpha,\omega}(-\infty,0],\mathcal{D})$ is defined through

$$\mathcal{B} = \Big\{v\in\mathcal{C}^{\alpha,\omega}((-\infty,0],\mathcal{D}) : Pv(0)=p, \|v-\overline{u}\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{D})}\le\varrho\Big\}$$

and endowed with the norm of $\mathcal{C}^{\alpha,\omega}((-\infty,0],\mathcal{D})$.

The Lemmas E.4 and D.8 show that $G(v)$ belongs to $\mathcal{C}^{\alpha,\omega}((-\infty,0],\mathcal{X})$ and further that $\Gamma_p(v)\in\mathcal{C}^{\alpha,\omega}((-\infty,0],\mathcal{D})$, i.e. Γ_p is well-defined on $\mathcal{B}$. In a similar way to the proof of Theorem 7.1, it follows from the auxiliary results Lemmas D.8 and E.5 that Γ_p is contractive and that $\Gamma(\mathcal{B})\subset\mathcal{B}$ holds for $0<\varrho\le R$ and $\|p-\overline{u}\|_{\mathcal{X}}\le\delta$ sufficiently small. Thus, the fixed point theorem of Banach can be applied and yields the existence of a unique fixed point u_p of Γ_p in $\mathcal{B}$.

2. It remains to show that the map φ given by (7.9) is Lipschitz continuous on $\mathcal{B}_\delta(\overline{u}) = \{p\in P(\ mX) : \|p-\overline{u}\|_{\mathcal{X}}\le\delta\}$. For this, we apply the above result with $\omega=0$. We choose $p,\widetilde{p}\in\mathcal{B}_\delta(\overline{u})$. Hence, the associated solutions u_p and $u_{\widetilde{p}}$ of (7.7) are the fixed

points of Γ_p and $\Gamma_{\widetilde{p}}$, respectively, and fulfil $\|u_p - \overline{u}\|_{\mathcal{C}^\alpha(\mathcal{D})} \le \varrho$ and $\|u_{\widetilde{p}} - \overline{u}\|_{\mathcal{C}^\alpha(\mathcal{D})} \le \varrho$. Notice that the contraction factor κ of $\Gamma_{\widetilde{p}}$ depends on the quantities L, R, ϱ, α and on the bounds for the analytical semigroup but not on $\widetilde{p}$. Therefore we receive the bound

$$\begin{aligned}\|u_p - u_{\widetilde{p}}\|_{\mathcal{C}^\alpha(\mathcal{D})} &\le \big\|\Gamma_p(u_p) - \Gamma_{\widetilde{p}}(u_p)\big\|_{\mathcal{C}^\alpha(\mathcal{D})} + \big\|\Gamma_{\widetilde{p}}(u_p) - \Gamma_{\widetilde{p}}(u_{\widetilde{p}})\big\|_{\mathcal{C}^\alpha(\mathcal{D})}\\ &\le \big\|E\big(p-\widetilde{p}\big)\big\|_{\mathcal{C}^\alpha(\mathcal{D})} + \kappa\|u_p - u_{\widetilde{p}}\|_{\mathcal{C}^\alpha(\mathcal{D})}.\end{aligned}$$

Now the following estimate follows easily with the help of Relation (6.8c)

$$\|Qu_p(0) - Qu_{\widetilde{p}}(0)\|_{\mathcal{D}} \le M_0^Q\|u_p - u_{\widetilde{p}}\|_{\mathcal{C}^\alpha(\mathcal{D})} \le C\|p-\widetilde{p}\|_{\mathcal{X}}.$$

The constant C depends on α and on the bounds for the analytic semigroup. This proves the Lipschitz continuity of φ and concludes the proof. □

For the unstable manifold of the Runge-Kutta time discretization for (7.1)

$$\begin{aligned}\mathcal{U}_\varrho^h = \big\{u_0 \in \mathcal{O} :\ &\text{there exists a Runge-Kutta solution } (u_n)_{n\le 0} \text{ for } h>0 \text{ and } u_0\\ &\text{such that } \|u_n - \overline{u}\|_{\mathcal{D}} \le \varrho \text{ for all } t_n \le 0 \text{ and } u_n \to \overline{u} \text{ as } n \to -\infty\big\}\end{aligned}$$

the corresponding result is true.

Theorem 7.6. *Assume that for the equilibrium point $\overline{u}$ of (7.1) the spectrum of $F'(\overline{u})$ fulfils Assumption 6.2, and fix $0 < \alpha < 1$. Suppose further that the Runge-Kutta method (7.2) is strongly $A(\vartheta)$-stable with $\vartheta > \pi - \varphi$ where φ denotes the constant from (6.5). Then, there is some $\varrho > 0$ and $\overline{h} > 0$ such that for $0 < h \le \overline{h}$, for a certain $\delta > 0$ and a Lipschitz continuous map*

$$\varphi : \mathcal{B}_\delta(\overline{u}) = \{p \in P(\mathcal{X}) : \|p - \overline{u}\|_{\mathcal{X}} \le \delta\} \longrightarrow Q(\mathcal{D}) \qquad \textit{with} \quad \varphi(\overline{u}) = 0,$$

the discrete local unstable manifold $\mathcal{U}_\varrho^h$ of $\overline{u}$ is given by

$$\mathcal{U}_\varrho^h = \big\{u_0 \in \mathcal{O} : u_0 = p + \varphi(p), p \in \mathcal{B}_\delta(\overline{u})\big\}.$$

For any initial value $u_0 \in \mathcal{U}_\varrho^h$ a unique Runge-Kutta solution $\underline{u} = (u_n)_{n\le 0}$ with stepsize $h > 0$ exists that belongs to $\mathcal{C}^\alpha(\mathcal{N}, \mathcal{D})$ for $\mathcal{N} = \{t_n = nh : n \in \boldsymbol{Z}, n \le 0\}$ and satisfies

$$\|\underline{u} - \underline{\overline{u}}\|_{\mathcal{C}^\alpha(\mathcal{D})} \le \varrho \qquad \textit{and} \qquad \lim_{n\to-\infty} u_n = \overline{u}.$$

Moreover, the Runge-Kutta solution has the regularity property $\underline{u} \in \mathcal{C}^{\beta,\omega}(\mathcal{N}, \mathcal{D})$ for any $0 < \beta < 1$ and $\omega \le 0$ such that $-a_2 < \omega \le -\omega < a_1$.

Proof. The present proof is performed in a very similar manner to that of Theorem 7.5. For representing the Runge-Kutta solution of the boundary value problem (7.7) we consider the recursion

$$U_n' = F(U_n), \qquad U_n = \mathbb{1} u_n + h\mathcal{O}U_n', \qquad u_{n+1} = u_n + hb^T U_n', \tag{7.10a}$$

endowed with the following boundary conditions

$$Pu_0 = p \qquad \text{and} \qquad \limsup_{n\to -\infty} \|Qu_n\|_{\mathcal{D}} < \infty, \tag{7.10b}$$

and we apply the modified variations-of-constants formula (D.9) twice. This yields

$$\begin{aligned}
u_n = R(hA)^n p &+ h\sum_{\nu=0}^{n-1} b^T \mathcal{J} R(hA)^{n-\nu-1} P\big(G(U_\nu) - G(U_{n-1})\big) \\
&+ A^{-1} b^T \mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1}\big(R(hA)^n P - P\big)G(U_{n-1}) \\
&+ h\sum_{\nu=-\infty}^{n-1} b^T \mathcal{J} R(hA)^{n-\nu-1} Q\big(G(U_\nu) - G(U_{n-1})\big) \\
&- A^{-1} b^T \mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1} QG(U_{n-1})
\end{aligned}$$

for the numerical solution, and the following formula for the internal stages

$$\begin{aligned}
U_{n-1} = \mathcal{J}\mathbb{1}\, R(hA)^{n-1} p &+ h\sum_{\nu=0}^{n-2} \mathcal{J}\mathbb{1}\, b^T \mathcal{J} R(hA)^{n-\nu-2} P\big(G(U_\nu) - G(U_{n-2})\big) \\
&+ A^{-1}\mathcal{J}\mathbb{1}\, b^T \mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1}\big(R(hA)^{n-1} P - P\big)G(U_{n-2}) \\
&+ h\sum_{\nu=-\infty}^{n-2} \mathcal{J}\mathbb{1}\, b^T \mathcal{J} R(hA)^{n-\nu-2} Q\big(G(U_\nu) - G(U_{n-2})\big) \\
&- A^{-1}\mathcal{J}\mathbb{1}\, b^T \mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1} QG(U_{n-2}) + h\mathcal{O}\mathcal{J}G(U_{n-1}).
\end{aligned}$$

For the special case of the backward Euler method where

$$\begin{aligned}
u_n = (I-hA)^{-n} p &+ h\sum_{\nu=0}^{n-1} (I-hA)^{-(n-\nu)} P\big(G(u_{\nu+1}) - G(u_n)\big) \\
&+ A^{-1}\big((I-hA)^{-n}P - P\big)G(u_n) \\
&+ h\sum_{\nu=-\infty}^{n-1} (I-hA)^{-(n-\nu)} Q\big(G(u_{\nu+1}) - G(u_n)\big) - A^{-1}QG(u_n),
\end{aligned}$$

the statement of the theorem follows literally from the the proof of Theorem 7.5 by replacing the continuous quantities with the discrete ones.

In general, we choose $0 < \alpha < 1$ and $-a_1 < \omega \leq 0$, and we use a fixed-point iteration for the stages in a ball $\mathcal{B} \subset \mathcal{C}^{\alpha,\omega}((-\infty, 0], \mathcal{D}^s)$ around the equilibrium point with radius ϱ. With the help of the auxiliary results Lemmas D.8 and E.4, one can show that $\underline{U}$ and thus $\underline{u}$ exists for elements p lying sufficiently close to $\overline{u}$ and for constants $\varrho > 0$ and $0 < h \leq \overline{h}$ chosen small enough. By using the identity

$$
\begin{aligned}
u_n - \overline{u} = R(hA)^n P(p - \overline{u}) &+ h \sum_{\nu=0}^{n-1} b^T \mathcal{J} R(hA)^{n-\nu-1} P\big(G(U_\nu) - G(U_{n-1})\big) \\
&+ A^{-1} b^T \mathcal{J} \big(b^T \mathcal{J} \mathbb{1}\big)^{-1} \big(R(hA)^n P - P\big)\big(G(U_{n-1}) - G(\overline{U})\big) \\
&+ h \sum_{\nu=-\infty}^{n-1} b^T \mathcal{J} R(hA)^{n-\nu-1} Q\big(G(U_\nu) - G(U_{n-1})\big) \\
&- A^{-1} b^T \mathcal{J} \big(b^T \mathcal{J} \mathbb{1}\big)^{-1} Q\big(G(U_{n-1}) - G(\overline{U})\big)
\end{aligned}
$$

and applying the ideas arising in the proof of Theorem 7.4, we receive the following bound

$$
\|u_n - \overline{u}\|_{\mathcal{D}} \leq C e^{-\omega t_n} \|p - \overline{u}\|_{\mathcal{D}} \qquad \text{for } t_n \leq 0.
$$

which implies $u_n \to \overline{u}$ for $n \to -\infty$. We denote by

$$
\underline{u}^p = \big(u_n^p\big)_{n \leq 0}
$$

the Runge-Kutta solution of (7.10) and by $\underline{U}^p$ the associated stages. It remains to show that the map φ defined through

$$
\varphi : \mathcal{B}_\delta(\overline{u}) = \{p \in P(\mathcal{X}) : \|p - \overline{u}\|_{\mathcal{X}} \leq \delta\} \longrightarrow Q(\mathcal{D}) : p \longmapsto \varphi(p) = Q\underline{u}_0^p \tag{7.11}
$$

is Lipschitz continuous. The same arguments as in the proof of Theorem 7.5 yield

$$
\big\|\underline{U}^p - \underline{U}^{\widetilde{p}}\big\|_{\mathcal{C}^\alpha(\mathcal{D}^s)} \leq C \big\|p - \widetilde{p}\big\|_{\mathcal{D}}.
$$

From this, due to the fact that the identity

$$
\mathbb{1}\big(u_n^p - u_n^{\widetilde{p}}\big) = (\mathcal{I} - hA\mathcal{Q})\Big(U_n^p - U_n^{\widetilde{p}} - h\mathcal{Q}\mathcal{J}\big(G\big(U_n^p\big) - G\big(U_n^{\widetilde{p}}\big)\big)\Big)
$$

holds and that G is Lipschitz continuous, the estimate

$$
\big\|\underline{u}^p - \underline{u}^{\widetilde{p}}\big\|_{\mathcal{C}^\alpha(\mathcal{D})} \leq C \|p - \overline{u}\|_{\mathcal{D}}
$$

follows. This implies the Lipschitz continuity of φ and concludes the proof. $\square$

7.2.2. The local stable manifold and its discrete counterpart

In the sequel, we state the analogous results of the preceding section for the local stable manifold $\mathcal{S}_\varrho$ which is given by

$$\mathcal{S}_\varrho = \big\{u_0 \in \mathcal{O} : \text{ there exists a solution } u \text{ on } [0,\infty) \text{ with initial value } u_0 \\ \text{such that } \|u(t)-\overline{u}\|_{\mathcal{D}} \le \varrho \text{ for all } t \ge 0 \text{ and } u(t) \to \overline{u} \text{ as } t \to \infty\big\},$$

for some $\varrho > 0$.

Theorem 7.7. (Lunardi [25, Theorem 9.1.4]) *Let $\overline{u}$ denote an equilibrium point of equation* (7.1)*, and suppose that the spectrum of $F'(\overline{u})$ satisfies Assumption 6.2. Choose $0 < \alpha < 1$. Then, there is some $\varrho > 0$ such that for some $\delta > 0$ and a Lipschitz continuous map*

$$\psi : \mathcal{B}_\delta(\overline{u}) = \{q \in Q(\mathcal{D}) : \|q-\overline{u}\|_{\mathcal{D}} \le \delta\} \longrightarrow P(\mathcal{X})$$

satisfying $\psi(\overline{u}) = 0$, the local stable manifold $\mathcal{S}_\varrho$ of $\overline{u}$ is given by

$$\mathcal{S}_\varrho = \big\{u_0 \in \mathcal{O} : u_0 = q + \psi(q), q \in \mathcal{B}_\delta(\overline{u})\big\}.$$

For every element $u_0 \in \mathcal{S}_\varrho$ a unique solution u of (7.1) *with initial value u_0 exists which belongs to $\mathcal{C}^\alpha_\alpha((0,\infty),\mathcal{D})$ and fulfils*

$$\|u-\overline{u}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} \le \varrho \qquad \text{and} \qquad \lim_{t\to\infty} u(t) = \overline{u}.$$

Furthermore, for all $0 < \beta < 1$ and $\omega \ge 0$ such that $-a_2 < -\omega \le \omega < a_1$ it holds $u \in \mathcal{C}^{\beta,\omega}_\beta((0,\infty),\mathcal{D})$.

Proof. In the present situation, we consider the boundary value problem

$$u'(t) = F(u(t)), \quad t > 0, \qquad Qu(0) = q, \qquad \limsup_{t\to\infty} \|Pu(t)\|_{\mathcal{D}} < \infty,$$

instead of (7.7). We use a fixed-point iteration in $\mathcal{C}^{\alpha,\omega}_\alpha((0,\infty),\mathcal{D})$ for some $0 < \alpha < 1$ and $0 \le \omega < a_2$ relying on the following representation of u

$$\begin{aligned} u(t) = e^{tA}q &+ \int_0^t e^{(t-\tau)A} Q\big(G(u(\tau)) - G(u(t))\big)\,\mathrm{d}\tau + A^{-1}\big(e^{tA}Q - Q\big)G(u(t)) \\ &- \int_t^\infty e^{(t-\tau)A} P\big(G(u(\tau)) - G(u(t))\big)\,\mathrm{d}\tau - A^{-1}PG(u(t)), \qquad t \ge 0. \end{aligned}$$

Due to the fact that the auxiliary results are appropriated by the Lemmas E.4 and D.8 and that the proof is now very close to that of Theorem 7.7, we omit a detailed derivation. □

The corresponding result for the discrete counterpart of $\mathcal{S}_\varrho$

$$\mathcal{S}_\varrho^h = \big\{u_0 \in \mathcal{O} : \text{ there exists a Runge-Kutta solution } (u_n)_{n=0}^\infty \text{ for } h > 0 \text{ and } u_0 \\ \text{such that } \|u_n - \overline{u}\|_{\mathcal{D}} \le \varrho \text{ for all } t_n \ge 0 \text{ and } u_n \to \overline{u} \text{ as } n \to \infty\big\}.$$

is the content of the following theorem.

Theorem 7.8. *Assume that for the equilibrium point of* (7.1) *the spectrum of* $F'(\overline{u})$ *fulfils Assumption 6.2, and choose* $0 < \alpha < 1$. *Suppose further that the Runge-Kutta method* (7.2) *is strongly* $A(\vartheta)$*-stable with* $\vartheta > \pi - \varphi$ *where* φ *denotes the constant from* (6.5). *Then, there is some* $\varrho > 0$ *and* $\overline{h} > 0$ *such that for* $0 < h \le \overline{h}$, *for some* $\delta > 0$ *and a Lipschitz continuous map*

$$\psi : \mathcal{B}_\delta(\overline{u}) = \{q \in Q(\mathcal{D}) : \|q - \overline{u}\|_{\mathcal{D}} \le \delta\} \longrightarrow P(\mathcal{X}) \qquad \textit{with} \quad \psi(\overline{u}) = 0,$$

the discrete local stable manifold $\mathcal{S}_\varrho^h$ *of* $\overline{u}$ *is given by*

$$\mathcal{S}_\varrho^h = \big\{u_0 \in \mathcal{O} : u_0 = q + \psi(q), q \in \mathcal{B}_\delta(\overline{u})\big\}.$$

For every element $u_0 \in \mathcal{S}_\varrho^h$ *a unique Runge-Kutta solution* $\underline{u} = (u_n)_{n=0}^\infty$ *with stepsize* $h > 0$ *and initial value* u_0 *exists which belongs to* $\mathcal{C}_\alpha^\alpha(\mathcal{N}, \mathcal{D})$ *where* $\mathcal{N} = \{t_n = nh : n \in \boldsymbol{N}\}$ *and satisfies*

$$\|\underline{u} - \overline{u}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} \le \varrho \qquad \textit{and} \qquad \lim_{t\to\infty} u(t) = \overline{u}.$$

Moreover, for any $0 < \beta < 1$ *and* $\omega \ge 0$ *such that* $-a_2 < -\omega \le \omega < a_1$ *the numerical solution lies in* $\mathcal{C}_\beta^{\beta,\omega}(\mathcal{N}, \mathcal{D})$.

Proof. In the present situation, we consider the Runge-Kutta scheme

$$U_n' = F(U_n), \qquad U_n = \mathbb{1} u_n + h\mathcal{Q}U_n', \qquad u_{n+1} = u_n + hb^T U_n',$$

on the positive half line and endowed it with the boundary conditions

$$Qu_0 = q \qquad \text{and} \qquad \limsup_{n\to\infty} \|Pu_n\|_{\mathcal{D}} < \infty.$$

An application of the modified variations-of-constants formula yields the following representation for the Runge-Kutta solution

$$\begin{aligned} u_n = R(hA)^n q &+ h\sum_{\nu=0}^{n-1} b^T \mathcal{J} A R(hA)^{n-\nu-1} Q\big(G(U_\nu) - G(U_{n-1})\big) \\ &+ A^{-1} b^T \mathcal{J}\big(b^T \mathcal{J}\mathbb{1}\big)^{-1}\big(R(hA)^n Q - Q\big) G(U_{n-1}) \\ &\quad - h\sum_{\nu=n-1}^{\infty} b^T \mathcal{J} R(hA)^{n-\nu-1} P\big(G(U_\nu) - G(U_{n-1})\big) \\ &\quad - A^{-1} b^T \mathcal{J}\big(b^T \mathcal{J}\mathbb{1}\big)^{-1} PG(U_{n-1}). \end{aligned}$$

For the internal stages, we use the identity

$$U_{n-1} = \mathcal{J}\mathbb{1}u_n + h\mathcal{A}\mathcal{J}G(U_{n-1}).$$

Now, essentially the same ideas apply as in the proof of Theorem 7.6. We omit the derivation. $\square$

Part IV.

Auxiliary results

A. Sectorial operators and analytic semigroups

In the present chapter, our field of interest is the study of the following homogeneous initial value problem involving an unbounded linear operator on a Banach space

$$x'(t) = Ax(t), \quad t > 0, \qquad x(0) = x_0. \tag{A.1}$$

Here, one is tempted to write the analytical solution by means of the exponential as

$$x(t) = e^{tA} x_0. \tag{A.2}$$

However, it is not obvious that this representation has a meaning for an abstract setting of Banach spaces.

For the special case of an ordinary differential equation where $\mathcal{X}$ is the m-dimensional Euclidian space $\boldsymbol{R}^m$ and A a $m \times m$-matrix, a common definition for the exponential is

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}, \qquad t \in \boldsymbol{R}.$$

For an arbitrary linear and bounded operator $A : \mathcal{X} \to \mathcal{X}$ on a Banach space $\mathcal{X}$, this power series is alike convergent in the operator norm for any $t \in \boldsymbol{R}$.

If $\mathcal{X}$ is a Hilbert space with scalar product $(\cdot, \cdot)$ and A a negative definite self-adjoint operator in $\mathcal{X}$, then e^{tA} is defined through the spectral decomposition

$$e^{tA} x_0 = \int_{-\infty}^{\infty} e^{-\lambda t} dE_\lambda(x_0), \qquad t \geq 0,$$

where E_λ is the spectral resolution of the identity generated by $-A$. Whenever A has a pure point spectrum, e^{tA} is therefore given by

$$e^{tA} x_0 = \sum_{k=1}^{\infty} e^{\lambda_k t} (x_0, \varphi_k) \varphi_k, \qquad t \geq 0,$$

where the real numbers $\lambda_k \leq 0$ are the eigenvalues of A and the φ_k the corresponding eigenfunctions which form an orthonormal basis, $k \geq 1$, see Krein [17, Chapter I.4], Triebel [34, Chapter IV] and Yosida [36, Chapter XI.5].

The appropriate setting where (A.2) has a precise meaning for an unbounded linear operator A on a Banach space $\mathcal{X}$, is the framework of sectorial operators and analytic semigroups. By means of the integral formula of Cauchy

$$e^{tA} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda I - A)^{-1}\,\mathrm{d}\lambda, \qquad t > 0,$$

for all positive times, one can define a bounded linear operator $e^{tA} : \mathcal{X} \to \mathcal{X}$ which generalizes the exponential, see Definition A.3.

In Section A.1 below we state the definition of sectorial operators and analytic semigroups. There, we further review several fundamental properties of analytical semigroups generated by sectorial operators used throughout. In view of Part III where the existence of backward solutions for equation (A.1) is studied, we introduce in Section A.2 projections associated with spectral sets and determine the behaviour of analytic semigroups in connection with spectral projections. The specified results are found in the monographs Henry [15] and Lunardi [25].

A.1. Analytic semigroups generated by sectorial operators

Let $(\mathcal{X}, \|\cdot\|_\mathcal{X})$ denote a real or complex Banach space and let A be a linear operator in $\mathcal{X}$ with dense domain $\mathcal{D}(A) = \{x \in \mathcal{X} : Ax \in \mathcal{X}\}$. In the sequel, we give the definition of sectorial operators and analytical semigroups generated by sectorial operators. For this, we first recall some notation.

The space of linear and continuous (and thus bounded) operators from a Banach space $\mathcal{X}$ to another Banach space $\mathcal{Y}$, denoted by

$$\mathcal{L}(\mathcal{X}, \mathcal{Y}) = \big\{A : \mathcal{X} \to \mathcal{Y} \text{ linear and continuous}\big\},$$

is endowed with the operator norm $\|\cdot\|_{\mathcal{Y}\leftarrow\mathcal{X}}$. As usual, we set

$$\mathcal{L}(\mathcal{X}) = \mathcal{L}(\mathcal{X}, \mathcal{X}).$$

For a linear operator A in $\mathcal{X}$, the *resolvent set* contains all elements $\lambda \in \boldsymbol{C}$ such that the range of $\lambda I - A : \mathcal{D}(A) \to \mathcal{X}$ is dense in $\mathcal{X}$ and that $\lambda I - A$ has a continuous inverse

$$\varrho(A) = \big\{\lambda \in \boldsymbol{C} : (\lambda I - A)^{-1} \in \mathcal{L}(\mathcal{X})\big\}.$$

All complex numbers lying in the complement of the resolvent set

$$\sigma(A) = \boldsymbol{C}\backslash\varrho(A)$$

form the *spectrum* of A.

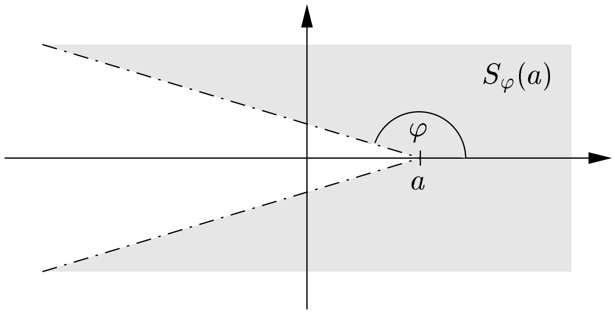

Figure A.1.: Condition (A.3) holds for all elements $\lambda \in S_\varphi(a)$

Definition A.1. (Henry [15, Definition 1.3.3], Lunardi [25, Section 2]) The family $\big(S(t)\big)_{t\geq 0}$ of linear operators on $\mathcal{X}$ forms a *semigroup* if the relations $S(0) = I$ and

$$S(s)S(t) = S(s+t), \qquad s,t \geq 0,$$

are satisfied. The semigroup is called *analytic* if the map $(0,\infty) \to \mathcal{L}(\mathcal{X}) : t \mapsto S(t)$ is analytic, and it is said to be *strongly continuous* if $[0,\infty) \to \mathcal{X} : t \mapsto S(t)x$ is continuous for every $x \in \mathcal{X}$.

We next recall the notion of sectorial operators and the definition of e^{tA}.

Definition A.2. (Henry [15, Definition 1.3.1], Lunardi [25, Definition 2.0.1]) A densely defined and closed linear operator A in $\mathcal{X}$ is called *sectorial* if there are constants $a \in \boldsymbol{R}$, $\pi/2 < \varphi < \pi$ and $M > 0$ such that the sector $S_\varphi(a) = \{\lambda \in \boldsymbol{C} : \lambda \neq a, |\arg(\lambda - a)| < \varphi\}$ is contained in the resolvent set of A and that further the estimate

$$\big\|(\lambda I - A)^{-1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \leq \frac{M}{|\lambda - a|}, \qquad \lambda \in S_\varphi(a), \tag{A.3}$$

is valid.

The sector $S_\varphi(a)$ of Definition A.2 is displayed in Figure A.1.

Definition A.3. (Henry [15, Definition 1.3.4], Lunardi [25, Definition 2.0.2]) Let A be a sectorial operator in $\mathcal{X}$, and, for some radius $r > 0$ and an angle $\pi/2 < \eta < \varphi$, let the counterclockwise oriented curve Γ consist of the half lines γ_+ and γ_-, given by

$$\gamma_\pm = a + \{\lambda \in \boldsymbol{C} : |\lambda| \geq r, \arg(\lambda) = \pm\eta\},$$

and the arc $\gamma_r = a + \{\lambda \in \boldsymbol{C} : |\lambda| = r, |\arg(\lambda)| \leq \eta\}$, see Figure A.2. Then, the family $\big(e^{tA}\big)_{t\geq 0}$ is defined through

$$e^{tA} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda I - A)^{-1}\,\mathrm{d}\lambda, \quad t > 0, \qquad e^{0A} = I, \tag{A.4}$$

and called the *analytic semigroup generated by* A in $\mathcal{X}$.

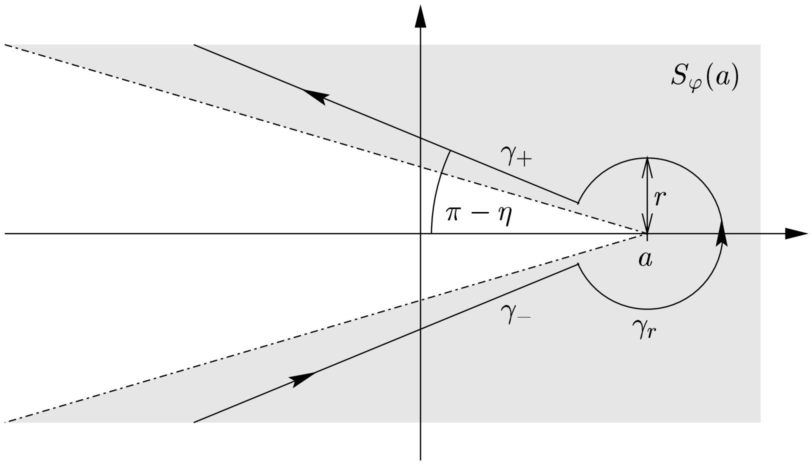

Figure A.2.: Illustration of the curve Γ from Definition A.3

Note that the above definition of the analytic semigroup generated by A does not depend on the choice of r and η, since the map

$$\lambda \longmapsto e^{\lambda t}(\lambda I - A)^{-1}$$

is holomorphic in $S_\varphi(a)$. For each $t \geq 0$, due to the resolvent bound (A.3), the operator $e^{tA} : \mathcal{X} \to \mathcal{X}$ is linear and bounded.

The following result summarizes the fundamental properties of e^{tA}. The second and the fourth statement imply that for every sectorial operator A the family $\left(e^{tA}\right)_{t\geq 0}$ defined by (A.4) actually forms an analytic semigroup.

Result A.1. (Henry [15, Theorem 1.3.4], Lunardi [25, Proposition 2.1.1]) *The analytic semigroup generated by the sectorial operator $A : \mathcal{D}(A) \to \mathcal{X}$ has the following properties.*

(i) The element $e^{tA}x$ belongs to $\mathcal{D}(A^k)$ for every $t > 0$, $x \in \mathcal{X}$ and $k \in \boldsymbol{N}$. Moreover, the identity

$$A^k e^{tA} x = e^{tA} A^k x$$

holds whenever $x \in \mathcal{D}(A^k)$.

(ii) The family $\left(e^{tA}\right)_{t\geq 0}$ forms a semigroup, i.e. the relation

$$e^{sA} e^{tA} = e^{(s+t)A}$$

is valid for all times $s, t \geq 0$.

(iii) Let a denote the quantity from the resolvent condition (A.3). *Then, for any integer $k \geq 0$, the estimate*

$$\left\|(A-aI)^k e^{tA}\right\|_{\mathcal{X}\leftarrow\mathcal{X}} \leq C e^{at} t^{-k}, \qquad t>0,$$

holds with constant C depending on k. In particular, for every $\omega > a$, there are constants M_k, $k \geq 0$, such that the following bounds are valid

$$\begin{aligned}\left\|e^{tA}\right\|_{\mathcal{X}\leftarrow\mathcal{X}} &\leq M_0 e^{at}, \quad t \geq 0,\\ \left\|A^k e^{tA}\right\|_{\mathcal{X}\leftarrow\mathcal{X}} &\leq M_k e^{\omega t} t^{-k}, \quad t>0.\end{aligned}$$

(iv) The operator e^{tA} is arbitrarily often differentiable with respect to the variable t on the positive half line $(0,\infty)$, and its k-th derivative equals

$$\frac{d^k}{dt^k}\, e^{tA} = A^k e^{tA}.$$

Moreover, on the sector $\{\lambda \in \boldsymbol{C} : \lambda \neq 0, |\arg(\lambda)| < \varphi - \pi/2\}$ the map $t \mapsto e^{tA}$ possesses an analytic extension.

The first statement of the following result implies that the analytic semigroup generated by A is strongly continuous whenever

$$\overline{\mathcal{D}(A)} = \mathcal{X}.$$

Due to our persistent requirements stated in Section 1.1.1, this condition is always satisfied.

Result A.2. (LUNARDI [25, Proposition 2.1.4]) *Let A be sectorial in $\mathcal{X}$.*

(i) Then, the analytic semigroup generated by A is strongly continuous if and only if $\mathcal{D}(A)$ is dense in $\mathcal{X}$. More precisely, it holds

$$\lim_{t\to 0_+} e^{tA}x = x \quad \Longleftrightarrow \quad x \in \overline{\mathcal{D}(A)}.$$

(ii) For all elements $x \in \mathcal{X}$ and for positive times $t \geq 0$ the identity

$$A\int_0^t e^{\tau A}x\,\mathrm{d}\tau = (e^{tA} - I)x \tag{A.5a}$$

is valid. Furthermore, the relation

$$\int_0^t A e^{\tau A}x\,\mathrm{d}\tau = A\int_0^t e^{\tau A}x\,\mathrm{d}\tau = (e^{tA} - I)x \tag{A.5b}$$

holds whenever the map $\tau \mapsto Ae^{\tau A}x$ lies in $L^1((0,t),\mathcal{X})$.

Now we are in the position to give (A.2) a precise meaning. From the above statements follows that the map x, defined through

$$x(t) = e^{tA}x_0, \qquad t \geq 0,$$

is a *classical solution* of the homogeneous initial value problem

$$x'(t) = Ax(t), \quad t > 0, \qquad x(0) = x_0,$$

for every initial value $x_0 \in \mathcal{X}$, i.e. the map x possesses the regularity properties

$$x \in \mathcal{C}^1((0,\infty),\mathcal{X}) \cap \mathcal{C}((0,\infty),\mathcal{D}(A)) \cap \mathcal{C}([0,\infty),\mathcal{X})$$

and satisfies $x'(t) = Ax(t)$ for all $t > 0$ and $x(0) = x_0$, see also Definition B.1.

A.2. Projections associated with spectral sets

Up to now, the analytic semigroup generated by a sectorial operator and likewise the linear equation (A.1) were merely studied on the positive line. Now, we proceed further and investigate the behaviour of e^{tA} in connection with spectral projections allowing negative times $t \leq 0$. Such results are important in view of Part III for analyzing the dynamical behaviour of an evolution equation near a steady state solution. There, we apply projections associated with the positive and the negative part of the spectrum, see Section 6.3 and Chapter 7. In the present section, we consider a slightly more general situation involving spectral sets.

Definition A.4. (Henry [15, Definition 1.5.1]) For a linear operator A in $\mathcal{X}$, a *spectral set* σ_1 is a subset of $\sigma(A) \cup \{\infty\}$ such that σ_1 and $(\sigma(A) \cup \{\infty\}) \backslash \sigma_1$ are closed in the extended plane $\boldsymbol{C} \cup \{\infty\}$.

We henceforth assume that $A : \mathcal{D}(A) \to \mathcal{X}$ is sectorial and that the spectrum of A can be decomposed into two spectral sets

$$\sigma(A) = \sigma_1 \cup \sigma_2 \qquad \text{with } \sigma_1 \text{ bounded.}$$

As the distance between σ_1 and σ_2 is positive there exists an open and bounded set Ω such that $\sigma_1 \subset \Omega$ and $\sigma_2 \cap \overline{\Omega} = \emptyset$ where $\overline{\Omega}$ denotes the closure of Ω. We further suppose that the boundary of $\partial\Omega$ consists of a finite number of rectifiable closed Jordan curves which are counterclockwise oriented. Then, the following statements hold, see Lunardi [25, Proposition A.1.2 and Section 2.3.1].

The linear and bounded operator $P : \mathcal{X} \to \mathcal{X}$ defined by

$$P = \frac{1}{2\pi i} \int_{\partial\Omega} (\lambda I - A)^{-1} \, \mathrm{d}\lambda,$$

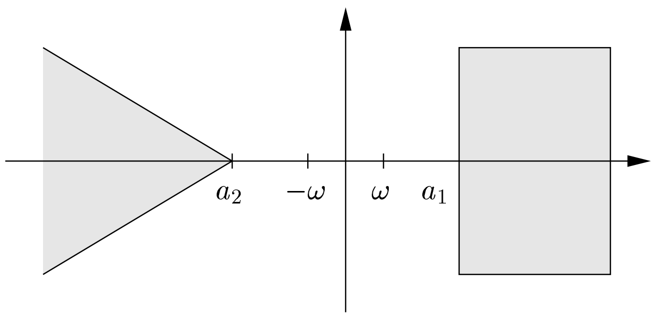

Figure A.3.: The spectrum of A is contained in the shaded area

is a projection on $\mathcal{X}$, i.e. $P^2 = P$. Moreover, the relation $P(\mathcal{X}) \subset \mathcal{D}(A^k)$ holds for every integer $k \geq 0$ and AP is a bounded operator on $\mathcal{X}$. The decomposition

$$\mathcal{X} = P(\mathcal{X}) \oplus Q(\mathcal{X}) \qquad \text{with} \quad Q = I - P$$

induces a splitting of A into $A_1 \in \mathcal{L}(P(\mathcal{X}))$ and A_2 where

$$\begin{aligned} &A_1 : P(\mathcal{X}) \longrightarrow P(\mathcal{X}) : x \longmapsto Ax, \\ &A_2 : \mathcal{D}(A) \cap Q(\mathcal{X}) \longrightarrow Q(\mathcal{X}) : x \longmapsto Ax. \end{aligned}$$

Then, it follows $\sigma(A_1) = \sigma_1$ and $\sigma(A_2) = \sigma_2$ and further

$$(\lambda - A_1)^{-1} = (\lambda - A)^{-1}\big|_{P(\mathcal{X})} \qquad \text{and} \qquad (\lambda - A_2)^{-1} = (\lambda - A)^{-1}\big|_{Q(\mathcal{X})}$$

for all elements $\lambda \in \varrho(A)$. The sectorial operators A_1 and A_2 generate analytic semigroups in $P(\mathcal{X})$ and $Q(\mathcal{X})$, respectively, such that the relations

$$e^{tA_1} = e^{tA}\big|_{P(\mathcal{X})} = e^{tA}P\big|_{P(\mathcal{X})} \qquad \text{and} \qquad e^{tA_2} = e^{tA}\big|_{Q(\mathcal{X})} = e^{tA}Q\big|_{Q(\mathcal{X})}$$

are valid.

The following result provides bounds for the composition $e^{tA}P$ on the negative half line and for $e^{tA}Q$ on the positive half line. Let

$$a_1 = \inf\{\Re(\lambda) : \lambda \in \sigma_1\} \qquad \text{and} \qquad a_2 = \sup\{\Re(\lambda) : \lambda \in \sigma_2\}.$$

Result A.3. (Lunardi [25, Proposition 2.3.3]) *Under the above assumptions, the map*

$$e^{(\cdot)A}P : [0,\infty) \longrightarrow \mathcal{L}(\mathcal{X}) : t \longmapsto e^{tA}P$$

has an analytic extension to the whole real line $\boldsymbol{R}$. *Further, for every* $\varepsilon > 0$ *and* $k \geq 0$ *there exist constants* $M^P > 0$ *and* $M^Q > 0$ *such that the bounds*

$$\begin{aligned} &\big\|A^k e^{tA} P\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \leq M^P e^{(a_1-\varepsilon)t}, \qquad t \leq 0, \\ &\big\|A^k e^{tA} Q\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \leq M^Q e^{(a_2+\varepsilon)t} t^{-k}, \qquad t > 0, \end{aligned}$$

are valid.

In Section 7.2 we apply this result to the situation where σ_1 is contained in the positive half plane and where σ_2 belongs to the negative half plane. In this case, for every integer $k \geq 0$ and for any choice of $\omega \geq 0$ fulfiling

$$a_2 < -\omega \qquad \text{and} \qquad \omega < a_1,$$

there are constants $M_k^P > 0$ and $M_k^Q > 0$ such that the following relations hold

$$\begin{aligned} \left\| A^k e^{tA} P \right\|_{\mathcal{X} \leftarrow \mathcal{X}} &\leq M_k^P e^{\omega t}, \qquad t \leq 0, \\ \left\| A^k e^{tA} Q \right\|_{\mathcal{X} \leftarrow \mathcal{X}} &\leq M_k^Q e^{-\omega t} t^{-k}, \qquad t > 0. \end{aligned}$$

This situation is represented in Figure A.3.

B. Properties of the analytical solution

In the sequel, we investigate the existence and uniqueness of the analytical solution of the following nonlinear problem under the basic assumptions stated in Section 1.1.1.

$$u'(t) = F(t, u(t)), \quad t > 0, \qquad u(0) \text{ given.} \tag{B.1}$$

For our purposes, it is useful to employ the notion of a *strict solution* u of (B.1), i.e. the map u satisfies certain smoothness requirements and the relation $u'(t) = F(t, u(t))$ on some interval as well as the initial condition. In order to construct such a strict solution, we linearize the equation around the initial value $u(0)$. This yields the following initial value problem

$$u'(t) = Au(t) + G(t, u(t)), \quad t > 0, \qquad u(0) \text{ given.} \tag{B.2}$$

Here, the linear operator A and the map G are defined through

$$A = D_v F(t, u(0)) \qquad \text{and} \qquad G(t, v) = F(t, v) - Av.$$

The solution of the formally semilinear problem (B.2) can be represented by means of the variation-of-constants formula

$$u(t) = e^{tA} u(0) + \int_0^t e^{(t-\tau)A} G(\tau, u(\tau)) \, \mathrm{d}\tau. \tag{B.3}$$

A solution of such an integral equation is called a *mild solution* of (B.2). In Section 4.2 we prove existence and uniqueness of a function u satisfying (B.3) on some interval $[0, \delta]$ with $\delta > 0$ by means of a fixed point iteration in (B.3). It further turns out that this solution is also a strict solution of the differential equation (B.1), see Proof of Theorem 4.1.

In Section B.1 we review the definitions of classical, strict and mild solutions for linear problems, and we give sufficient conditions for a mild solution to be a classical or a strict solution. For nonlinear problems, these types of solutions are defined in a similar way. In Section B.2 we then state a local existence and uniqueness result for problem (B.1), see Result B.2, and we introduce the notion of the *maximally defined solution*. Besides, we state an auxiliary result needed in Section 5.2.2 for proving the Theorems 5.1 and 5.2.

The specified definitions and results are found in LUNARDI [25, Chapters 4 and 8]. In addition, a proof of Result B.2 in the lines of LUNARDI [25, Theorem 8.1.1] is given in Section 4.2.

B.1. Classical, strict and mild solutions

We henceforth assume that $A : \mathcal{D}(A) \subset \mathcal{X} \to \mathcal{X}$ is a densely defined sectorial operator on a Banach space $\mathcal{X}$ and that the map g defined on some interval $[0, T]$ where $T > 0$ takes values in $\mathcal{X}$. In the sequel we consider the nonhomogeneous initial value problem

$$u'(t) = Au(t) + g(t), \quad t > 0, \qquad u(0) = u_0. \tag{B.4}$$

Depending on the smoothness requirements on u, it is common in literature to distinguish between several types of solutions such as classical, strict or mild solutions.

Definition B.1. (Lunardi [25, Definition 4.1.1 and 4.1.4]) Choose $u_0 \in \mathcal{X}$.

(i) Assume that $g : (0, T] \to \mathcal{X}$ is continuous. Then, a *classical solution* of (B.4) on $[0, T]$ is a map $u \in \mathcal{C}^1((0, T], \mathcal{X}) \cap \mathcal{C}((0, T], \mathcal{D}(A)) \cap \mathcal{C}([0, T], \mathcal{X})$ satisfying the relation $u'(t) = Au(t) + g(t)$ for all $t \in (0, T]$ and $u(0) = u_0$.

(ii) If g is continuous on $[0, T]$ with values in $\mathcal{X}$, a *strict solution* of (B.4) on $[0, T]$ is a map $u \in \mathcal{C}^1([0, T], \mathcal{X}) \cap \mathcal{C}([0, T], \mathcal{D}(A))$ such that $u'(t) = Au(t) + g(t)$ holds for all $t \in [0, T]$ and $u(0) = u_0$.

(iii) Suppose that $g \in L^1((0, T), \mathcal{X})$ is Lebesgue integrable on $(0, T)$. Then, a *mild solution* of (B.4) on $[0, T]$ is defined through

$$u(t) = e^{tA} u_0 + \int_0^t e^{(t-\tau)A} g(\tau) \, \mathrm{d}\tau, \qquad 0 \le t \le T. \tag{B.5}$$

Provided that g is defined for all positive times, a map $u : [0, \infty) \to \mathcal{X}$ is called a *classical, strict or mild solution of* (B.4) *on* $[0, \infty)$, respectively, if for every $T > 0$ the restriction of u to $[0, T]$ is a strict, classical or mild solution of (B.4) in $[0, T]$.

Clearly, every strict solution of (B.4) is also a classical solution of (B.4). The above definition implies that for a strict solution of (B.4) the initial value u_0 belongs to $\mathcal{D}(A)$. Besides, the mild solution of (B.4) fulfils the estimate

$$\|u(t)\|_{\mathcal{X}} \le C \left(\|u_0\|_{\mathcal{X}} + \int_0^t \|g(\tau)\|_{\mathcal{X}} \, \mathrm{d}\tau \right), \qquad 0 \le t \le T,$$

and lies in $\mathcal{C}([0, T], \mathcal{X})$, see Statement (iii) of Result A.1 and Statement (i) of Result A.2. From Lunardi [25, Proposition 4.1.2] follows that the classical solution of (B.4) is uniquely determined and represented by formula (B.5) whenever $g \in \mathcal{C}((0, T], \mathcal{X})$. Alike, this is true for the strict solution provided that $g \in \mathcal{C}([0, T], \mathcal{X})$.

The following result gives sufficient conditions for a mild solution to be a classical or a strict solution.

Result B.1. (LUNARDI [25, Lemma 4.1.6]) *Assume* $g \in \mathcal{C}((0,T],\mathcal{X})$, *and for some* $u_0 \in \mathcal{X}$ *let* u *denote the mild solution of* (B.4) *on* $[0,T]$.

(i) Then the following statements are equivalent.

  (a) The function u *is a classical solution of* (B.4) *on* $[0,T]$.

  (b) $u \in \mathcal{C}((0,T],\mathcal{D}(A))$.

  (c) $u \in \mathcal{C}^1((0,T],\mathcal{X})$.

(ii) Suppose $g \in \mathcal{C}([0,T],\mathcal{X})$. *Then the following statements are equivalent.*

  (a) The function u *is a strict solution of* (B.4) *on* $[0,T]$.

  (b) $u \in \mathcal{C}([0,T],\mathcal{D}(A))$.

  (c) $u \in \mathcal{C}^1([0,T],\mathcal{X})$.

For nonlinear evolution equations, classical and strict solutions are defined in a similar way as for linear problems. We consider the initial value problem (B.1) where the right-hand side $F : [0,T] \times \mathcal{O} \to \mathcal{X}$ is defined on an open subset $\mathcal{O}$ of the densely embedded Banach space $\mathcal{D} \subset \mathcal{X}$. For initial values $u(0) \in \mathcal{O}$, a *strict solution* of (B.1) on $[0,T]$ is a map

$$u \in \mathcal{C}^1([0,T],\mathcal{X}) \cap \mathcal{C}([0,T],\mathcal{D})$$

such that $u(t) \in \mathcal{O}$ and $u'(t) = F(t,u(t))$ holds for all times $t \in [0,T]$ and further the initial condition. Analogously, a *classical solution* of (B.1) on $[0,T]$ is a map

$$u \in \mathcal{C}^1((0,T],\mathcal{X}) \cap \mathcal{C}((0,T],\mathcal{D}) \cap \mathcal{C}([0,T],\mathcal{X})$$

satisfying $u(t) \in \mathcal{O}$ and $u'(t) = F(t,u(t))$ for all $t \in (0,T]$ and the initial condition.

B.2. The maximally defined solution

In the present section, we collect several regularity properties of the analytical solution of the nonlinear problem

$$u'(t) = F(t,u(t)), \quad t > 0, \qquad u(0) \text{ given.} \tag{B.6}$$

We henceforth suppose that the Banach space $\mathcal{D}$ is densely embedded in another Banach space $\mathcal{X}$ and that the right-hand side $F : [0,T] \times \mathcal{O} \to \mathcal{X}$ of (B.6) which is defined on an open subset $\mathcal{O} \subset \mathcal{D}$, satisfies the requirements of Assumption 1.1.

In the first place, we state the main result on existence and uniqueness. The space $\mathcal{C}^\alpha_\alpha((0,\delta],\mathcal{D})$ is introduced in Section 1.2.1, and for a definition of the real interpolation space $(\mathcal{X},\mathcal{D})_{\alpha,\infty}$ we refer to LUNARDI [25, Definition 1.2.2].

Result B.2. (LUNARDI [25, Theorem 8.1.1], Theorem 4.1) *Suppose that the above requirements are satisfied, and choose* $u(0) \in \mathcal{O}$*. Then, there is some* $\delta > 0$ *such that there exists a strict solution*

$$u \in \mathcal{C}([0,\delta],\mathcal{D}) \cap \mathcal{C}^1([0,\delta],\mathcal{X})$$

of problem (B.6) *on* $[0,\delta]$*. Moreover, it holds*

$$u \in \mathcal{C}^{\alpha}_{\alpha}((0,\delta],\mathcal{D}), \quad t^{\alpha} u' \in \mathcal{B}((0,\delta],(\mathcal{X},\mathcal{D})_{\alpha,\infty}) \quad \textit{and} \quad \lim_{\varepsilon \to 0_+} \varepsilon^{\alpha} [u]_{\mathcal{C}^{\alpha}([\varepsilon,2\,\varepsilon],\mathcal{D})} = 0.$$

The map u *is the unique solution of* (B.6) *belonging to the set*

$$\mathcal{C}([0,\delta],\mathcal{D}) \cap \bigcup_{0<\beta<1} \mathcal{C}^{\beta}_{\beta}((0,\delta],\mathcal{D}).$$

Whenever the initial value is more regular, i.e. the element $F(0,u(0))$ lies in the real interpolation space $(\mathcal{X},\mathcal{D})_{\alpha,\infty}$, then the analytical solution $u : [0,\delta] \to \mathcal{D}$ of (B.6), given by the previous result, is α-Hölder continuous. More precisely, the map u fulfils

$$u \in \mathcal{C}^{\alpha}([0,\delta],\mathcal{D}) \qquad \text{and} \qquad u' \in \mathcal{C}^{\alpha}([0,\delta],\mathcal{X}),$$

see LUNARDI [25, Theorem 8.1.3]. Due to this result, the local solution u can be continued to $[\delta,\delta_1]$ for some $\delta_1 > \delta$ such that the extended function belongs to $\mathcal{C}^{\alpha}_{\alpha}((0,\delta_1],\mathcal{D})$. On the interval

$$\begin{aligned} I(u_0) = \bigcup \Big\{ [0,\delta] :\ & \text{for some } 0 < \beta < 1 \text{ the initial value problem (B.6) has} \\ & \text{a solution } u_\delta \text{ belonging to } \mathcal{C}([0,\delta],\mathcal{D}]) \cap \mathcal{C}^{\beta}_{\beta}((0,\delta),\mathcal{D}) \Big\} \end{aligned} \tag{B.7a}$$

we define the *maximally defined solution* of (B.6) through

$$u(\cdot,u_0) : I(u_0) \longrightarrow \mathcal{D} : t \longmapsto u_\delta(t) \quad \text{for } t \in [0,\delta]. \tag{B.7b}$$

From Result B.2 follows that $u(\cdot,u_0)$ is well-defined, and LUNARDI [25, Proposition 8.2.1] implies that the maximally defined solution has no continuous extension in $\mathcal{D}$.

Result B.3. (LUNARDI [25, Proposition 8.2.1]) *If the map* $u(\cdot,u_0) : I(u_0) \to \mathcal{D}$ *given by* (B.7) *is uniformly continuous in* $\mathcal{D}$*, then it holds either*

$$\lim_{t \to \tau(u_0)} u(t,u_0) \in \partial\mathcal{O} \qquad \textit{or} \qquad I(u_0) = [0,T]$$

where $\tau(u_0) = \sup I(u_0)$*.*

The maximally defined solution depends continuously on the initial value. The precise formulation is as follows.

Result B.4. (LUNARDI [25, Proposition 8.2.3]) *Choose* $u_0 \in \mathcal{O}$ *and* $0 < \overline{\tau} < \tau(u_0)$. *Then there exist constants* $\delta > 0$ *and* $C > 0$ *such that for all elements* $v_0 \in \mathcal{O}$ *satisfying* $\|v_0 - u_0\|_{\mathcal{D}} \leq \delta$ *the following relations*

$$\tau(v_0) \geq \overline{\tau} \qquad \text{and} \qquad \|u(\cdot, u_0) - u(\cdot, v_0)\|_{\mathcal{C}^\alpha_\alpha((0,\overline{\tau}],\mathcal{D})} \leq C\|u_0 - v_0\|_{\mathcal{D}}$$

are valid.

We close this section with an auxiliary result needed in the proofs of the convergence results Theorems 5.1 and 5.2.

Lemma B.1. *Assume that the analytical solution of* (B.6) *fulfils* $u \in \mathcal{C}([0,T],\mathcal{D})$. *Then, for every* $\varepsilon > 0$, *there exists a partition* $0 = T_0 < T_1 < \cdots < T_J = T$ *of the interval* $[0,T]$ *such that the bound*

$$\|u - u(T_j)\|_{\mathcal{C}^\alpha_\alpha((T_j,T_{j+1}],\mathcal{D})} \leq \varepsilon \tag{B.8}$$

holds for all $0 \leq j \leq J-1$.

Proof. Due to the local existence result stated in Result B.2 and proven in Section 4.2, the existence of a strict solution u of the initial value problem (B.6) is ensured on some interval $[0,\delta] \subset [0,T]$. This solution is just the fixed point of the nonlinear operator Γ defined by (4.8). Consequently, we have

$$\|u - u(0)\|_{\mathcal{C}^\alpha_\alpha((0,\delta],\mathcal{D})} \leq \frac{1}{1-\kappa}\,\|\Gamma(u(0)) - u(0)\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})}$$

with κ denoting the contraction factor κ of Γ. As in the proof of Theorem 4.1, see Relation (4.14) and below, it follows from the Lemmas F.1 and F.2 that the estimate

$$\|u - u(0)\|_{\mathcal{C}^\alpha_\alpha((0,T_1],\mathcal{D})} \leq \varepsilon$$

holds for some $T_1 \leq \delta$ chosen sufficiently small. The same arguments apply in order to construct a local solution with initial value $u(T_1)$ which satisfies (B.8) for $j = 1$ on some interval $[T_1, T_2]$ with $T_2 > T_1$. By repeating this process, the endpoint T is reached after a finite number of steps. Due to compactness of $[0,T]$, the arising constants can be chosen uniformly. $\square$

C. Implicit Runge-Kutta methods

In the present chapter, Runge-Kutta time discretizations for nonlinear parabolic evolution equations of the form

$$u'(t) = F(t, u(t)), \qquad t > 0, \tag{C.1}$$

are considered. We show in Part II that for a strongly stable Runge-Kutta method, applied to the abstract problem (C.1), the order of convergence is essentially the stage order of the method, see Theorem 5.2 in Section 5.2.2 and Theorem 5.4 in Section 5.3.2. For proving the results on existence and convergence there, we employ two different approaches. In Section 5.2.2 we linearize the right-hand side of (C.1) around a fixed point which yields a formally semilinear equation

$$u'(t) = Au(t) + G(t, u(t)), \qquad t > 0, \tag{C.2}$$

whereas in Section 5.3.2 our main technique is to linearize along the exact solution which leads to a non-autonomous problem of the form

$$u'(t) = A(t)u(t) + G(t, u(t)), \qquad t > 0. \tag{C.3}$$

Important tools for deriving the error estimates of the Theorems 5.2 and 5.4 are stability bounds for Runge-Kutta time discretizations. The needed results for (C.2) are provided by Lubich & Nevanlinna [20] or the recent work Bakaev & Ostermann [4] on rational one-step approximations of analytic semigroups. Similar stability estimates are also given in Le Roux [18], Lubich & Ostermann [21, 24] and González et al. [11]. For problem (C.3) involving a time-dependent linear operator the employed stability results are proven in González & Palencia [12].

At first, we state in Section C.1 the scheme of an implicit Runge-Kutta method, and we introduce some useful notation which allows us to write the numerical method in compact vector form. In Section C.2 we then review the notions of order and stability from classical analysis. For a detailed description of implicit Runge-Kutta methods we refer to the monographs Hairer et al. [13] and Hairer & Wanner [14]. The fundamental stability results for (C.2) and (C.3) are collected in Section C.3.

C.1. Numerical scheme

An s-stage *Runge-Kutta method* with constant stepsize $h > 0$, applied to the initial value problem

$$u'(t) = F(t, u(t)), \quad t > 0, \qquad u(0) \text{ given}, \tag{C.4}$$

yields an approximation u_n to the value of the exact solution of (C.4) at $t_n = nh$, and is given by the following scheme for the stages

$$U'_{ni} = F(t_n + c_i h, U_{ni}), \qquad U_{ni} = u_n + h \sum_{j=1}^{s} a_{ij} U'_{nj}, \qquad 1 \le i \le s, \tag{C.5a}$$

and by the recursion for the numerical solution

$$u_{n+1} = u_n + h \sum_{i=1}^{s} b_i U'_{ni}, \qquad u_0 \text{ given}. \tag{C.5b}$$

The real numbers a_{ij}, b_i and c_i, $1 \le i, j \le s$, are the *coefficients* of the method. For our purposes, it is convenient to suppose

$$c_i \in [0, 1] \qquad \text{for all } 1 \le i \le s,$$

since then the following relation is valid

$$t_n \le t_n + c_i h \le t_{n+1}, \qquad 1 \le i \le s.$$

In particular, this assumption is fulfiled by Runge-Kutta methods that are based on quadrature formulas.

In each step, the internal stages $(U_{n1}, \dots, U_{ns})^T$ are obtained from (C.5a) by solving, in general, a system of coupled nonlinear equations

$$U_{ni} = u_n + h \sum_{j=1}^{s} a_{ij} F(t_n + c_j h, U_{nj}), \qquad 1 \le i \le s,$$

wherefore global existence and uniqueness results have to be investigated specifically. The Runge-Kutta solution u_{n+1} is then computed from the one-step recursion in (C.5b).

In order to write the numerical scheme (C.5) more compactly, we introduce the following matrix and vector notation

$$\mathcal{O}\!\iota = (a_{ij})_{1 \le i,j \le s}, \qquad b = (b_1, \dots, b_s)^T, \qquad c = (c_1, \dots, c_s)^T,$$
$$U_n = (U_{n1}, \dots, U_{ns})^T, \qquad U'_n = (U'_{n1}, \dots, U'_{ns})^T, \qquad \mathbb{1} = (1, \dots, 1)^T \in \boldsymbol{R}^s.$$

Moreover, we define $F(t_n + ch, U_n)$ through

$$F(t_n + ch, U_n) = \big(F(t_n + c_1 h, U_{n1}), \ldots, F(t_n + c_s h, U_{ns})\big)^T.$$

With these abbreviations, the Runge-Kutta method (C.5) takes the form

$$U_n' = F(t_n + ch, U_n), \qquad U_n = \mathbb{1}\, u_n + h\mathcal{A}U_n', \qquad u_{n+1} = u_n + hb^T U_n'. \tag{C.6}$$

Here, the matrix $\mathcal{A}$ is considered as a linear operator and therefore the i-th component of $\mathcal{A}U_n'$ is given by

$$\big(\mathcal{A}U_n'\big)_i = \sum_{j=1}^{s} a_{ij} U_{nj}'.$$

The quantity $b^T U_n'$ is defined in an analogous way.

Often, the coefficients of the Runge-Kutta method (C.6) are represented by the *Butcher tableau*, see Table C.1. Whenever $\mathcal{A}$ is a lower triangular matrix, i.e. it holds

$$a_{ij} = 0 \qquad \text{for all } 1 \le i \le j \le s,$$

the method is called *explicit*, and *implicit* otherwise.

Due to its elementariness, the *implicit* or *backward Euler method* which is given by the recursion

$$\frac{u_{n+1} - u_n}{h} = F(t_{n+1}, u_{n+1})$$

is of particular interest for our analysis of Runge-Kutta time discretizations for nonlinear parabolic problems. Further examples for implicit Runge-Kutta methods are the *Gauss-Legendre methods*, the *Radau methods* or the *Lobatto methods* which are based on quadrature formulas. For the construction and a detailed description of these methods we refer to Hairer & Wanner [14, Chapter IV.5].

$$\begin{array}{c|c} c & \mathcal{A} \\ \hline & b^T \end{array}$$

Table C.1.: Butcher tableau for the Runge-Kutta scheme (C.6)

C.2. Order and stability

In the present section, we review the basic notions of order and stability for Runge-Kutta methods of the form (C.6). For this purpose, we interpret (C.4) as an ordinary differential equation.

Definition C.1. (HAIRER ET AL. [13, Definition 1.2]) A Runge-Kutta method has *order p* if for problems (C.4) with sufficiently differentiable right-hand side, the Taylor series for the exact solution $u(h)$ and for the numerical approximation u_1 coincide up to the term h^p

$$u_1 - u(h) = \mathcal{O}\big(h^{p+1}\big) \qquad \text{for } h \to 0.$$

The method has *stage order q* whenever the following relation

$$U_{0i} - u(c_i h) = \mathcal{O}\big(h^{q+1}\big) \qquad \text{as } h \to 0$$

is fulfiled for all $1 \leq i \leq s$.

The classical theory of stability studies the question how certain damping properties of the exact solution of the *Dahlquist test equation*

$$x'(t) = \lambda x(t), \quad \Re(\lambda) \leq 0, \quad t > 0, \qquad x(0) = x_0, \tag{C.7}$$

are captured by the numerical method. Applying the Runge-Kutta method (C.6) with initial value x_0 to this homogeneous initial value problem on $\boldsymbol{C}$, yields

$$x_n = R(h\lambda)^n x_0, \quad n \geq 0, \qquad \text{where} \quad R(h\lambda) = 1 + h\lambda b^T(\mathcal{I} - h\lambda\mathcal{Q})^{-1}\mathbb{1}.$$

Here, $\mathcal{I}$ denotes the identity matrix on $\boldsymbol{R}^s$. Since the complex number λ has negative real part, the analytical solution of (C.7) which is given by

$$x(t) = e^{t\lambda} x_0, \qquad t \geq 0,$$

is bounded for all positive times. More precisely, it follows

$$|x(t)| \leq |x_0| \qquad \text{for all } t \geq 0. \tag{C.8a}$$

The analogous property is valid for the numerical solution if the estimate

$$|R(h\lambda)| \leq 1$$

holds for all stepsizes h such that $h\lambda$ belongs to the negative complex half plane $\boldsymbol{C}_{\leq 0}$. A numerical method satisfying the above relation under no restrictions on the stepsize is called *A-stable*. Besides, the exact solution of (C.7) converges to zero

$$\lim_{t\to\infty} x(t) = 0, \tag{C.8b}$$

and the Runge-Kutta solution shows the same asymptotic behaviour whenever it holds

$$|R(\infty)| < 1.$$

The above considerations motivate the following definition.

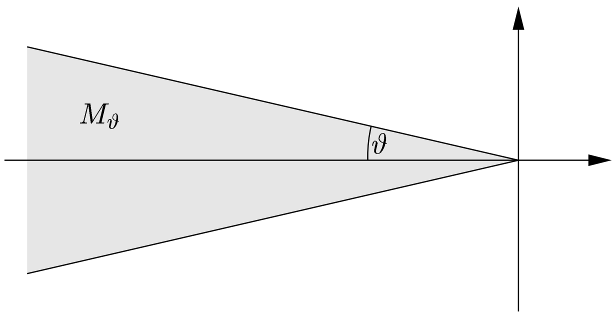

Figure C.1.: Stability region M_ϑ of an $A(\vartheta)$-stable Runge-Kutta method

Definition C.2. (Hairer & Wanner [14, Chapter IV.3]) A Runge-Kutta method of the form (C.6) is $A(\vartheta)$-*stable* if $I - z\mathcal{A}$ is invertible on $M_\vartheta = \{z \in \boldsymbol{C} : |\arg(-z)| \le \vartheta\}$ and if the absolute value of the *stability function*,

$$R(z) = 1 + zb^T(\mathcal{I} - z\mathcal{A})^{-1}\mathbb{1},$$

is bounded by one for all $z \in M_\vartheta$. The method is called *strongly* $A(\vartheta)$-*stable* if in addition $\mathcal{A}$ is invertible and the module of R at infinity,

$$R(\infty) = 1 - b^T\mathcal{A}^{-1}\mathbb{1},$$

is strictly smaller than one. An $A(\vartheta)$-stable method is called A-*stable* if $\vartheta = \pi/2$.

For some angle $0 < \vartheta < \pi/2$ the sector M_ϑ arising in Definition C.2 is displayed in Figure C.1.

Notice that for an explicit Runge-Kutta method the stability function $R(z)$ is a polynomial in z of degree less or equal s

$$R(z) = 1 + zb^T\big(\mathcal{I} + (z\mathcal{A}) + \cdots + (z\mathcal{A})^{s-1}\big)\mathbb{1}.$$

Thus, the requirement of $A(\vartheta)$-stability excludes all explicit Runge-Kutta methods.

We close this section with some examples of implicit Runge-Kutta methods possessing good stability properties. A brief overview on important strongly stable Runge-Kutta method is found in Table C.3.

1. The *backward Euler method* applied to (C.4) is given by the scheme

$$u_{n+1} = u_n + hF(t_{n+1}, u_{n+1})$$

The method has order one and is strongly A-stable, see also Table C.2. This is an easy consequence of the fact that its stability function

$$R(z) = \frac{1}{1-z}$$

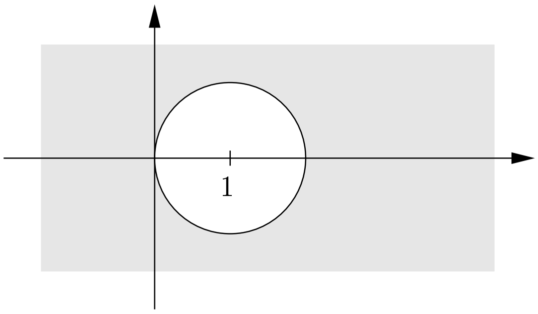

Figure C.2.: Stability region of the backward Euler method

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array} \qquad\qquad \begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}$$

Table C.2.: Backward Euler and implicit midpoint rule

is bounded in the complement of the unit circle around one, see Figure C.2, and further fulfils $R(\infty) = 0$.

2. A simple example for a Gauss-Legendre method is the *implicit midpoint rule* of order two, defined through

$$u_{n+1} = u_n + \frac{h}{2} F\Big(t_n + \frac{h}{2}, \frac{u_{n+1} + u_n}{2}\Big)$$

when applied to (C.4), see also Table C.2. The method is A-stable, but not strongly stable. This follows at once from the identity

$$R(z) = \frac{1 + z/2}{1 - z/2}$$

which implies $R(\infty) = -1$.

3. Further classes of A-stable Runge-Kutta methods are the *Gauss-Legendre methods* of order $2s$, the *Radau methods* IA and IIA of order $2s-1$ and the *Lobatto methods* IIIA, IIIB and IIIC of order $2s-2$, see Hairer & Wanner [14, Section IV.5].

C.3. Stability estimates

In the sequel, we come back to the abstract situation considered in the preceding Chapters A and B. Let henceforth A denote a sectorial operator on a Banach space $\mathcal{X}$ with

Method	Order	Stage order	Stability properties
Backward Euler	$p = 1$	$q = 1$	A-stable, $R(\infty) = 0$
s-stage Radau IIA	$p = 2s - 1$	$q = s$	A-stable, $R(\infty) = 0$
s-stage Lobatto IIIC	$p = 2s - 2$	$q = s - 1$	A-stable, $R(\infty) = 0$

Table C.3.: Examples of strongly stable Runge-Kutta methods

dense domain $\mathcal{D}(A)$, and let $a \in \boldsymbol{R}$ and $\pi/2 < \varphi < \pi$ be such that the resolvent condition

$$\left\|(\lambda I - A)^{-1}\right\|_{\mathcal{X}\leftarrow\mathcal{X}} \leq \frac{M}{|\lambda - a|}, \qquad |\arg(\lambda - a)| < \varphi, \tag{C.9}$$

is satisfied, see also Definition A.2. For discretizing the homogeneous initial value problem

$$x'(t) = Ax(t), \quad t > 0, \qquad x(0) \text{ given}, \tag{C.10}$$

in time, we apply an implicit Runge-Kutta method of the form (C.6) with initial value x_0.

Roughly spoken, the consideration of such a parabolic evolution equation involving an unbounded linear operator corresponds to the case where the parameter λ arising in the Dahlquist test equation (C.10) tends to $-\infty$. In this situation, only strongly stable Runge-Kutta methods capture the damping behaviour (C.8) of the exact solution correctly. Hence, for discretizing an abstract in time initial value problem the use of implicit Runge-Kutta methods is indispensable.

A short calculation shows that the Runge-Kutta solution of the linear problem (C.10) with initial value x_0 is given by the following relation involving the stability function of the method

$$x_n = R(hA)^n x_0, \quad n \geq 0, \qquad \text{where} \quad R(hA) = 1 + hAb^T(\mathcal{I} - hA\mathcal{Q})^{-1}\mathbb{1}.$$

For the backward Euler method this formula specializes to

$$x_n = (I - hA)^{-n} x_0, \qquad n \geq 0.$$

The following result provides several estimates for $(I - hA)^n$ and $R(hA)^n$, respectively, which show that the behaviour of the stability function and the analytic semigroup are strongly connected, see also Result A.1.

Result C.1. *Let $A : \mathcal{D}(A) \to \mathcal{X}$ be sectorial and let $a \in \boldsymbol{R}$ and $\pi/2 < \varphi < \pi$ denote the constants arising in* (C.9). *Assume further that the Runge-Kutta method* (C.6) *is strongly $A(\vartheta)$-stable with $\vartheta > \pi - \varphi$. Then, the following statements hold.*

(i) For all $h > 0$ the operator $\mathcal{I} - hA\mathcal{Q}$ has a bounded inverse on $\mathcal{X}^s$. Moreover, for stepsizes $0 < h \leq \overline{h}$ sufficiently small, the operator $I - R(hA)$ is invertible and the identity

$$(I - R(hA))^{-1} = (1 - R(\infty))^{-1} I + h^{-1} B A^{-1} \tag{C.11}$$

holds with $B : \mathcal{X} \to \mathcal{X}$ bounded.

(ii) *For any $\omega > a$ there exist constants $\overline{h} > 0$ and $C > 0$ such that for $0 < h \leq \overline{h}$ the following bounds*

$$
\begin{aligned}
\left\|(I - hA)^{-n}\right\|_{\mathcal{X}\leftarrow\mathcal{X}} &\leq C e^{\omega t_n}, && n \geq 0,\\
\left\|A(I - hA)^{-n}\right\|_{\mathcal{X}\leftarrow\mathcal{X}} &\leq C e^{\omega t_n} t_n^{-1}, && n \geq 1,\\
\left\|A^2(I - hA)^{-n}\right\|_{\mathcal{X}\leftarrow\mathcal{X}} &\leq C e^{\omega t_n} t_n^{-2}, && n \geq 2,
\end{aligned}
\tag{C.12}
$$

are valid.

Whenever $a < 0$ holds, then for every $\overline{h} > 0$ there exist constants $a < \omega < 0$ and $C > 0$ such that Relation (C.12) *holds for $0 < h \leq \overline{h}$.*

(iii) *For every $\omega > a$ there exist constants $\overline{h} > 0$ and $C > 0$ such that the relations*

$$
\begin{aligned}
\left\|R(hA)^n\right\|_{\mathcal{X}\leftarrow\mathcal{X}} &\leq C e^{\omega t_n}, && n \geq 0,\\
\left\|A\big(R(hA)^n - R(\infty)^n\big)\right\|_{\mathcal{X}\leftarrow\mathcal{X}} &\leq C e^{\omega t_n} t_n^{-1}, && n \geq 1,\\
\left\|A^2\big(R(hA)^n - R(\infty)^n\big)\right\|_{\mathcal{X}\leftarrow\mathcal{X}} &\leq C e^{\omega t_n} t_n^{-2}, && n \geq 2,
\end{aligned}
\tag{C.13}
$$

holds for all $0 < h \leq \overline{h}$.

If $a < 0$ then for any $\overline{h} > 0$ there are constants $a < \omega < 0$ and $C > 0$ such that (C.13) *holds for $0 < h \leq \overline{h}$.*

Besides, the relation $\omega/a = 1 + \mathcal{O}((ha)^p)$ is valid where p denotes the order of the method.

Proof. (i) The first statement is derived in NAKAGUCHI & YAGI [26, Section 3], e.g., and for the special case where $\mathcal{O}\!\iota$ has a lower block triangular structure a proof is also given in SCHWITZER [30, Lemma 2.1]. From GONZÁLEZ & OSTERMANN [10, Lemma 4.1] Relation (C.11) follows.

(ii) In GONZÁLEZ ET AL. [11, Lemma 5.1] the second result is proven for variable stepsizes backward Euler discretizations.

(iii) By using similar techniques as in LUBICH & NEVANLINNA [20, Theorem 3.5] or LE ROUX [18, Theorem 1.2] the bound

$$
\left\|A^k\big(R(hA)^n - R(\infty)^n\big)\right\|_{\mathcal{X}\leftarrow\mathcal{X}} \leq C t_n^{-k}, \qquad n \geq k, \quad 0 \leq k \leq 2,
$$

follows. Considerations as in the proofs of LUBICH & OSTERMANN [21, Lemma 2.3], LUBICH & OSTERMANN [24, Lemma 3.1] and BAKAEV & OSTERMANN [4, Theorem 4.1] yield the exponential term in (C.13). □

For the remainder of this section we consider the following problem involving a time-dependent linear operator $A(t)$ on $\mathcal{X}$ with fixed domain $\mathcal{D}$

$$
x'(t) = A(t)x(t), \quad t \geq 0, \qquad x(0) \text{ given.} \tag{C.14}
$$

We assume that $A(t)$ is uniformly sectorial for $t \in [0,T]$, and we denote by $a \in \boldsymbol{R}$ and $\pi/2 < \varphi < \pi$ the constants arising in the resolvent condition

$$\big\|(\lambda I - A(t))^{-1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \frac{M}{|\lambda - a|}, \qquad |\arg(\lambda - a)| < \varphi, \tag{C.15}$$

from Definition A.2. Furthermore, we suppose that A possesses the regularity property

$$A \in \mathcal{C}^{\alpha}\big((0,T], \mathcal{L}(\mathcal{D},\mathcal{X})\big).$$

Under these assumptions, stability bounds for strongly $A(\vartheta)$-stable Runge-Kutta time discretizations of (C.14) with $\vartheta > \pi - \varphi$ are provided by González & Palencia [12]. For summarizing these results, it is helpful to introduce first some notation.

The backward Euler solution of the homogeneous equation (C.14) with initial value x_0 is given by the relation

$$\frac{x_{n+1} - x_n}{h} = A(t_{n+1})x_{n+1}.$$

For the discrete transition operator we use the following abbreviation

$$r(t_n,t_k) = \big(I - hA(t_n)\big)^{-1} \dots \big(I - hA(t_{k+1})\big)^{-1}, \quad n > k \ge 0, \qquad r(t_n,t_n) = I.$$

Solving the above recursion then yields

$$x_n = r(t_n,0)x_0, \qquad n \ge 0.$$

For a general Runge-Kutta method with coefficients $\mathcal{O}\!\iota$, b and c, we employ the notation

$$\mathcal{A}(t_n + ch) = \operatorname{diag}\big(A(t_n + c_1h), \dots, A(t_n + c_sh)\big),$$
$$\mathcal{J}(t_n + ch) = \big(\mathcal{I} - h\mathcal{O}\!\iota\mathcal{A}(t_n + ch)\big)^{-1}, \qquad \mathcal{K}(t_n + ch) = \big(\mathcal{I} - h\mathcal{A}(t_n + ch)\mathcal{O}\!\iota\big)^{-1}.$$

Moreover, we define the stability function through

$$R\big(h\mathcal{A}(t_n + ch)\big) = 1 + hb^T\mathcal{A}(t_n + ch)\mathcal{J}(t_n + ch)\mathbb{1},$$

and we denote by $R(t_n,t_k)$ the discrete transition operator

$$R(t_n,t_k) = R\big(h\mathcal{A}(t_{n-1} + ch)\big) \dots R\big(h\mathcal{A}(t_k + ch)\big), \quad n > k \ge 0, \qquad R(t_n,t_n) = \mathcal{I}.$$

With these abbreviations, the Runge-Kutta scheme (C.6), applied to (C.14), takes the form

$$X_n' = \mathcal{A}(t_n + ch)X_n, \qquad X_n = \mathbb{1}x_n + h\mathcal{O}\!\iota X_n', \qquad x_{n+1} = x_n + hb^TX_n',$$

and a short calculation shows that the numerical solution is given by

$$x_n = R(t_n,0)x_0, \qquad n \ge 0.$$

Now we are ready, to state the result on stability.

Result C.2. *In the above situation, the following statements hold.*

(i) The operators $\mathcal{J}(t_n + ch)$ and $\mathcal{K}(t_n + ch)$ are well-defined and bounded on $\mathcal{X}^s$ for stepsizes $0 < h \leq \overline{h}$ small enough.

(ii) There exist constants $C > 0$ and $\overline{h} > 0$ such that for all $0 < h \leq \overline{h}$ the following stability bounds are valid for the backward Euler method

$$\begin{aligned} \|r(t_n, t_k)\|_{\mathcal{X}\leftarrow\mathcal{X}} &\leq C, \qquad 0 \leq k \leq n \leq N, \\ \|r(t_n, t_k)\|_{\mathcal{D}\leftarrow\mathcal{X}} &\leq C t_{n-k}^{-1}, \qquad 0 \leq k < n \leq N. \end{aligned}$$

The constants $C > 0$ in general depend on t_N, but not on n or h.

(iii) For a general Runge-Kutta method, there are constants $C > 0$ and $\overline{h} > 0$ such that for all $0 < h \leq \overline{h}$ the estimates

$$\begin{aligned} \left\|R(t_n, t_k)\right\|_{\mathcal{X}\leftarrow\mathcal{X}} &\leq C, \qquad 0 \leq k \leq n \leq N, \\ \left\|R(t_n, t_k) - R(\infty)^{n-k}\right\|_{\mathcal{D}\leftarrow\mathcal{X}} &\leq C t_{n-k}^{-1}, \qquad 0 \leq k < n \leq N, \end{aligned} \tag{C.16}$$

hold with constants $C > 0$ depending in general on t_N, but not on n or h.

Proof. Due to the resolvent condition (C.15) and the strong stability of the Runge-Kutta method, the first result follows, see also Result C.1. The remaining statements are the contents of González & Palencia [12, Theorem 1.1]. □

Notice that the term $R(\infty)$ in (C.16) can be omitted by means of an additional factor $\mathcal{J}$ or $\mathcal{K}$. Consequently, bounds such as

$$\left\|\mathcal{J}(t_n + ch)\mathbb{1} R(t_n, t_{k+1})\right\|_{\mathcal{D}^s\leftarrow\mathcal{X}^s} \leq \hat{M}_1 t_{n-k}^{-1}, \qquad 0 \leq k < n \leq N,$$

are valid for $0 < h \leq \overline{h}$ small enough. This estimate follows by using similar ideas as in Section D.2.1, see Formula (D.16) and below.

D. The nonhomogeneous problem

A first step towards the study of time discretizations for a nonlinear parabolic problem

$$u'(t) = F(t, u(t)), \quad t > 0, \qquad u(0) \text{ given},$$

is the analysis of numerical methods applied to less involved problems such as linear equations. Our concern in the present chapter is hence the investigation of the nonhomogeneous problem

$$u'(t) = Au(t) + g(t), \quad t > 0, \qquad u(0) \text{ given}. \tag{D.1}$$

In order to discretize this abstract initial value problem in time, we apply an implicit s-stage Runge-Kutta method with coefficients a_{ij}, b_i and c_i, $1 \leq i, j \leq s$, and initial value u_0 to (D.1). By using the matrix and vector notation introduced in Section C.1 and further the abbreviation

$$g(t_n + ch) = \big(g(t_n + c_1 h), \dots, g(t_n + c_s h)\big)^T,$$

the Runge-Kutta scheme takes the form

$$U_n' = AU_n + g(t_n + ch), \qquad U_n = \mathbb{1} u_n + h\mathcal{O}\!\!\!\!\mathcal{Q} U_n', \qquad u_{n+1} = u_n + hb^T U_n'. \tag{D.2}$$

In accordance with the requirements of Section 1.1.1, we henceforth assume that A is a sectorial operator on a Banach space $\mathcal{X}$ with domain $\mathcal{D}(A) \subset \mathcal{X}$. Let further $a \in \boldsymbol{R}$, $\pi/2 < \varphi < \pi$ and $M > 0$ be such that the resolvent estimate

$$\big\|(\lambda I - A)^{-1}\big\|_{\mathcal{X} \leftarrow \mathcal{X}} \leq \frac{M}{|\lambda - a|}, \qquad |\arg(\lambda - a)| < \varphi, \tag{D.3}$$

is fulfiled, see Definition A.2. If not stated otherwise we suppose that the map g in (D.1) is (α, α)-Hölder continuous with values in $\mathcal{X}$. Furthermore, we assume that the Runge-Kutta method (D.2) is strongly $A(\vartheta)$-stable with $\vartheta > \pi - \varphi$.

The contents of the present chapter are the following. In Section D.1 we represent the analytical solution of (D.1) by the modified variation-of-constants formula, and we further derive the discrete version of this relation for the backward Euler method and for a general Runge-Kutta method. These formulas are of use in Section D.2 where we study the regularity of the exact and the numerical solution on bounded and unbounded time intervals. There, it is shown that certain properties of the map g arising on the right-hand side of (D.1) such as the (α,α)-Hölder continuity and the exponentially weighted Hölder continuity go over to the analytical solution and to the Runge-Kutta solution of (D.1). These statements are needed in order to prove the existence and convergence results from Chapters 5 and 7. For the definition of the weighted Hölder spaces and the corresponding sequence spaces, we refer to the Tables 1.1 and 1.2. Auxiliary estimates for Section D.2 are collected in Section D.3.

The results for the analytical solution are partially found in Lunardi [25, Section 4]. Nevertheless, proving them in a more direct way is useful for our purposes, since then the results for the discrete case follow in a similar manner.

D.1. Modified variation-of-constants formulas

The solution of the nonhomogeneous initial value problem (D.1) is formally given by the *variation-of-constants formula*

$$u(t) = e^{tA}u(0) + \int_0^t e^{(t-\tau)A} g(\tau)\,\mathrm{d}\tau. \tag{D.4}$$

By substracting and adding $g(t)$ under the integral, we receive

$$u(t) = e^{tA}u(0) + \int_0^t e^{(t-\tau)A}\big(g(\tau)-g(t)\big)\,\mathrm{d}\tau + \int_0^t e^{(t-\tau)A} g(t)\,\mathrm{d}\tau.$$

Multiplying with A and using (A.5), yields

$$Au(t) = e^{tA}Au(0) + \int_0^t Ae^{(t-\tau)A}\big(g(\tau)-g(t)\big)\,\mathrm{d}\tau + \big(e^{tA}-I\big)g(t), \tag{D.5}$$

the *modified variation-of-constants formula*. This relation is well-defined in $\mathcal{X}$ under the assumption that the initial value lies in $\mathcal{D}(A)$ and that g is (α,α)-Hölder continuous with values in $\mathcal{X}$.

The backward Euler discretization of (D.1) with initial value u_0 is given by the scheme

$$\frac{u_{n+1}-u_n}{h} = Au_{n+1} + g(t_{n+1}).$$

Solving this recursion yields, analogously to (D.4), the relation

$$u_n = (I-hA)^{-n}u_0 + h\sum_{\nu=0}^{n-1}(I-hA)^{-(n-\nu)}g(t_{\nu+1}) \tag{D.6}$$

as the *discrete version of the variation-of-constants formula* for the backward Euler method. By adding $g(t_n)$, multiplying with A, and applying the identity

$$h\sum_{\nu=0}^{n-1} A(I-hA)^{-(n-\nu)} = (I-hA)^{-n} - I,$$

we receive the analogue to (D.5), namely

$$\begin{aligned} Au_n = (I-hA)^{-n}Au_0 + h\sum_{\nu=0}^{n-1} A(I-hA)^{-(n-\nu)}\big(g(t_{\nu+1}) - g(t_n)\big) \\ + \big((I-hA)^{-n} - I\big)g(t_n), \end{aligned} \tag{D.7}$$

the *discrete modified variation-of-constants formula* for the implicit Euler method.

In the sequel, we derive the discrete modified variation-of-constants formula for a general Runge-Kutta method of the form (D.2) with initial value u_0. Here, it is useful to introduce the abbreviation

$$\mathcal{J} = (\mathcal{I} - hA\mathcal{Q})^{-1}.$$

With this notation, a straightforward calculation in (D.2) yields the following relations

$$\begin{aligned} U_n &= \mathcal{J}\mathbb{1}\,u_n + h\mathcal{Q}\mathcal{J}g(t_n + ch), \\ u_{n+1} &= R(hA)u_n + hb^T\mathcal{J}g(t_n + ch), \end{aligned}$$

involving the stability function $R(z) = 1 + zb^T(\mathcal{I} - z\mathcal{Q})^{-1}\mathbb{1}$ of the method. Solving the above recursion for u_n yields the *discrete version of the variation-of-constants formula*

$$\begin{aligned} u_n &= R(hA)^n u_0 + h\sum_{\nu=0}^{n-1} b^T\mathcal{J}R(hA)^{n-\nu-1}g(t_\nu + ch), \\ U_{n-1} &= \mathcal{J}\mathbb{1}\,R(hA)^{n-1}u_0 + h\mathcal{Q}\mathcal{J}g(t_{n-1} + ch) \\ &\quad + h\sum_{\nu=0}^{n-2} \mathcal{J}\mathbb{1}\,b^T\mathcal{J}R(hA)^{n-\nu-2}g(t_\nu + ch). \end{aligned} \tag{D.8}$$

By means of the identity

$$\sum_{\nu=0}^{n-1} R(hA)^{n-\nu-1} = \big(R(hA) - I\big)^{-1}\big(R(hA)^n - I\big),$$

as generalization of (D.7), we thus receive the relation

$$\begin{aligned}
Au_n &= R(hA)^n Au_0 + h\sum_{\nu=0}^{n-1} b^T\mathcal{J}AR(hA)^{n-\nu-1}\big(g(t_\nu+ch)-g(t_{n-1}+ch)\big)\\
&\qquad + b^T\mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1}\big(R(hA)^n - I\big)g(t_{n-1}+ch),\\
AU_{n-1} &= \mathcal{J}\mathbb{1}R(hA)^{n-1}Au_0 + hA\mathcal{O}\mathcal{J}g(t_{n-1}+ch)\\
&\qquad + h\sum_{\nu=0}^{n-2}\mathcal{J}\mathbb{1}b^T\mathcal{J}AR(hA)^{n-\nu-2}\big(g(t_\nu+ch)-g(t_{n-2}+ch)\big)\\
&\qquad\quad + \mathcal{J}\mathbb{1}b^T\mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1}\big(R(hA)^{n-1}-I\big)g(t_{n-2}+ch),
\end{aligned} \tag{D.9}$$

which we refer to as the *discrete modified variation-of-constants formula* for a general Runge-Kutta method. Notice that the operator

$$b^T\mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1} = b^T\mathcal{J}hA\big(R(hA)-I\big)^{-1}$$

is bounded for stepsizes $0 < h \le \overline{h}$ small enough, see (C.11). Therefore, it follows by means of the stability bounds collected in Result C.1, that Formula (D.9) is well-defined whenever u_0 lies in $\mathcal{D}(A)$ and g is (α,α)-Hölder continuous in $\mathcal{X}$.

D.2. Regularity properties of exact and numerical solution

In the sequel, we derive bounds for the analytical solution of the nonhomogeneous problem (D.1) in terms of the initial value and the map g. Furthermore, we prove the corresponding results for a strongly stable Runge-Kutta method of the form (D.2). In Section D.2.1 we study the behaviour of the exact and the numerical solution on bounded time intervals, and in Sections D.2.2 and D.2.3 we consider unbounded time intervals under certain assumptions on the spectrum of the sectorial operator A.

Our main tools for the following analysis are estimates for the analytic semigroup and the stability function stated in Chapters A and C. For convenience, we briefly review the needed results in each section. We tacitly suppose that the basic assumptions on the evolution equation and the Runge-Kutta method stated at the beginning of this chapter are always satisfied.

D.2.1. The nonhomogeneous problem on bounded time intervals

In the present section, we show that the analytical and the Runge-Kutta solution of the nonhomogeneous initial value problem

$$u'(t) = Au(t) + g(t), \quad 0 < t \le T, \qquad u(0) \text{ given}, \tag{D.10}$$

are (α,α)-Hölder continuous on the interval $(0,T]$ with values in $D(A)$ whenever the initial value lies in $\mathcal{D}(A)$ and g belongs to $\mathcal{C}^\alpha_\alpha((0,T],\mathcal{X})$.

For some $h>0$ we define $N \in \boldsymbol{N}$ through $t_N \le T \le t_{N+1}$ and we henceforth set $\mathcal{N} = \{t_n = nh : n \in \boldsymbol{N},\, 0 < n \le N\}$ and $\mathcal{N}_0 = \{t_n = nh : n \in \boldsymbol{N}, 0 < n \le N-1\}$. For the backward Euler method, we define the sequence $\underline{g}$ through

$$\underline{g} = \big(g(t_n)\big)_{t_n \in \mathcal{N}} \in \mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{X}), \tag{D.11a}$$

whereas for a general Runge-Kutta method of the form (D.2), we denote by $\underline{g}$ the sequence

$$\underline{g} = \big(g(t_n + ch)\big)_{t_n \in \mathcal{N}_0} \in \mathcal{C}^\alpha_\alpha(\mathcal{N}_0,\mathcal{X}^s) \quad \text{where} \quad g(t_n+ch) = \big(g(t_n + c_i h)\big)_{i=1}^s. \tag{D.11b}$$

Moreover, we employ the sequence notation

$$\underline{u} = \big(u_n\big)_{n\in\mathcal{N}} \qquad \text{and} \qquad \underline{U} = \big(U_n\big)_{n\in\mathcal{N}_0}$$

for the solution and the associated internal stages of the Runge-Kutta method (D.2).

Now we are in the position to state the following result.

Lemma D.1. *Choose initial values $u(0)$ and u_0 in $\mathcal{D}(A)$, and suppose that the map g belongs to $\mathcal{C}^\alpha_\alpha((0,T],\mathcal{X})$ for some $0<\alpha<1$. Then, the following statements hold.*

(i) The analytical solution u of equation (D.10) *belongs to $\mathcal{C}^\alpha_\alpha((0,T],\mathcal{D}(A))$ and satisfies the estimate*

$$\|u\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} \le C\big(\|u(0)\|_{\mathcal{D}(A)} + \|g\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})}\big).$$

(ii) For $0<h\le\overline{h}$ sufficiently small, the backward Euler solution $\underline{u} \in \mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{D}(A))$ of (D.10) *fulfils*

$$\|\underline{u}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} \le C\big(\|u_0\|_{\mathcal{D}(A)} + \|\underline{g}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})}\big).$$

(iii) For the Runge-Kutta solution $\underline{u} \in \mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{D}(A))$ and the stages $\underline{U} \in \mathcal{C}^\alpha_\alpha(\mathcal{N}_0,\mathcal{D}(A)^s)$ the bound

$$\|\underline{u}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} + \|\underline{U}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A)^s)} \le C\big(\|u_0\|_{\mathcal{D}(A)} + \|\underline{g}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X}^s)}\big)$$

is valid for $0<h\le\overline{h}$ small enough.

The constants C are specified in Lemma D.2 and Lemma D.3 below. In statement (i), the constant C depends on α and on the bounds for the analytical semigroup for $0\le t\le T$. The constants C from statements (ii) and (iii) further depend on $\overline{h}$ and on the method, in particular on the bounds for the stability function.

Proof. This result follows at once from the Lemmas D.2 and D.3 below. A proof of the first statement is also found in LUNARDI [25, Theorem 4.3.5]. □

For proving Lemma D.1, it is convenient to decompose the exact solution of (D.10) into

$$u = x + y.$$

Here, the map x denotes the solution of the homogeneous initial value problem

$$x'(t) = Ax(t), \quad 0 < t \le T, \qquad x(0) = u(0), \tag{D.12a}$$

and y the analytical solution of the nonhomogeneous problem

$$y'(t) = Ay(t) + g(t), \quad 0 < t \le T, \qquad y(0) = 0. \tag{D.12b}$$

At first, we consider the exact and the numerical solution of the homogenous equation (D.12a). For this, we need the following auxiliary results.

On bounded time intervals, Result A.1 implies that there are constants $M_0 > 0$, $M_1 > 0$ and $M_2 > 0$ such that the following estimates are satisfied

$$\big\|e^{tA}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le M_0, \quad 0 \le t \le T, \tag{D.13a}$$
$$\big\|Ae^{tA}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le M_1 t^{-1}, \quad 0 < t \le T, \tag{D.13b}$$
$$\big\|A^2e^{tA}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le M_2 t^{-2}, \quad 0 < t \le T. \tag{D.13c}$$

Notice that the map $\tau \mapsto Ae^{\tau A}x$ lies in $L^1((s,t),\mathcal{X})$ for all $0 < \varepsilon \le s < t \le T$ and $x \in \mathcal{X}$. Therefore, we obtain from Result A.2 the identity

$$\int_s^t Ae^{\tau A}x \,\mathrm{d}\tau = \big(e^{tA} - e^{sA}\big)x. \tag{D.14}$$

Since for all $x \in \mathcal{X}$ also $\tau \mapsto A^2e^{\tau A}x$ belongs to $L^1((s,t),\mathcal{X})$, we further have

$$A\int_s^t Ae^{\tau A}x \,\mathrm{d}\tau = \int_s^t A^2e^{\tau A}x \,\mathrm{d}\tau. \tag{D.15}$$

The backward Euler method fulfils the stability bounds

$$\big\|(I-hA)^{-n}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \hat{M}_0, \quad 0 \le t_n \le T, \tag{D.16a}$$
$$\big\|A(I-hA)^{-n}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \hat{M}_1 t_n^{-1}, \quad 0 < t_n \le T, \tag{D.16b}$$
$$\big\|A^2(I-hA)^{-n}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \hat{M}_2 t_n^{-2}, \quad 0 < t_n \le T, \quad n \ge 2, \tag{D.16c}$$

and for a general Runge-Kutta method the corresponding estimates

$$\begin{aligned} \big\|R(hA)^n\big\|_{\mathcal{X}\leftarrow\mathcal{X}} &\le \hat{M}_0, \quad 0 \le t_n \le T, \\ \big\|A\big(R(hA)^n - R(\infty)^n\big)\big\|_{\mathcal{X}\leftarrow\mathcal{X}} &\le \hat{M}_1 t_n^{-1}, \quad 0 < t_n \le T, \\ \big\|A^2\big(R(hA)^n - R(\infty)^n\big)\big\|_{\mathcal{X}\leftarrow\mathcal{X}} &\le \hat{M}_2 t_n^{-2}, \quad 0 < t_n \le T, \quad n \ge 2, \end{aligned}$$

are valid whenever $0 < h \le \overline{h}$ is sufficiently small, see Result C.1. In particular, the linear operator $\mathcal{J} = (\mathcal{I} - hA\mathcal{Q})^{-1}$ is bounded on $\mathcal{X}^s$, and we further have

$$\big\|\mathcal{J}\big\|_{\mathcal{X}^s\leftarrow\mathcal{X}^s} + h\big\|A\mathcal{J}\big\|_{\mathcal{X}^s\leftarrow\mathcal{X}^s} \le C_{\mathcal{J}}$$

for some constant $C_{\mathcal{J}} > 0$. Remark that in the above relations the term $R(\infty)^n$ can be omitted by means of an additional factor of $\mathcal{J}$. This is shown as follows. From the strong stability of the method, i.e. $\gamma = |R(\infty)| < 1$, it namely follows

$$h^{-1}|R(\infty)|^n = t_n^{-1} \cdot n e^{-n|\log(\gamma)|} \le c_1 t_n^{-1}, \qquad n \ge 1,$$

for some constant $c_1 > 0$. Due to the fact that the following relation is valid for $t_n \le T$

$$\begin{aligned} \big\|b^T\mathcal{J}AR(hA)^{n-1}\mathbb{1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} &= \big\|b^T\mathcal{J}A\big(R(hA)^{n-1} - R(\infty)^{n-1}\big)\mathbb{1} + b^T\mathcal{J}AR(\infty)^{n-1}\mathbb{1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \\ &\le \Big(\big\|b^T\mathcal{J}\mathbb{1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\hat{M}_1 + c_1\big\|b^T\big\|_{\boldsymbol{R}\leftarrow\boldsymbol{R}^n} C_{\mathcal{J}}\Big) t_{n-1}^{-1} \le C t_{n-1}^{-1} \le 2C t_n^{-1}, \qquad n \ge 2, \end{aligned}$$

after changing the constant $\hat{M}_1 > 0$, we therefore get the estimates

$$\big\|R(hA)^n\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \hat{M}_0, \quad 0 \le t_n \le T, \tag{D.17a}$$

$$\big\|b^T\mathcal{J}AR(hA)^{n-1}\mathbb{1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \hat{M}_1 t_n^{-1}, \quad 0 < t_n \le T, \tag{D.17b}$$

for stepsizes $0 < h \le \overline{h}$ small enough. Similar considerations further show that

$$\big\|b^T\mathcal{J}b^T\mathcal{J}A^2R(hA)^{n-2}\mathbb{1}\big\|_{\mathcal{X}^s\leftarrow\mathcal{X}^s} \le \hat{M}_2 t_n^{-2}, \quad 0 < t_n \le T, \quad n \ge 2, \tag{D.17c}$$

holds as well for sufficiently small stepsizes $0 < h \le \overline{h}$.

Lemma D.2. *Choose initial values $u(0)$ and u_0 in $\mathcal{D}(A)$ and fix $0 < \alpha < 1$.*

(i) Then, the analytical solution $x = e^{tA}u(0)$ of (D.12a) *belongs to $\mathcal{C}^\alpha_\alpha((0,T],\mathcal{D}(A))$, and further the following estimate holds*

$$\|x\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} \le \big(M_0 + \alpha^{-1}M_1\big)\|u(0)\|_{\mathcal{D}(A)}.$$

(ii) For sufficiently small stepsizes $0 < h \le \overline{h}$, the implicit Euler solution $\underline{x} = (x_n)_{n\in\mathcal{N}}$ with components $x_n = (I - hA)^{-n}u_0$ lies in $\mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{D}(A))$ and satisfies

$$\|\underline{x}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} \le \big(\hat{M}_0 + \alpha^{-1}\hat{M}_1\big)\|u_0\|_{\mathcal{D}(A)}.$$

(iii) The Runge-Kutta solution $\underline{x} = (x_n)_{n\in\mathcal{N}}$ with components $x_n = R(hA)^n u_0$ belongs to $\mathcal{C}^\alpha_\alpha(\mathcal{N}, \mathcal{D}(A))$, and further the estimate

$$\|\underline{x}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} \le \big(\hat{M}_0 + \alpha^{-1}\hat{M}_1\big)\|u_0\|_{\mathcal{D}(A)}$$

is valid for $0 < h \le \overline{h}$ small enough. The stages $\underline{X} \in \mathcal{C}^\alpha_\alpha(\mathcal{N}_0, \mathcal{D}(A)^s)$ are bounded by

$$\|\underline{X}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A)^s)} \le C\|u_0\|_{\mathcal{D}(A)}.$$

The constant C depends on $\alpha^{-1}, \hat{M}_0, \hat{M}_1$ and on the bound for $\mathcal{J} = (\mathcal{I} - hA\mathcal{O\!\!\iota})^{-1}$.

Proof. In the first place, we derive the desired result for the exact solution of (D.12a). For proving the statements for the numerical solution, we henceforth assume that the stepsize $0 < h \le \overline{h}$ is chosen sufficiently small such that the stability bounds (D.16) and (D.17) and the bounds from Estimate D.2 remain valid.

(i)(a) By using (D.13a) we have

$$\|x(t)\|_{\mathcal{D}(A)} = \big\|e^{tA}Au(0)\big\|_{\mathcal{X}} \le \big\|e^{tA}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|u(0)\|_{\mathcal{D}(A)} \le M_0\|u(0)\|_{\mathcal{D}(A)}$$

for every $0 \le t \le T$ and thus

$$\|x\|_{\mathcal{B}(\mathcal{D}(A))} \le M_0\|u(0)\|_{\mathcal{D}(A)}.$$

(b) For $0 < \varepsilon \le s < t \le T$ the relation

$$A\big(x(t) - x(s)\big) = \big(e^{tA} - e^{sA}\big)Au(0) = \int_s^t Ae^{\tau A}Au(0)\,\mathrm{d}\tau$$

follows from (D.14). Applying (D.13b) and (D.35) from Estimate D.1, further yields

$$\begin{aligned}\|x(t) - x(s)\|_{\mathcal{D}(A)} &\le \int_s^t \big\|Ae^{\tau A}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|u(0)\|_{\mathcal{D}(A)}\,\mathrm{d}\tau \le M_1\|u(0)\|_{\mathcal{D}(A)}\int_s^t \tau^{-1}\,\mathrm{d}\tau\\ &\le \alpha^{-1}M_1\|u(0)\|_{\mathcal{D}(A)}\cdot\varepsilon^{-\alpha}(t-s)^\alpha.\end{aligned}$$

By using the definition of the (α,α)-Hölder norm, see Table 1.1, we get

$$[x]_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} \le \alpha^{-1}M_1\|u(0)\|_{\mathcal{D}(A)},$$

and consequently the bound

$$\|x\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} = \|x\|_{\mathcal{B}(\mathcal{D}(A))} + [x]_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} \le \big(M_0 + \alpha^{-1}M_1\big)\|u(0)\|_{\mathcal{D}(A)}$$

follows.

(ii) (a) Similarly to before, by (D.16a) we have

$$\|x_n\|_{\mathcal{D}(A)} = \big\|(I-hA)^{-n}Au_0\big\|_{\mathcal{X}} \le \big\|(I-hA)^{-n}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|u_0\|_{\mathcal{D}(A)} \le \hat{M}_0\|u_0\|_{\mathcal{D}(A)}$$

for all $0 \le t_n \le T$.

(b) For the implicit Euler method the identity

$$\begin{aligned}A(x_n - x_m) &= \big((I-hA)^{-n} - (I-hA)^{-m}\big)Au_0 = \big((I-hA)^{-1} - I\big)\sum_{\nu=m}^{n-1}(I-hA)^{-\nu}Au_0 \\ &= h\sum_{\nu=m}^{n-1} A(I-hA)^{-(\nu+1)}Au_0\end{aligned}$$

is valid for all $0 < t_{n_1} \le t_m < t_n \le T$. Due to (D.16b) and (D.40) from Estimate D.2, this implies

$$\begin{aligned}\|x_n - x_m\|_{\mathcal{D}(A)} &\le h\sum_{\nu=m}^{n-1}\big\|A(I-hA)^{-(\nu+1)}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|u_0\|_{\mathcal{D}(A)} \le \hat{M}_1\|u_0\|_{\mathcal{D}(A)}\cdot h\sum_{\nu=m}^{n-1} t_{\nu+1}^{-1} \\ &\le \alpha^{-1}\hat{M}_1\|u_0\|_{\mathcal{D}(A)}\cdot t_{n_1}^{-\alpha}(t_n - t_m)^{\alpha}.\end{aligned}$$

Applying the definition of $\|\cdot\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{D}(A))}$, see Table 1.2, thus yields

$$\|\underline{x}\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{D}(A))} = \|\underline{x}\|_{\mathcal{B}(\mathcal{D}(A))} + [\underline{x}]_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{D}(A))} \le \big(\hat{M}_0 + \alpha^{-1}\hat{M}_1\big)\|u_0\|_{\mathcal{D}(A)}.$$

(iii)(a) For a general Runge-Kutta solution the bound

$$\|x_n\|_{\mathcal{D}(A)} = \big\|R(hA)^n Au_0\big\|_{\mathcal{X}} \le \big\|R(hA)^n\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|u_0\|_{\mathcal{D}(A)} \le \hat{M}_0\|u_0\|_{\mathcal{D}(A)}$$

follows from (D.17a) for all $0 \le t_n \le T$.

(b) Applying (D.17b) and (D.40) to

$$\begin{aligned}A(x_n - x_m) &= \big(R(hA)^n - R(hA)^m\big)Au_0 = \big(R(hA) - I\big)\sum_{\nu=m}^{n-1} R(hA)^{\nu}Au_0 \\ &= h\sum_{\nu=m}^{n-1} b^T\mathcal{J}AR(hA)^{\nu}\mathbb{1}Au_0\end{aligned}$$

yields the following estimate for $0 < t_{n_1} \le t_m < t_n \le T$

$$\begin{aligned}\|x_n - x_m\|_{\mathcal{D}(A)} &\le h\sum_{\nu=m}^{n-1}\big\|b^T\mathcal{J}AR(hA)^{\nu}\mathbb{1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|u_0\|_{\mathcal{D}(A)} \le \hat{M}_1\|u_0\|_{\mathcal{D}(A)}\cdot h\sum_{\nu=m}^{n-1} t_{\nu+1}^{-1} \\ &\le \alpha^{-1}\hat{M}_1\|u_0\|_{\mathcal{D}(A)}\cdot t_{n_1}^{-\alpha}(t_n - t_m)^{\alpha}.\end{aligned}$$

Now the estimate for the internal stages $\underline{X}$ follows at once from the identity

$$\underline{X} = \mathcal{J}\mathbb{1}\,\underline{x}.$$

This concludes the proof. $\square$

In Lemma D.3 below, we give the corresponding result to Lemma D.2 for the initial value problem (D.12b). For representing the exact and the numerical solution, we apply the modified variation-of-constants formulas, derived in Section D.1. For the analytical solution y of (D.12b) this yields

$$Ay(t) = \int_0^t Ae^{(t-\tau)A}\big(g(\tau) - g(t)\big)\,\mathrm{d}\tau + \big(e^{tA} - I\big)g(t).$$

Due to (D.7), the components of the backward Euler solution $\underline{y} = (y_n)_{n\in\mathcal{N}}$ are given by

$$Ay_n = h\sum_{\nu=0}^{n-1} A(I-hA)^{-(n-\nu)}\big(g(t_{\nu+1}) - g(t_n)\big) + \big((I-hA)^{-n} - I\big)g(t_n).$$

Besides, we have the following representation

$$\begin{aligned} Ay_n = h\sum_{\nu=0}^{n-1} & b^T\mathcal{J}AR(hA)^{n-\nu-1}\big(g(t_\nu + ch) - g(t_{n-1}+ch)\big) \\ & + b^T\mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1}\big(R(hA)^n - I\big)g(t_{n-1}+ch) \end{aligned}$$

for the general Runge-Kutta solution of (D.12b) with initial value zero, see (D.9).

Now, we are ready to state the following result for y and $\underline{y}$. Recall the sequence notation $\underline{g}$ introduced in (D.11).

Lemma D.3. *Assume that the map g belongs to $\mathcal{C}^\alpha_\alpha((0,T],\mathcal{X})$ for some $0<\alpha<1$. Then the following statements holds.*

(i) The analytical solution y of equation (D.12b) belongs to $\mathcal{C}^\alpha_\alpha((0,T],\mathcal{D}(A))$ and fulfils

$$\|y\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} \le C\|g\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})}$$

with constant $C = 2 + 4M_0 + \big(2\alpha^{-1} + B(\alpha,1-\alpha)\big)M_1 + (\alpha(1-\alpha))^{-1}M_2$.

(ii) For $0 < h \le \overline{h}$ sufficiently small, the backward Euler solution $\underline{y} \in \mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{D}(A))$ satisfies the estimate

$$\|\underline{y}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} \le C\|\underline{g}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})}$$

with constant $C = 2 + 4\hat{M}_0 + \big(2\alpha^{-1} + B(\alpha,1-\alpha)\big)\hat{M}_1 + (\alpha(1-\alpha))^{-1}\hat{M}_2$.

(iii) For $0 < h \le \overline{h}$ sufficiently small, the Runge-Kutta solution $\underline{y} \in \mathcal{C}_\alpha^\alpha(\mathcal{N}, \mathcal{D}(A))$ and the internal stages $\underline{Y} \in \mathcal{C}_\alpha^\alpha(\mathcal{N}_0, \mathcal{D}(A)^s)$ are bounded by

$$\|\underline{y}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}(A))} + \|\underline{Y}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}(A))} \le C\|\underline{g}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{X}^s)}.$$

In particular, the constant C depends on α^{-1}, the constants $\hat{M}_0, \hat{M}_1, \hat{M}_2$ and further on the bound for $\mathcal{J} = (\mathcal{I} - hA\mathcal{O}\!\!\!\mathcal{L})^{-1}$ and the coefficients of the method.

Proof. Notice first that for g the following relations hold

$$\begin{aligned}
\|g(t) - g(s)\| &\le [g]_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}\varepsilon^{-\alpha}(t-s)^\alpha, \quad 0 < \varepsilon \le s \le t \le T,\\
\|g(t) - g(\tau)\| &\le [g]_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}(t-\tau)^\alpha\tau^{-\alpha}, \quad 0 < \tau < t \le T.
\end{aligned}$$

For any integer $1 \le k \le n-1$, we therefore have

$$\begin{aligned}
\|g(t_k + ch) - g(t_{n-1} + ch)\|_{\mathcal{X}^s} &= \max_{1\le i\le s} \|g(t_{n-1} + c_ih) - g(t_k + c_ih)\|_{\mathcal{X}}\\
&\le [g]_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}(t_{n-1} - t_k)^\alpha \max_{1\le i\le s}(t_k + c_ih)^{-\alpha} \le 2^\alpha[g]_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}(t_n - t_{k+1})^\alpha t_{k+1}^{-\alpha}.
\end{aligned}$$

We denote by $c_0 > 0$ a constant such that the bound

$$\big\|b^T\mathcal{J}\big(b^T\mathcal{J}1\!\!1\big)^{-1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}^s} \le c_0, \qquad 0 < h \le \overline{h},$$

is fulfiled for $\overline{h} > 0$ small enough, see Result C.1.

(i) By the modified variation-of-constants formula (D.5), we have the following representation of the analytical solution

$$Ay(t) = \int_0^t Ae^{(t-\tau)A}\big(g(\tau) - g(t)\big)\,\mathrm{d}\tau + \big(e^{tA} - I\big)g(t), \qquad 0 \le t \le T.$$

(a) With the help of the Estimates (D.13a), (D.13b) and the Beta-Function, defined by Formula (D.36) of Estimate D.1, we thus receive

$$\begin{aligned}
\|y(t)\|_{\mathcal{D}(A)} &\le \int_0^t \big\|Ae^{(t-\tau)A}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|g(\tau) - g(t)\|_{\mathcal{X}}\,\mathrm{d}\tau + \big\|e^{tA} - I\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|g(t)\|_{\mathcal{X}}\\
&\le M_1[g]_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}\int_0^t (t-\tau)^{-1+\alpha}\tau^{-\alpha}\,\mathrm{d}\tau + (1 + M_0)\|g\|_{\mathcal{B}(\mathcal{X})}\\
&\le B(\alpha, 1-\alpha)M_1[g]_{\mathcal{C}_\alpha^\alpha(\mathcal{X})} + (1 + M_0)\|g\|_{\mathcal{B}(\mathcal{X})}
\end{aligned}$$

for $0 \le t \le T$.

(b) For all elements $0 < \varepsilon \le s < t \le T$ the following relation

$$
\begin{aligned}
A(y(t) - y(s)) &= \int_0^s A\big(e^{(t-\tau)A} - e^{(s-\tau)A}\big)\big(g(\tau) - g(s)\big)\,\mathrm{d}\tau \\
&\quad + \int_0^s Ae^{(t-\tau)A}\big(g(s) - g(t)\big)\,\mathrm{d}\tau + \int_s^t Ae^{(t-\tau)A}\big(g(\tau) - g(t)\big)\,\mathrm{d}\tau \\
&\qquad + \big(e^{tA} - e^{sA}\big)g(t) + \big(e^{sA} - I\big)\big(g(t) - g(s)\big) \\
&= \int_0^s \int_{s-\tau}^{t-\tau} A^2 e^{\sigma A}\big(g(\tau) - g(s)\big)\,\mathrm{d}\sigma\,\mathrm{d}\tau + \big(e^{tA} - e^{(t-s)A}\big)\big(g(s) - g(t)\big) \\
&\quad + \int_s^t Ae^{(t-\tau)A}\big(g(\tau) - g(t)\big)\,\mathrm{d}\tau + \int_s^t Ae^{\tau A} g(t)\,\mathrm{d}\tau + \big(e^{sA} - I\big)\big(g(t) - g(s)\big),
\end{aligned}
$$

together with (D.13a) to (D.13c) and the bounds (D.35), (D.37) and (D.38) from Estimate D.1, implies

$$
\begin{aligned}
\|y(t) - y(s)\|_{\mathcal{D}(A)} &\le \int_0^s \int_{s-\tau}^{t-\tau} \big\|A^2 e^{\sigma A}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \|g(\tau) - g(s)\|_{\mathcal{X}}\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\quad + \int_s^t \big\|Ae^{(t-\tau)A}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \|g(\tau) - g(t)\|_{\mathcal{X}}\,\mathrm{d}\tau + \int_s^t \big\|Ae^{\tau A}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \|g(t)\|_{\mathcal{X}}\,\mathrm{d}\tau \\
&\qquad + \big\|e^{(t-s)A} - e^{tA} + e^{sA} - I\|_{\mathcal{X}\leftarrow\mathcal{X}} \|g(t) - g(s)\|_{\mathcal{X}} \\
&\le M_2[g]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \int_0^s \int_{s-\tau}^{t-\tau} \sigma^{-2}(s-\tau)^\alpha \tau^{-\alpha}\,\mathrm{d}\sigma\,\mathrm{d}\tau + M_1[g]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \int_s^t (t-\tau)^{-1+\alpha}\tau^{-\alpha}\,\mathrm{d}\tau \\
&\quad + M_1\|g\|_{\mathcal{B}(\mathcal{X})} \int_s^t \tau^{-1}\,\mathrm{d}\tau + (1 + 3M_0)[g]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \cdot \varepsilon^{-\alpha}(t-s)^\alpha \\
&\le \Big(\big(1 + 3M_0 + \alpha^{-1}M_1 + (\alpha(1-\alpha))^{-1}M_2\big)[g]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} + \alpha^{-1}M_1\|g\|_{\mathcal{B}(\mathcal{X})}\Big)\varepsilon^{-\alpha}(t-s)^\alpha.
\end{aligned}
$$

(ii) Due to (D.7), the backward Euler solution writes as

$$
Ay_n = h\sum_{\nu=0}^{n-1} A(I - hA)^{-(n-\nu)}\big(g(t_{\nu+1}) - g(t_n)\big) + \big((I - hA)^{-n} - I\big)g(t_n).
$$

(a) By using (D.16a), (D.16b) and Estimate D.2, we have

$$
\begin{aligned}
\|y_n\|_{\mathcal{D}(A)} &\le h\sum_{\nu=0}^{n-1} \big\|A(I - hA)^{-(n-\nu)}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \|g(t_{\nu+1}) - g(t_n)\|_{\mathcal{X}} \\
&\quad + \big\|(I - hA)^{-n} - I\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \|g(t_n)\|_{\mathcal{X}} \\
&\le \hat{M}_1[\underline{g}]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \cdot h\sum_{\nu=0}^{n-1} (t_n - t_\nu)^{-1}(t_n - t_{\nu+1})^\alpha t_{\nu+1}^{-\alpha} + (1 + \hat{M}_0)\|\underline{g}\|_{\mathcal{B}(\mathcal{X})} \\
&\le B(\alpha, 1-\alpha)\hat{M}_1[\underline{g}]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} + (1 + \hat{M}_0)\|\underline{g}\|_{\mathcal{B}(\mathcal{X})}.
\end{aligned}
$$

(b) Similarly to the analytical case, the relation

$$
\begin{aligned}
A(y_n - y_m) &= h \sum_{\nu=0}^{m-1} A\big((I-hA)^{-(n-\nu)} - (I-hA)^{-(m-\nu)}\big)\big(g(t_{\nu+1}) - g(t_m)\big) \\
&\quad + h \sum_{\nu=0}^{m-1} A(I-hA)^{-(n-\nu)}\big(g(t_m) - g(t_n)\big) + h \sum_{\nu=m}^{n-1} A(I-hA)^{-(n-\nu)}\big(g(t_{\nu+1}) - g(t_n)\big) \\
&\quad + \big((I-hA)^{-n} - (I-hA)^{-m}\big) g(t_n) + \big((I-hA)^{-m} - I\big)\big(g(t_n) - g(t_m)\big)
\end{aligned}
$$

is valid. Furthermore, it follows

$$
\begin{aligned}
A(y_n - y_m) &= h^2 \sum_{\nu=0}^{m-1} \sum_{\mu=m-\nu}^{n-\nu-1} A^2 (I-hA)^{-(\mu+1)}\big(g(t_{\nu+1}) - g(t_m)\big) \\
&\quad + h \sum_{\nu=m}^{n-1} A(I-hA)^{-(n-\nu)}\big(g(t_{\nu+1}) - g(t_n)\big) + h \sum_{\nu=m}^{n-1} A(I-hA)^{-(\nu+1)} g(t_n) \\
&\quad + \big((I-hA)^{-(n-m)} - (I-hA)^{-n} + (I-hA)^{-m} - I\big)\big(g(t_n) - g(t_m)\big).
\end{aligned}
$$

This identity implies

$$
\begin{aligned}
\|y_n - y_m\|_{\mathcal{D}(A)} &\le h^2 \sum_{\nu=0}^{m-1} \sum_{\mu=m-\nu}^{n-\nu-1} \big\|A^2 (I-hA)^{-(\mu+1)}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \|g(t_{\nu+1}) - g(t_m)\|_{\mathcal{X}} \\
&\quad + \big\|(I-hA)^{-(n-m)} - (I-hA)^{-n} + (I-hA)^{-m} - I\big\|_{\mathcal{X}} \|g(t_n) - g(t_m)\|_{\mathcal{X}} \\
&\quad + h \sum_{\nu=m}^{n-1} \big\|A(I-hA)^{-(\nu+1)}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \|g(t_n)\|_{\mathcal{X}} \\
&\quad + h \sum_{\nu=m}^{n-1} \big\|A(I-hA)^{-(n-\nu)}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \|g(t_{\nu+1}) - g(t_n)\|_{\mathcal{X}}.
\end{aligned}
$$

We apply (D.16a) to (D.16c), and the bounds (D.40), (D.43), and (D.44) from Estimate D.2. Therefore, we receive

$$
\begin{aligned}
\|y_n - y_m\|_{\mathcal{D}(A)} &\le \hat{M}_2 [\underline{g}]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \cdot h^2 \sum_{\nu=0}^{m-1} \sum_{\mu=m-\nu}^{n-\nu-1} t_{\mu+1}^{-2} (t_m - t_{\nu+1})^\alpha t_{\nu+1}^{-\alpha} \\
&\quad + (1 + 3\hat{M}_0)[\underline{g}]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \cdot t_{n_1}^{-\alpha} (t_n - t_m)^\alpha + \hat{M}_1 \|\underline{g}\|_{\mathcal{B}(\mathcal{X})} \cdot h \sum_{\nu=m}^{n-1} t_{\nu+1}^{-1} \\
&\quad + \hat{M}_1 [\underline{g}]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \cdot h \sum_{\nu=m}^{n-1} (t_n - t_\nu)^{-1} (t_n - t_{\nu+1})^\alpha t_{\nu+1}^{-\alpha},
\end{aligned}
$$

and, moreover, we obtain the estimate

$$\|y_n - y_m\|_{\mathcal{D}(A)} \le \Big(\big(1 + 3\hat{M}_0 + \alpha^{-1}\hat{M}_1 + (\alpha(1-\alpha))^{-1}\hat{M}_2\big)[\underline{g}]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} + \alpha^{-1}\hat{M}_1\|\underline{g}\|_{\mathcal{B}(\mathcal{X})}\Big) \cdot t_{n_1}^{-\alpha}(t_n - t_m)^\alpha.$$

(iii) For a general Runge-Kutta method of the form (C.6), we have the representation

$$\begin{aligned} Ay_n = h\sum_{\nu=0}^{n-1} b^T \mathcal{J} AR(hA)^{n-\nu-1}\big(g(t_\nu + ch) - g(t_{n-1}+ch)\big) \\ + b^T\mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1}\big(R(hA)^n - I\big)g(t_{n-1}+ch). \end{aligned}$$

(a) By splitting the sum and considering the term for $\nu = 0$ seperately, the bounds (D.17a), (D.17b) and (D.42) from Estimate D.2 imply

$$\begin{aligned} \|y_n\|_{\mathcal{D}(A)} &\le h\sum_{\nu=0}^{n-1}\big\|b^T\mathcal{J}AR(hA)^{n-\nu-1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}^s}\|g(t_\nu+ch) - g(t_{n-1}+ch)\|_{\mathcal{X}^s} \\ &\quad + \big\|b^T\mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}^s}\big\|R(hA)^n - I\|_{\mathcal{X}\leftarrow\mathcal{X}}\|g(t_{n-1}+ch\|_{\mathcal{X}^s} \\ &\le 2\hat{M}_1\|\underline{g}\|_{\mathcal{B}(\mathcal{X})} + 2^\alpha\hat{M}_1[\underline{g}]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \cdot h\sum_{\nu=1}^{n-1}(t_n - t_\nu)^{-1}(t_n - t_{\nu+1})^\alpha t_{\nu+1}^{-\alpha} \\ &\quad + c_0(1+\hat{M}_0)\|\underline{g}\|_{\mathcal{B}(\mathcal{X})} \\ &\le 2^\alpha\hat{M}_1 B(\alpha, 1-\alpha)[\underline{g}]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} + \big(c_0(1+\hat{M}_0) + 2\hat{M}_1\big)\|\underline{g}\|_{\mathcal{B}(\mathcal{X})}. \end{aligned}$$

(b) For a general Runge-Kutta method the identity

$$\begin{aligned} A(y_n - y_m) &= h\sum_{\nu=0}^{m-1} b^T\mathcal{J}A\big(R(hA)^{n-\nu-1} - R(hA)^{m-\nu-1}\big)\big(g(t_\nu+ch) - g(t_{m-1}+ch)\big) \\ &\quad + h\sum_{\nu=0}^{m-1} b^T\mathcal{J}AR(hA)^{n-\nu-1}\big(g(t_{m-1}+ch) - g(t_{n-1}+ch)\big) \\ &\quad + h\sum_{\nu=m}^{n-1} Ab^T\mathcal{J}R(hA)^{n-\nu-1}\big(g(t_\nu+ch) - g(t_{n-1}+ch)\big) \\ &\quad + b^T\mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1}\Big(\big(R(hA)^n - R(hA)^m\big)g(t_{n-1}+ch) \\ &\qquad + \big(R(hA)^m - I\big)\big(g(t_{n-1}+ch) - g(t_{m-1}+ch)\big)\Big) \end{aligned}$$

holds. Without any loss of generality, we may assume $m \ge 0$. Therefore, we receive further for $0 < t_{n_1} \le t_m < t_n \le T$

$$\begin{aligned}A(y_n - y_m) = h^2 \sum_{\nu=0}^{m-1} \sum_{\mu=m-\nu}^{n-\nu-1} & b^T \mathcal{J} b^T \mathcal{J} A^2 R(hA)^{\mu-1} \mathbb{1}\big(g(t_\nu + ch) - g(t_{m-1} + ch)\big) \\ & + h \sum_{\nu=m}^{n-1} b^T \mathcal{J} A R(hA)^{n-\nu-1}\big(g(t_\nu + ch) - g(t_{n-1} + ch)\big) \\ & + h \sum_{\nu=m}^{n-1} b^T \mathcal{J}\big(b^T \mathcal{J} \mathbb{1}\big)^{-1} b^T \mathcal{J} A R(hA)^{\nu} \mathbb{1}\, g(t_{n-1} + ch) \\ & + b^T \mathcal{J}\big(b^T \mathcal{J} \mathbb{1}\big)^{-1}\big(R(hA)^{n-m} - R(hA)^n + R(hA)^m - I\big) \\ & \cdot \big(g(t_{n-1} + ch) - g(t_{m-1} + ch)\big).\end{aligned}$$

An application of (D.17) and Estimate D.2 yields

$$\begin{aligned}\|y_n - y_m\|_{\mathcal{D}(A)} \le h^2 \sum_{\nu=0}^{m-1} \sum_{\mu=m-\nu}^{n-\nu-1} & \big\|b^T \mathcal{J} b^T \mathcal{J} A^2 R(hA)^{\mu-1} \mathbb{1}\big\|_{\mathcal{X}^s \leftarrow \mathcal{X}^s} \\ & \cdot \|g(t_\nu + ch) - g(t_{m-1} + ch)\|_{\mathcal{X}^s} \\ & + h \sum_{\nu=m}^{n-1} \big\|b^T \mathcal{J} A R(hA)^{n-\nu-1}\|_{\mathcal{X} \leftarrow \mathcal{X}^s} \|g(t_\nu + ch) - g(t_{n-1} + ch)\|_{\mathcal{X}^s} \\ & + h \sum_{\nu=m}^{n-1} \big\|b^T \mathcal{J}\big(b^T \mathcal{J} \mathbb{1}\big)^{-1}\big\|_{\mathcal{X} \leftarrow \mathcal{X}^s} \big\|b^T \mathcal{J} A R(hA)^{\nu} \mathbb{1}\big\|_{\mathcal{X} \leftarrow \mathcal{X}} \|g(t_{n-1} + ch)\|_{\mathcal{X}^s} \\ & + \big\|b^T \mathcal{J}\big(b^T \mathcal{J} \mathbb{1}\big)^{-1}\big\|_{\mathcal{X} \leftarrow \mathcal{X}^s} \big\|R(hA)^{n-m} - R(hA)^n + R(hA)^m - I\big\|_{\mathcal{X} \leftarrow \mathcal{X}} \\ & \cdot \|g(t_{n-1} + ch) - g(t_{m-1} + ch)\|_{\mathcal{X}^s}\end{aligned}$$

and finally the desired result

$$\begin{aligned}\|y_n - y_m\|_{\mathcal{D}(A)} & \le 2\hat{M}_2 \|\underline{g}\|_{\mathcal{B}(\mathcal{X})} \cdot h^2 \sum_{\mu=m}^{n-1} t_{\mu+1}^{-2} \\ & + 2^\alpha \hat{M}_2 [\underline{g}]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \cdot h^2 \sum_{\nu=1}^{m-1} \sum_{\mu=m-\nu}^{n-\nu-1} t_{\mu+1}^{-2} (t_m - t_{\nu+1})^\alpha t_{\nu+1}^{-\alpha} \\ & + c_0 2^\alpha \hat{M}_1 [\underline{g}]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \cdot h \sum_{\nu=m}^{n-1} (t_n - t_\nu)^{-1} (t_n - t_{\nu+1})^\alpha t_{\nu+1}^{-\alpha} \\ & + c_0 \hat{M}_1 \|\underline{g}\|_{\mathcal{B}(\mathcal{X})} \cdot h \sum_{\nu=m}^{n-1} t_{\nu+1}^{-1} + c_0 (1 + 3\hat{M}_0) [\underline{g}]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} t_{n_1}^{-\alpha} (t_n - t_m)^\alpha \\ & \le C \|\underline{g}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \cdot t_{n_1}^{-\alpha} (t_n - t_m)^\alpha\end{aligned}$$

with a constant C depending $c_0, \hat{M}_0, \hat{M}_1, \hat{M}_2$ and α^{-1}. In order to prove the estimate for the internal stages we use the identity

$$A\underline{Y} = \mathcal{J}\mathbb{1}\, A\underline{y} + hA\mathfrak{O}\mathcal{J}\underline{g}.$$

Together with the previous bound for the Runge-Kutta solution this yields the desired result and finally concludes the proof of the lemma. □

D.2.2. The nonhomogeneous problem on unbounded time intervals

In the following, we study the behaviour of the nonhomogeneous problem

$$u'(t) = Au(t) + g(t), \quad t > 0, \qquad u(0) \text{ given}, \tag{D.18}$$

on unbounded time intervals, under the assumption that the spectrum of A is contained in the negative complex plane. In this situation, the exact and the numerical solution exists for all positive times, and the analogous result to Lemma D.1 is provided by Lemma D.4 below.

For some $h > 0$ we henceforth denote by $\mathcal{N}$ the set $\mathcal{N} = \{t_n = nh : n \in \boldsymbol{N}\}$, and we further on employ the sequence notation introduced in Section D.2.1, see Relation (D.11) and below.

Lemma D.4. *Assume that the constant a arising in the resolvent estimate* (D.3) *for the sectorial operator A fulfils $a < 0$. Suppose further that the initial values $u(0)$ and u_0 lie in $\mathcal{D}(A)$ and that $g \in \mathcal{C}^\alpha_\alpha((0,\infty), \mathcal{X})$ holds for some $\alpha \in (0,1)$. Then, the following statements are valid.*

(i) The analytical solution u of equation (D.18) *belongs to $\mathcal{C}^\alpha_\alpha((0,\infty), \mathcal{D}(A))$ and satisfies the estimate*

$$\|u\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} \le C\big(\|u(0)\|_{\mathcal{D}(A)} + \|g\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})}\big).$$

The constant C depends on α and on the bounds for the analytical semigroup on the positive half line $(0,\infty)$.

(ii) For $0 < h \le \overline{h}$ sufficiently small, the backward Euler solution $\underline{u}$ of (D.18) *lies in $\mathcal{C}^\alpha_\alpha(\mathcal{N}, \mathcal{D}(A))$ and fulfils*

$$\|\underline{u}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} \le C\big(\|u_0\|_{\mathcal{D}(A)} + \|\underline{g}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})}\big)$$

with constant C depending on α and on the bounds for the stability function.

(iii) The Runge-Kutta solution $\underline{u}$ of equation (D.18) *belongs to $\mathcal{C}^\alpha_\alpha(\mathcal{N}, \mathcal{D}(A))$ and the internal stages fulfil $\underline{U} \in \mathcal{C}^\alpha_\alpha(\mathcal{N}, \mathcal{D}(A)^s)$. Moreover, the bound*

$$\|\underline{u}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} + \|\underline{U}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A)^s)} \le C\big(\|u_0\|_{\mathcal{D}(A)} + \|\underline{g}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X}^s)}\big)$$

is valid for $0 < h \le \overline{h}$ small enough. In particular, the constant C depends on α and on the bounds for the stability function of the method.

Proof. (i) In order to prove the first statement for the analytical solution of (D.18), we follow the proofs of the Lemmas D.2 and D.3. In the present situation, the analytic semigroup decays exponentially, see Result A.1. Thus, for any $0 < \omega_0 < -a$ there are constants $M_0 > 0$, $M_1 > 0$ and $M_2 > 0$ such that the following relations hold

$$\left\|e^{tA}\right\|_{\mathcal{X}\leftarrow\mathcal{X}} \leq M_0 e^{-\omega_0 t}, \quad t \geq 0, \tag{D.19a}$$

$$\left\|Ae^{tA}\right\|_{\mathcal{X}\leftarrow\mathcal{X}} \leq M_1 e^{-\omega_0 t} t^{-1}, \quad t > 0, \tag{D.19b}$$

$$\left\|A^2 e^{tA}\right\|_{\mathcal{X}\leftarrow\mathcal{X}} \leq M_2 e^{-\omega_0 t} t^{-2}, \quad t > 0. \tag{D.19c}$$

Again, we split u into the solution of the homogeneous problem

$$x'(t) = Ax(t), \quad t > 0, \qquad x(0) = u(0), \tag{D.20}$$

and the solution of the nonhomogeneous problem

$$y'(t) = Ay(t) + g(t), \quad t > 0, \qquad y(0) = 0, \tag{D.21}$$

with vanishing initial value. In the sequel, we derive estimates for the (α, α)-norms of x and y in $\mathcal{D}(A)$ by means of the modified variation-of-constants formula (D.5). Auxiliary estimates for integrals are collected in Estimate D.1. Notice that the present proof is very close to that of the Lemmas D.2 and D.3.

(a) The assumption $\omega_0 > 0$ implies $e^{-\omega_0 t} \leq 1$ for all $t \geq 0$. From (D.19a) we thus get

$$\begin{aligned}\|x(t)\|_{\mathcal{D}(A)} = \left\|e^{tA}Au(0)\right\|_{\mathcal{X}} &\leq \left\|e^{tA}\right\|_{\mathcal{X}\leftarrow\mathcal{X}}\|u(0)\|_{\mathcal{D}(A)} \leq M_0 e^{-\omega_0 t}\|u(0)\|_{\mathcal{D}(A)} \\ &\leq M_0\|u(0)\|_{\mathcal{D}(A)}\end{aligned}$$

for every $t \geq 0$.

(b) For all times $0 < \varepsilon \leq s < t$ the bound

$$\begin{aligned}\|x(t) - x(s)\|_{\mathcal{D}(A)} &= \left\|\left(e^{tA} - e^{sA}\right)Au(0)\right\|_{\mathcal{X}} \leq \int_s^t \left\|Ae^{\tau A}\right\|_{\mathcal{X}\leftarrow\mathcal{X}}\|u(0)\|_{\mathcal{D}(A)}\,\mathrm{d}\tau \\ &\leq M_1\|u(0)\|_{\mathcal{D}(A)}\int_s^t e^{-\omega_0\tau}\tau^{-1}\,\mathrm{d}\tau \leq M_1\|u(0)\|_{\mathcal{D}(A)}\int_s^t \tau^{-1}\,\mathrm{d}\tau \\ &\leq \alpha^{-1}M_1\|u(0)\|_{\mathcal{D}(A)}\cdot\varepsilon^{-\alpha}(t-s)^{\alpha}\end{aligned}$$

follows with the help of (D.19b) and (D.35) from Estimate D.1.

(c) By applying the bounds from (D.19) and Relation (D.36), for all $t \geq 0$ we receive

$$\begin{aligned}\|y(t)\|_{\mathcal{D}(A)} &\leq \int_0^t \left\|Ae^{(t-\tau)A}\right\|_{\mathcal{X}\leftarrow\mathcal{X}}\|g(\tau) - g(t)\|_{\mathcal{X}}\,\mathrm{d}\tau + \left\|e^{tA} - I\right\|_{\mathcal{X}\leftarrow\mathcal{X}}\|g(t)\|_{\mathcal{X}} \\ &\leq M_1[g]_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}\int_0^t e^{-\omega_0(t-\tau)}(t-\tau)^{-1+\alpha}\tau^{-\alpha}\,\mathrm{d}\tau + \left(1 + M_0 e^{-\omega_0 t}\right)\|g\|_{\mathcal{B}(\mathcal{X})} \\ &\leq B(\alpha, 1-\alpha)M_1[g]_{\mathcal{C}_\alpha^\alpha(\mathcal{X})} + (1 + M_0)\|g\|_{\mathcal{B}(\mathcal{X})}.\end{aligned}$$

(d) In order to estimate the difference $y(t)-y(s)$, we use the following bound which was derived in Part (ii)(b) of Lemma D.3

$$\begin{aligned}\|y(t)-y(s)\|_{\mathcal{D}(A)} \le \int_0^s \int_{s-\tau}^{t-\tau} \big\|A^2 e^{\sigma A}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \|g(\tau)-g(s)\|_{\mathcal{X}} \,\mathrm{d}\sigma\,\mathrm{d}\tau \\ + \int_s^t \big\|A e^{(t-\tau)A}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \|g(\tau)-g(t)\|_{\mathcal{X}} \,\mathrm{d}\tau + \int_s^t \big\|A e^{\tau A}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \|g(t)\|_{\mathcal{X}} \,\mathrm{d}\tau \\ + \big\|e^{(t-s)A} - e^{tA} + e^{sA} - I\|_{\mathcal{X}\leftarrow\mathcal{X}} \|g(t)-g(s)\|_{\mathcal{X}}.\end{aligned}$$

For $0<\varepsilon\le s<t$ this further gives

$$\begin{aligned}\|y(t)-y(s)\|_{\mathcal{D}(A)} &\le M_2 [g]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} (s-\tau)^\alpha \tau^{-\alpha} \,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &\quad + M_1 [g]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \int_s^t e^{-\omega_0(t-\tau)} (t-\tau)^{-1+\alpha} \tau^{-\alpha} \,\mathrm{d}\tau + M_1 \|g\|_{\mathcal{B}(\mathcal{X})} \int_s^t e^{-\omega_0 \tau} \tau^{-1} \,\mathrm{d}\tau \\ &\quad + \Big(1 + M_0 \big(e^{-\omega_0 t} + e^{-\omega_0(t-s)} + e^{-\omega_0 s}\big)\Big) [g]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \cdot \varepsilon^{-\alpha} (t-s)^\alpha \\ &\le M_2 [g]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \int_0^s \int_{s-\tau}^{t-\tau} \sigma^{-2} (s-\tau)^\alpha \tau^{-\alpha} \,\mathrm{d}\sigma\,\mathrm{d}\tau + (1+3M_0) [g]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \cdot \varepsilon^{-\alpha} (t-s)^\alpha \\ &\quad + M_1 [g]_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \int_s^t (t-\tau)^{-1+\alpha} \tau^{-\alpha} \,\mathrm{d}\tau + M_1 \|g\|_{\mathcal{B}(\mathcal{X})} \int_s^t \tau^{-1} \,\mathrm{d}\tau \\ &\le \Big(1 + 3M_0 + 2\alpha^{-1} M_1 + (\alpha(1-\alpha))^{-1} M_2\Big) \|g\|_{\mathcal{C}^\alpha_\alpha(\mathcal{X})} \cdot \varepsilon^{-\alpha} (t-s)^\alpha\end{aligned}$$

see (D.19) and Estimate D.1, and thus yields the desired result for the analytical solution.

(ii) Likewise to the analytic semigroup, the stability function of the backward Euler method applied to (D.18) shows an exponentially decaying behaviour whenever the spectrum of the sectorial operator A lies in the negative complex plane. On the one hand, for every $0<\omega_0<-a$ there exist constants $\overline{h}>0$ and $\hat{M}_k>0$ such that the bounds

$$\big\|(I-hA)^{-n}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \hat{M}_0 e^{-\omega_0 t_n}, \quad t_n \ge 0, \tag{D.22a}$$

$$\big\|A(I-hA)^{-n}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \hat{M}_1 e^{-\omega_0 t_n} t_n^{-1}, \quad t_n > 0, \tag{D.22b}$$

$$\big\|A^2(I-hA)^{-n}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \hat{M}_2 e^{-\omega_0 t_n} t_n^{-2}, \quad t_n > 0, \quad n \ge 2, \tag{D.22c}$$

are valid for all stepsizes $0<h\le\overline{h}$, and, furthermore, for every $\overline{h}>0$ there exist constants $0<\omega_0<-a$ and $M_k>0$ such that the above relations hold for $0<h\le\overline{h}$, see Result C.1. By means of these stability results the proof of the second statement for the backward Euler solution of (D.18) is now very close to that of the first statement. The needed auxiliary results are provided by Estimate D.2.

(iii) In order to derive the statement for a general Runge-Kutta method, we follow the proofs of the Lemmas D.2 and D.3, and apply the stability bounds

$$\big\|R(hA)^n\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \hat{M}_0 e^{-\omega_0 t_n}, \quad t_n \ge 0, \tag{D.23a}$$
$$\big\|b^T\mathcal{J}AR(hA)^{n-1}\mathbb{1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \hat{M}_1 e^{-\omega_0 t_n} t_n^{-1}, \quad t_n > 0, \tag{D.23b}$$
$$\big\|b^T\mathcal{J}b^T\mathcal{J}A^2R(hA)^{n-2}\mathbb{1}\big\|_{\mathcal{X}^s\leftarrow\mathcal{X}^s} \le \hat{M}_2 e^{-\omega_0 t_n} t_n^{-2}, \quad t_n > 0, \quad n \ge 2. \tag{D.23c}$$

Due to the fact that $\gamma = |R(\infty)|$ satisfies $\gamma = e^{-\omega_0 h - \varepsilon}$ for suitable choices of $0 < \omega_0 < -a$, $0 < h \le \overline{h}$ and $\varepsilon > 0$, the above relations follow from Result C.1. We omit a detailed derivation and refer to the proof of Lemma D.5 where a more general result is shown. $\square$

In the following result, we generalize the statements of the preceding Lemma D.4 to exponentially weighted (α,α)-Hölder continuous functions.

Lemma D.5. *Assume that the constant a arising in the resolvent estimate* (D.3) *for the sectorial operator A fulfils $a < 0$. Suppose further that the initial values $u(0)$ and u_0 lie in $\mathcal{D}(A)$ and that $g \in \mathcal{C}_\alpha^{\alpha,\omega}((0,\infty),\mathcal{X})$ holds for some $0 \le \omega < -a$ and $0 < \alpha < 1$. Then, the following statements are valid.*

(i) The analytical solution u of equation (D.18) *belongs to $\mathcal{C}_\alpha^{\alpha,\omega}((0,\infty),\mathcal{D}(A))$ and satisfies the estimate*

$$\|u\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D}(A))} \le C\big(\|u(0)\|_{\mathcal{D}(A)} + \|g\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{X})}\big).$$

The constant C depends on the choices of ω and α and further on the bounds for the analytical semigroup on the positive half line $(0,\infty)$.

(ii) For $0 < h \le \overline{h}$ sufficiently small, the backward Euler solution $\underline{u}$ of (D.18) *lies in $\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{N},\mathcal{D}(A))$ and fulfils*

$$\|\underline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D}(A))} \le C\big(\|u_0\|_{\mathcal{D}(A)} + \|\underline{g}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{X})}\big)$$

with constant C depending on ω, α and on the bounds for the stability function.

(iii) The Runge-Kutta solution $\underline{u}$ of equation (D.18) *belongs to $\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{N},\mathcal{D}(A))$ and the internal stages fulfil $\underline{U} \in \mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{N},\mathcal{D}(A)^s)$. Moreover, the bound*

$$\|\underline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D}(A))} + \|\underline{U}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D}(A)^s)} \le C\big(\|u_0\|_{\mathcal{D}(A)} + \|\underline{g}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{X}^s)}\big)$$

is valid for $0 < h \le \overline{h}$ small enough. In particular, the constant C depends on the choices of ω and α and on the bounds for the stability function of the method.

Proof. For the subsequent considerations, it is useful to introduce the abbreviation

$$A_\omega = \omega I + A$$

for the sectorial operator $\omega I + A : \mathcal{D}(A) \to \mathcal{X}$. Besides, we define the map g_ω through

$$g_\omega(t) = e^{\omega t} g(t), \qquad t \geq 0.$$

Notice that the first statement of the lemma for the analytical solution of (D.18) is a direct consequence of Lemma D.4. It is easy to see that the map $u_\omega(\cdot) = e^{(\cdot)\omega} u$ satisfies the following initial value problem

$$u_\omega'(t) = A_\omega u_\omega(t) + g_\omega(t), \quad t > 0, \qquad u_\omega(0) = u(0).$$

Due to our requirement $\omega < -a$, the spectrum of A_ω is still contained in the negative complex plane. Therefore, Lemma D.4 is applicable and yields

$$\begin{aligned} \|u\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D}(A))} &= \|u_\omega\|_{\mathcal{C}_\alpha^{\alpha}(\mathcal{D}(A))} \leq C\big(\|u(0)\|_{\mathcal{D}(A)} + \|g_\omega\|_{\mathcal{C}_\alpha^{\alpha}(\mathcal{X})}\big) \\ &= C\big(\|u(0)\|_{\mathcal{D}(A)} + \|g\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{X})}\big). \end{aligned}$$

Unfortunately, these ideas cannot be applied directly to the backward Euler method or a general Runge-Kutta method. For this reason, we derive the above bound for the analytical solution in a more direct way, and we show then the corresponding result for the Runge-Kutta solution. First of all, several preliminary remarks are useful.

As in the proof of Lemma D.4, we decompose the solution $u = x + y$ of the nonhomogeneous problem (D.18) into the solution of a homogeneous and a nonhomogeneous problem with vanishing initial value

$$x'(t) = Ax(t), \quad t > 0, \qquad x(0) = u(0), \tag{D.24a}$$

$$y'(t) = Ay(t) + g(t), \quad t > 0, \qquad y(0) = 0. \tag{D.24b}$$

For representing the analytical solution of (D.24b) we apply the variation-of-constants formula

$$y(t) = \int_0^t e^{(t-\tau)A} g(\tau)\,\mathrm{d}\tau, \qquad t \geq 0.$$

Multiplying this identity with $e^{\omega t}$ and A_ω yields

$$\begin{aligned} A_\omega e^{\omega t} y(t) &= \int_0^t A_\omega e^{(t-\tau)A_\omega} g_\omega(\tau)\,\mathrm{d}\tau \\ &= \int_0^t A_\omega e^{(t-\tau)A_\omega} \big(g_\omega(\tau) - g_\omega(t)\big)\,\mathrm{d}\tau + \big(e^{tA_\omega} - I\big) g_\omega(t), \quad t \geq 0. \end{aligned}$$

For the backward Euler solution of (D.24b) we obtain the following relation

$$\begin{aligned}
Ae^{\omega t_n} y_n &= e^{-\omega h} \cdot h \sum_{\nu=0}^{n-1} A\big(e^{\omega h}(I-hA)^{-1}\big)^{(n-\nu)} g_\omega(t_{\nu+1}) \\
&= e^{-\omega h} \cdot h \sum_{\nu=0}^{n-1} A\big(e^{\omega h}(I-hA)^{-1}\big)^{(n-\nu)} \big(g_\omega(t_{\nu+1}) - g_\omega(t_n)\big) \\
&\qquad + e^{-\omega h} \cdot hA \sum_{\nu=0}^{n-1} \big(e^{\omega h}(I-hA)^{-1}\big)^{(n-\nu)} g_\omega(t_n) \\
&= e^{-\omega h} \cdot h \sum_{\nu=0}^{n-1} A\big(e^{\omega h}(I-hA)^{-1}\big)^{(n-\nu)} \big(g_\omega(t_{\nu+1}) - g_\omega(t_n)\big) \\
&\qquad + e^{-\omega h} hA\big(\big(1-e^{-\omega h}\big)I + e^{-\omega h} hA\big)^{-1} \big(e^{\omega t_n}(I-hA)^{-n} - I\big) g_\omega(t_n),
\end{aligned}$$

due to the discrete variation-of-constants formula

$$y_n = h \sum_{\nu=0}^{n-1} (I-hA)^{-(n-\nu)} g(t_{\nu+1}).$$

Note that the linear operator

$$c_\omega(hA) = e^{-\omega h} hA\big(\big(1-e^{-\omega h}\big)I + e^{-\omega h} hA\big)^{-1}$$

is bounded on $\mathcal{X}$. For a general Runge-Kutta method of the form (D.2) it holds

$$y_n = h \sum_{\nu=0}^{n-1} b^T \mathcal{J} R(hA)^{n-\nu-1} g(t_\nu + ch).$$

From this, we further receive the relation

$$\begin{aligned}
Ae^{\omega t_n} y_n &= h \sum_{\nu=0}^{n-1} b^T e^{\omega h} \mathcal{J} A\big(e^{\omega h} R(hA)\big)^{n-\nu-1} D\, g_\omega(t_\nu + ch) \\
&= h \sum_{\nu=0}^{n-1} b^T e^{\omega h} \mathcal{J} A\big(e^{\omega h} R(hA)\big)^{n-\nu-1} D\big(g_\omega(t_\nu + ch) - g_\omega(t_{n-1} + ch)\big) \\
&\quad + hAb^T \mathcal{J}\big(\big(1-e^{-\omega h}\big)I + hAb^T \mathcal{J}\mathbb{1}\big)^{-1} D\big(e^{\omega t_n} R(hA)^n - I\big) g_\omega(t_{n-1} + ch).
\end{aligned}$$

Here, the matrix D is given by $D = \operatorname{diag}\big(e^{-\omega c_1 h}, \ldots, e^{-\omega c_s h}\big)$. Similarly to before, we denote by $C_\omega(hA)$ the bounded linear operator

$$C_\omega(hA) = hAb^T \mathcal{J}\big(\big(1-e^{-\omega h}\big)I + hAb^T \mathcal{J}\mathbb{1}\big)^{-1}.$$

Altogether, we receive the following formulas for the exact solutions of (D.24)

$$
\begin{aligned}
Ae^{\omega t}x(t) &= e^{tA_\omega} Au(0),\\
A_\omega e^{\omega t}y(t) &= \int_0^t A_\omega e^{(t-\tau)A_\omega}\big(g_\omega(\tau)-g_\omega(t)\big)\,\mathrm{d}\tau + \big(e^{tA_\omega}-I\big)g_\omega(t).
\end{aligned}
\tag{D.25a}
$$

The corresponding relations for the backward Euler method are

$$
\begin{aligned}
Ae^{\omega t_n}x_n &= r_\omega(hA)^n Au_0 \qquad \text{where} \quad r_\omega(hA) = e^{\omega h}(I-hA)^{-1} \quad \text{and}\\
Ae^{\omega t_n}y_n &= e^{-\omega h}\cdot h\sum_{\nu=0}^{n-1} Ar_\omega(hA)^{n-\nu}\big(g_\omega(t_{\nu+1})-g_\omega(t_n)\big)\\
&\qquad + c_\omega(hA)\big(r_\omega(hA)^n - I\big)g_\omega(t_n).
\end{aligned}
\tag{D.25b}
$$

For a general Runge-Kutta method we introduce the operators

$$
\mathcal{J}_\omega = e^{\omega h}(\mathcal{I}-hA\mathcal{O\!\!\!\!\!\!\;\;\;C}\!\!\!\!\!\!\;\;\;\mathcal{A})^{-1} \qquad \text{and} \qquad R_\omega(hA) = e^{\omega h}R(hA).
$$

Therefore, we obtain the following representation

$$
\begin{aligned}
Ae^{\omega t_n}x_n &= R_\omega(hA)^n Au_0,\\
Ae^{\omega t_n}y_n &= h\sum_{\nu=0}^{n-1} b^T\mathcal{J}_\omega AR_\omega(hA)^{n-\nu-1}D\,\big(g_\omega(t_\nu+ch)-g_\omega(t_{n-1}+ch)\big)\\
&\qquad + C_\omega(hA)D\big(R_\omega(hA)^n - I\big)g_\omega(t_{n-1}+ch).
\end{aligned}
\tag{D.25c}
$$

We henceforth assume that $\omega > 0$ and $0 < h \le \overline{h}$ are chosen suitably such that the bounds from Estimate D.2 and the stability bounds (D.22) and (D.23) remain valid for some $0 < \omega_0 < -a$ such that

$$
\delta_\omega = \omega_0 - \omega > 0.
$$

Then, the estimates from (D.19) imply the bounds

$$
\big\|e^{tA_\omega}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le M_0 e^{-\delta_\omega t}, \quad t \ge 0, \tag{D.26a}
$$

$$
\big\|A^k e^{tA_\omega}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le M_k e^{-\delta_\omega t}t^{-k}, \quad t > 0, \quad 1 \le k \le 2, \tag{D.26b}
$$

for the analytic semigroup generated by A_ω. Likely, the modified stability function for the backward Euler method fulfils

$$
\big\|A^k r_\omega(hA)^n\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \hat{M}_k e^{-\delta_\omega t_n}t_n^{-k}, \quad t_n \ge t_k, \quad 0 \le k \le 2, \tag{D.27}
$$

see also (D.22). Due to (D.23), the relations

$$\big\|R_\omega(hA)^n\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \hat{M}_0 e^{-\delta_\omega t_n}, \quad t_n \ge 0, \tag{D.28a}$$

$$\big\|b^T\mathcal{J}_\omega A R_\omega(hA)^{n-1}\mathbb{1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \hat{M}_1 e^{-\delta_\omega t_n} t_n^{-1}, \quad t_n > 0, \tag{D.28b}$$

$$\big\|b^T\mathcal{J}_\omega b^T\mathcal{J}_\omega \mathbb{1} A^2 R_\omega(hA)^{n-2}\big\|_{\mathcal{X}^s\leftarrow\mathcal{X}^s} \le \hat{M}_2 e^{-\delta_\omega t_n} t_n^{-2}, \quad t_n > 0, \quad n \ge 2, \tag{D.28c}$$

are valid for a strongly stable Runge-Kutta method. Notice that the graph norms of the linear operators A_ω and A are equivalent. We denote by C_γ the constant which arises from interchanging these norms, i.e. we have

$$\|A_\omega v\|_{\mathcal{X}} \le C_\gamma \|Av\|_{\mathcal{X}} \qquad \text{for all } v \in \mathcal{D}(A).$$

In the sequel, we derive estimates for the (α,α)-norms of the exact and numerical solutions of (D.24) by means of the formulas (D.25). Auxiliary estimates for integrals are collected in Estimate D.1.

(i)(a) For every $t \ge 0$ we have

$$\begin{aligned}\big\|e^{\omega t}x(t)\big\|_{\mathcal{D}(A)} &= \big\|e^{tA_\omega}Au(0)\big\|_{\mathcal{X}} \le \big\|e^{tA_\omega}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|u(0)\|_{\mathcal{D}(A)}\\ &\le M_0 e^{-\delta_\omega t}\|u(0)\|_{\mathcal{D}(A)} \le M_0\|u(0)\|_{\mathcal{D}(A)}.\end{aligned}$$

Consequently, we obtain the following estimate

$$\|x\|_{\mathcal{B}^\omega(\mathcal{D}(A))} \le M_0\|u(0)\|_{\mathcal{D}(A)}.$$

(b) Due to the validity of

$$A\big(e^{\omega t}x(t) - e^{\omega s}x(s)\big) = \big(e^{tA_\omega} - e^{sA_\omega}\big)Au(0) = \int_s^t A_\omega e^{\tau A_\omega}Au(0)\,\mathrm{d}\tau,$$

for all times $0 < \varepsilon \le s < t$ the estimate

$$\begin{aligned}\big\|e^{\omega t}x(t) - e^{\omega s}x(s)\big\|_{\mathcal{D}(A)} &\le C_\gamma M_1\|u(0)\|_{\mathcal{D}(A)} \int_s^t e^{-\delta_\omega\tau}\tau^{-1}\,\mathrm{d}\tau\\ &\le C_\gamma M_1\|u(0)\|_{\mathcal{D}(A)}\int_s^t \tau^{-1}\,\mathrm{d}\tau \le \alpha^{-1}C_\gamma M_1\|u(0)\|_{\mathcal{D}(A)}\cdot\varepsilon^{-\alpha}(t-s)^\alpha\end{aligned}$$

follows. From this, we receive the bound

$$[x]_{\mathcal{C}^{\alpha,\omega}_\alpha(\mathcal{D}(A))} \le \alpha^{-1}C_\gamma M_1\|u(0)\|_{\mathcal{D}(A)}.$$

(c) Due to the fact that the map g_ω is (α,α)-Hölder continuous in $\mathcal{X}$, we have

$$\begin{aligned}
\big\|e^{\omega t}y(t)\big\|_{\mathcal{D}(A)} &\le C_\gamma\big\|A_\omega e^{\omega t}y(t)\big\|_{\mathcal{X}} \\
&\le C_\gamma^2\int_0^t \big\|Ae^{(t-\tau)A_\omega}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|g_\omega(\tau)-g_\omega(t)\|_{\mathcal{X}}\,\mathrm{d}\tau \\
&\qquad + C_\gamma\big\|e^{tA_\omega}-I\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|g_\omega(t)\|_{\mathcal{X}} \\
&\le C_\gamma^2 M_1[g_\omega]_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}\int_0^t e^{-\delta_\omega(t-\tau)}(t-\tau)^{-1+\alpha}\tau^{-\alpha}\,\mathrm{d}\tau \\
&\qquad + C_\gamma\big(M_0e^{-\delta_\omega t}+1\big)\|g_\omega\|_{\mathcal{B}(\mathcal{X})} \\
&\le C_\gamma^2M_1B(\alpha,1-\alpha)[g_\omega]_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}+C_\gamma(1+M_0)\|g_\omega\|_{\mathcal{B}(\mathcal{X})}
\end{aligned}$$

for all $t\ge 0$. Thus, the following estimate holds

$$\|y\|_{\mathcal{B}^\omega(\mathcal{D}(A))}\le C_\gamma\big(1+M_0+C_\gamma M_1B(\alpha,1-\alpha)\big)\|g\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{X})}.$$

(d) Notice first that the identity

$$\begin{aligned}
A_\omega\big(e^{\omega t}y(t)-e^{\omega s}y(s)\big) &= \int_0^s\int_{s-\tau}^{t-\tau}A_\omega^2e^{\sigma A_\omega}\big(g_\omega(\tau)-g_\omega(s)\big)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\quad+\int_s^t A_\omega e^{(t-\tau)A_\omega}\big(g_\omega(\tau)-g_\omega(t)\big)\,\mathrm{d}\tau+\int_s^t A_\omega e^{\tau A_\omega}g_\omega(t)\,\mathrm{d}\tau \\
&\qquad+\big(e^{(t-s)A_\omega}-e^{tA_\omega}+e^{sA_\omega}-I\big)\big(g_\omega(t)-g_\omega(s)\big)
\end{aligned}$$

is valid. For all $0<\varepsilon\le s<t$ this implies

$$\begin{aligned}
\big\|A_\omega\big(e^{\omega t}y(t)-e^{\omega s}y(s)\big)\big\|_{\mathcal{X}} &\le \int_0^s\int_{s-\tau}^{t-\tau}C_\gamma\big\|A^2e^{\sigma A_\omega}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|g_\omega(\tau)-g_\omega(s)\|_{\mathcal{X}}\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\quad+\int_s^t C_\gamma\big\|Ae^{(t-\tau)A_\omega}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|g_\omega(\tau)-g_\omega(t)\|_{\mathcal{X}}\,\mathrm{d}\tau+C_\gamma\int_s^t\big\|Ae^{\tau A_\omega}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|g_\omega(t)\|_{\mathcal{X}}\,\mathrm{d}\tau \\
&\qquad+\big\|e^{(t-s)A_\omega}-e^{tA_\omega}+e^{sA_\omega}-I\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|g_\omega(t)-g_\omega(s)\|_{\mathcal{X}} \\
&\le C_\gamma M_2[g_\omega]_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}\int_0^s\int_{s-\tau}^{t-\tau}e^{-\delta_\omega\sigma}\sigma^{-2}(s-\tau)^\alpha\tau^{-\alpha}\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\quad+C_\gamma M_1[g_\omega]_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}\int_s^t e^{-\delta_\omega(t-\tau)}(t-\tau)^{-1+\alpha}\tau^{-\alpha}\,\mathrm{d}\tau+C_\gamma M_1\|g_\omega\|_{\mathcal{B}(\mathcal{X})}\int_s^t e^{-\delta_\omega\tau}\tau^{-1}\,\mathrm{d}\tau \\
&\qquad+\Big(M_0\big(e^{-\delta_\omega(t-s)}+e^{-\delta_\omega t}+e^{-\delta_\omega s}\big)+1\Big)[g_\omega]_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}\varepsilon^{-\alpha}(t-s)^\alpha
\end{aligned}$$

and furthermore

$$\begin{aligned}
\big\|e^{\omega t}y(t)-e^{\omega s}y(s)\big\|_{\mathcal{D}(A)} &\le C_\gamma\Big(1+3M_0+2\alpha^{-1}C_\gamma M_1+(\alpha(1-\alpha))^{-1}C_\gamma M_2\Big) \\
&\qquad\cdot\|g_\omega\|_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}\cdot\varepsilon^{-\alpha}(t-s)^\alpha.
\end{aligned}$$

Thus, we receive the following bound

$$[y]_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D}(A))} \le C_\gamma\Big(1 + 3M_0 + 2\alpha^{-1}C_\gamma M_1 + (\alpha(1-\alpha))^{-1}C_\gamma M_2\Big)\|g\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{X})}.$$

This yields the desired result for the analytical solution.

(ii)(a) For the backward Euler method it holds

$$\begin{aligned}\big\|e^{\omega t_n}x_n\big\|_{\mathcal{D}(A)} &= \big\|r_\omega(hA)^n Au_0\big\|_{\mathcal{X}} \le \big\|r_\omega(hA)^n\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|u_0\|_{\mathcal{D}(A)} \\ &\le \hat{M}_0 e^{-\delta_\omega t_n}\|u_0\|_{\mathcal{D}(A)} \le \hat{M}_0\|u_0\|_{\mathcal{D}(A)}\end{aligned}$$

for $t_n \ge 0$.

(b) Remark that the linear operator

$$\big(I - r_\omega(hA)^{-1}\big)r_\omega(hA)^k = \big((1-e^{-\omega h})I + e^{-\omega h}hA\big)r_\omega(hA)^k$$

fulfils the following estimate for every $t_k > 0$

$$\big\|\big(I - r_\omega(hA)^{-1}\big)r_\omega(hA)^k\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le Ch\,e^{-\delta_\omega t_k}\big(1 + t_k^{-1}\big)$$

with some constant $C>0$ depending on ω, $\hat{M}_0$ and $\hat{M}_1$. From this and the identity

$$\begin{aligned}A\big(e^{\omega t_n}x_n - e^{\omega t_m}x_m\big) &= \big(r_\omega(hA)^n - r_\omega(hA)^m\big)Au_0 \\ &= \big(I - r_\omega(hA)^{-1}\big)\sum_{\nu=m}^{n-1} r_\omega(hA)^{\nu+1}Au_0,\end{aligned}$$

we thus obtain the bound

$$\begin{aligned}\big\|e^{\omega t_n}x_n - e^{\omega t_m}x_m\big\|_{\mathcal{D}(A)} &\le C\cdot h\sum_{\nu=m}^{n-1} e^{-\delta_\omega t_{\nu+1}}\big(1 + t_{\nu+1}^{-1}\big) \\ &\le C\int_{t_m}^{t_n} e^{-\delta_\omega\tau}\,\mathrm{d}\tau + C\int_{t_m}^{t_n}\tau^{-1}\,\mathrm{d}\tau \le \widetilde{C}\cdot\varepsilon^{-\alpha}(t_n - t_m)^\alpha\end{aligned}$$

with $\widetilde{C}$ depending on C and further on α and δ_ω.

(c) For all times $t \ge 0$ we have

$$\begin{aligned}\big\|e^{\omega t_n}y_n\big\|_{\mathcal{D}(A)} \le\;& h\sum_{\nu=0}^{n-1}\big\|Ar_\omega(hA)^{n-\nu}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\big\|g_\omega(t_{\nu+1}) - g_\omega(t_n)\big\|_{\mathcal{X}} \\ &+ \big\|c_\omega(hA)\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\big\|r_\omega(hA)^n - I\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\big\|g_\omega(t_n)\big\|_{\mathcal{X}} \\ \le\;& \hat{M}_1[g_\omega]_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}\cdot h\sum_{\nu=0}^{n-1} e^{-\delta_\omega t_{n-\nu}}(t_n - t_\nu)^{-1}(t_n - t_{\nu+1})^\alpha t_{\nu+1}^{-\alpha} \\ &+ \big\|c_\omega(hA)\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\big(\hat{M}_0 e^{-\delta_\omega t_n} + 1\big)\|g_\omega\|_{\mathcal{B}(\mathcal{X})} \\ \le\;& \Big(\hat{M}_1 B(\alpha, 1-\alpha) + \|c_\omega(hA)\|_{\mathcal{X}\leftarrow\mathcal{X}}\big(1 + \hat{M}_0\big)\Big)\|g_\omega\|_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}.\end{aligned}$$

(d) For the backward Euler a similar relation as for the exact solution holds, namely

$$
\begin{aligned}
A\big(e^{\omega t_n} y_n - e^{\omega t_m} y_m\big) = e^{-\omega h} \cdot h \sum_{\nu=0}^{m-1} A\big(r_\omega(hA)^{n-\nu} - r_\omega(hA)^{m-\nu}\big)\big(g_\omega(t_{\nu+1}) - g_\omega(t_m)\big)& \\
+ e^{-\omega h} \cdot h \sum_{\nu=0}^{m-1} A r_\omega(hA)^{n-\nu}\big(g_\omega(t_m) - g_\omega(t_n)\big)& \\
+ e^{-\omega h} \cdot h \sum_{\nu=m}^{n-1} A r_\omega(hA)^{n-\nu}\big(g_\omega(t_{\nu+1}) - g_\omega(t_n)\big)& \\
+ c_\omega(hA)\big(r_\omega(hA)^n - r_\omega(hA)^m\big) g_\omega(t_n)& \\
+ c_\omega(hA)\big(r_\omega(hA)^m - I\big)\big(g_\omega(t_n) - g_\omega(t_m)\big)&
\end{aligned}
$$

for $0 < t_{n_1} \le t_m < t_n$. From this, we get

$$
\begin{aligned}
A\big(e^{\omega t_n} y_n - e^{\omega t_m} y_m\big) = e^{-\omega h} \cdot h \sum_{\nu=0}^{m-1} \sum_{\mu=m-\nu}^{n-\nu-1} A\big(I - r_\omega(hA)^{-1}\big) r_\omega(hA)^{\mu+1}& \\
\cdot \big(g_\omega(t_{\nu+1}) - g_\omega(t_m)\big)& \\
+ e^{-\omega h} \cdot h \sum_{\nu=m}^{n-1} A r_\omega(hA)^{n-\nu}\big(g_\omega(t_{\nu+1}) - g_\omega(t_n)\big)& \\
+ c_\omega(hA)\big(I - r_\omega(hA)^{-1}\big) \sum_{\nu=m}^{n-1} r_\omega(hA)^{\nu+1} g_\omega(t_n)& \\
+ \Big(e^{-\omega h} \cdot hA\big(I - r_\omega(hA)^{-1}\big)^{-1}\big(r_\omega^{n-m}(hA) - r_\omega(hA)^n\big)& \\
+ c_\omega(hA)\big(r_\omega(hA)^m - I\big)\Big) \cdot \big(g_\omega(t_n) - g_\omega(t_m)\big).&
\end{aligned}
$$

In order to estimate this expression we apply the stability bounds (D.27). Note that the following relations

$$
\begin{aligned}
\big\|\big(I - r_\omega(hA)^{-1}\big) r_\omega(hA)^k\big\|_{\mathcal{X}\leftarrow\mathcal{X}} &\le Ch\, e^{-\delta_\omega t_k}\big(1 + t_k^{-1}\big), \qquad t_k > 0, \\
\big\|A\big(I - r_\omega(hA)^{-1}\big) r_\omega(hA)^k\big\|_{\mathcal{X}\leftarrow\mathcal{X}} &\le Ch\, e^{-\delta_\omega t_k}\big(t_k^{-1} + t_k^{-2}\big), \qquad t_k > 0,
\end{aligned}
$$

are valid. Further, we make use of the (α,α)-Hölder continuity of g_ω and estimate the arising sums by the corresponding integrals. This yields

$$
\begin{aligned}
\big\|e^{\omega t_n} y_n - e^{\omega t_m} y_m\big\|_{\mathcal{D}(A)} \le C\bigg(\int_0^{t_m} \int_{t_m-\tau}^{t_n-\tau} e^{-\delta_\omega \sigma}\big(\sigma^{-1} + \sigma^{-2}\big)(t_m - \tau)^\alpha \tau^{-\alpha}\, \mathrm{d}\sigma\, \mathrm{d}\tau& \\
+ \int_{t_m}^{t_n} (t_n - \tau)^{-1+\alpha} \tau^{-\alpha}\, \mathrm{d}\tau + \int_{t_m}^{t} e^{-\delta_\omega \tau}\big(1 + \tau^{-1}\big)\, \mathrm{d}\tau + t_{n_1}^{-\alpha}(t_n - t_m)^\alpha\bigg)\|g_\omega\|_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}&
\end{aligned}
$$

and finally shows the following estimate

$$\left\|e^{\omega t_n} y_n - e^{\omega t_m} y_m\right\|_{\mathcal{D}(A)} \le C \|g\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{X})} \cdot t_{n_1}^{-\alpha} (t_n - t_m)^\alpha$$

with constant C depending on α, δ_ω, ω and the bounds for the stability function.

(iii) For the Runge-Kutta solution the statement of the lemma follows in a similar way to part (ii) by using the stability bounds (D.28) and Estimate D.1. The bound for the internal stages then follows immediately from the relation

$$Ae^{\omega t_n} U_n = \mathcal{J}\mathbb{1}\, Ae^{\omega t_n} u_n + hA\mathcal{O}\!\mathcal{l}\mathcal{J} D g_\omega(t_n + ch).$$

We omit the derivation. □

D.2.3. The nonhomogeneous problem on half lines

In the present section, we study the exact solution and the Runge-Kutta solution of the nonhomogeneous problem

$$u'(t) = Au(t) + g(t), \quad t > 0, \qquad u(0) \text{ given}, \tag{D.29}$$

for negative and positive times. Our fundamental assumption for the following considerations is that the spectrum of the sectorial operator A does not intersect the imaginary axis. In accordance with the requirements of Assumption 6.2, we henceforth suppose that the positive part of the spectrum of A

$$\sigma_1(A) = \big\{\lambda \in \sigma(A) : \Re(\lambda) > 0\big\} \subset \mathcal{R} \tag{D.30a}$$

is contained in a rectangle $\mathcal{R}$ such that $\Re(\lambda) \ge a_1 > 0$ for $\lambda \in \mathcal{R}$ and that $\sigma_1(A)$ consists of a finite number of eigenvalues with finite multiplicity. We further assume that the set

$$\sigma_2(A) = \big\{\lambda \in \sigma(A) : \Re(\lambda) < 0\big\} \subset \mathcal{S} \tag{D.30b}$$

lies in a sector $\mathcal{S}$ with top $-a_2 < 0$ and opening angle smaller than $\pi/2$. This situation in illustrated in Figure D.1.

In the first place, we consider the nonhomogeneous problem on the negative half line. The following result states that for any initial value in $\mathcal{D}(A)$ such that

$$Qu(0) = \int_{-\infty}^{0} e^{-\tau A} Q g(\tau)\,\mathrm{d}\tau, \tag{D.31a}$$

a backward solution of (D.29) exists which is α-Hölder continuous on the negative half line whenever g belongs to $\mathcal{C}_\alpha^\alpha((-\infty, 0], \mathcal{X})$. Here, Q denotes the projection on the negative part of the spectrum of A defined by (6.7). Besides, a corresponding result is valid for

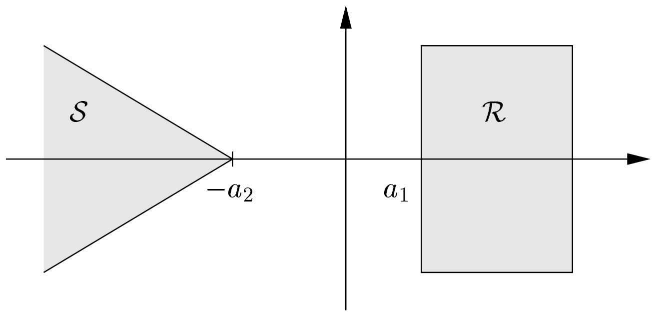

Figure D.1.: The spectrum of A is contained in the shaded area

the solution of a strongly stable Runge-Kutta method (D.2) with initial value $u_0 \in \mathcal{D}(A)$ satisfying the relation

$$Qu_0 = h \sum_{\nu=-\infty}^{-1} b^T \mathcal{J} R(hA)^{-(\nu+1)} Qg(t_\nu + ch), \tag{D.31b}$$

and, in particular, for the backward Euler method.

Lemma D.6. *Suppose that the sectorial operator A satisfies the above assumptions* (D.30)*. Assume further that the initial values $u(0)$ and u_0 fulfil* (D.31) *and that the map g belongs to $\mathcal{C}^\alpha((-\infty, 0], \mathcal{X})$ for some $0 < \alpha < 1$. For some fixed stepsize $h > 0$ let $\mathcal{N}$ denote the set $\mathcal{N} = \{t_n = nh : n \in \boldsymbol{Z},\, n \le 0\}$. Then, the following statements are valid.*

(i) The backward solution u of equation (D.29) *belongs to $\mathcal{C}^\alpha((-\infty, 0], \mathcal{D}(A))$ and satisfies the estimate*

$$\|u\|_{\mathcal{C}^\alpha(\mathcal{D}(A))} \le C\big(\|u(0)\|_{\mathcal{D}(A)} + \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\big).$$

The constant C depends on α and further on the bounds for the analytical semigroup.

(ii) For $0 < h \le \overline{h}$ sufficiently small, the backward Euler solution $\underline{u}$ of (D.29) *lies in $\mathcal{C}^\alpha(\mathcal{N}, \mathcal{D}(A))$ and fulfils*

$$\|\underline{u}\|_{\mathcal{C}^\alpha(\mathcal{D}(A))} \le C\big(\|u_0\|_{\mathcal{D}(A)} + \|\underline{g}\|_{\mathcal{C}^\alpha(\mathcal{X})}\big)$$

with constant C depending on α and on the bounds for the stability function.

(iii) The Runge-Kutta solution $\underline{u}$ of equation (D.29) *belongs to $\mathcal{C}^\alpha(\mathcal{N}, \mathcal{D}(A))$ and the stages fulfil $\underline{U} \in \mathcal{C}^\alpha(\mathcal{N}, \mathcal{D}(A)^s)$. Moreover, the bound*

$$\|\underline{u}\|_{\mathcal{C}^\alpha(\mathcal{D}(A))} + \|\underline{U}\|_{\mathcal{C}^\alpha(\mathcal{D}(A)^s)} \le C\big(\|u_0\|_{\mathcal{D}(A)} + \|\underline{g}\|_{\mathcal{C}^\alpha(\mathcal{X}^s)}\big)$$

is valid for $0 < h \le \overline{h}$ small enough. In particular, the constant C depends on the choice of α and on the bounds for the stability function of the method.

Proof. (i) For the subsequent considerations we fix constants

$$0 < -\omega_1 < a_1 \qquad \text{and} \qquad 0 < \omega_2 < a_2.$$

It is known from LUNARDI [25, Theorem 4.4.6] that a strict solution of (D.29) exists if the initial value $u(0)$ fulfils (D.31). In order to justify this condition, we perform a brief calculation involving the spectral projections P and Q on the positive and negative part of the spectrum of A, see (6.7). Due to Result (A.3), the analytic semigroup satisfies the bounds

$$\begin{aligned} \|A^k e^{tA} P\|_{\mathcal{X}\leftarrow\mathcal{X}} &\le M_k^P e^{-\omega_1 t}, \qquad t \le 0, \qquad 0 \le k \le 2, \\ \|A^k e^{tA} Q\|_{\mathcal{X}\leftarrow\mathcal{X}} &\le M_k^Q e^{-\omega_2 t} t^{-k}, \qquad t > 0, \qquad 0 \le k \le 2, \end{aligned} \tag{D.32}$$

with some constants $M_k^P > 0$ and $M_k^Q > 0$. We split the analytical solution u in the following way

$$u = p + q \qquad \text{where} \quad p = Pu \quad \text{and} \quad q = Qu.$$

Then, the map p solves the initial value problem

$$p'(t) = Ap(t) + Pg(t), \quad t \le 0, \qquad p(0) \text{ given}.$$

By the modified variation-of-constants formula (D.9) we thus obtain the representation

$$Ap(t) = e^{tA} Ap(0) + \int_0^t A e^{(t-\tau)A} P\big(g(\tau) - g(t)\big)\,\mathrm{d}\tau + \big(e^{tA}P - P\big) g(t), \quad t \le 0.$$

For some $T < 0$ we consider the initial value problem

$$q'(t) = Aq(t) + Pq(t), \quad t \le 0, \qquad q(T) \text{ given}.$$

Here, the solution q is given by the formula

$$Aq(t) = e^{(t-T)A} Aq(T) + \int_T^t A e^{(t-\tau)A} Q\big(g(\tau) - g(t)\big)\,\mathrm{d}\tau + \big(e^{(t-T)A}Q - Q\big) g(t), \quad t \le 0.$$

Taking the limit $T \to -\infty$ and applying (D.32) yields the identity

$$Aq(t) = \int_{-\infty}^t A e^{(t-\tau)A} Q\big(g(\tau) - g(t)\big)\,\mathrm{d}\tau - Qg(t), \quad t \le 0.$$

Consequently, the analytical solution of (D.29) has the representation

$$\begin{aligned} Au(t) = e^{tA} PAu(0) &+ \int_0^t A e^{(t-\tau)A} P\big(g(\tau) - g(t)\big)\,\mathrm{d}\tau + \big(e^{tA}P - P\big) g(t) \\ &+ \int_{-\infty}^t A e^{(t-\tau)A} Q\big(g(\tau) - g(t)\big)\,\mathrm{d}\tau - Qg(t), \qquad t \le 0. \end{aligned} \tag{D.33}$$

By setting $t = 0$ in the above formula, we essentially obtain the condition from (D.31) on the initial value $u(0)$.

In the following, we derive estimates for the α-Hölder norm of the exact solution u. For this, it is useful to decompose u into $u = x + y + z$ where

$$\begin{aligned} Ax(t) &= e^{tA} PAu(0), \quad t \le 0, \\ Ay(t) &= \int_0^t Ae^{(t-\tau)A} P\big(g(\tau) - g(t)\big)\,\mathrm{d}\tau + \big(e^{tA}P - P\big)g(t), \quad t \le 0, \\ Az(t) &= \int_{-\infty}^t Ae^{(t-\tau)A} Q\big(g(\tau) - g(t)\big)\,\mathrm{d}\tau - Qg(t), \quad t \le 0, \end{aligned}$$

and to treat each term separately. Our main tools are the relations from (D.32) and the bounds provided by Estimate D.3.

(a) For all $t \le 0$, due to $e^{-\omega_1 t} \le 1$, the following bound holds

$$\begin{aligned} \|x(t)\|_{\mathcal{D}(A)} = \big\|e^{tA}PAu(0)\big\|_{\mathcal{X}} &\le \big\|e^{tA}P\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \|u(0)\|_{\mathcal{D}(A)} \\ &\le M_0^P e^{-\omega_1 t}\|u(0)\|_{\mathcal{D}(A)} \le M_0^P\|u(0)\|_{\mathcal{D}(A)}. \end{aligned}$$

(b) Moreover, for $s < t \le 0$ we have

$$\begin{aligned} \|x(s) - x(t)\|_{\mathcal{D}(A)} &= \big\|\big(e^{sA} - e^{tA}\big)PAu(0)\big\|_{\mathcal{X}} \le \left| \int_t^s \big\|Ae^{\tau A}P\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \|u(0)\|_{\mathcal{D}(A)}\,\mathrm{d}\tau \right| \\ &\le M_1^P\|u(0)\|_{\mathcal{D}(A)} \left| \int_t^s e^{-\omega_1\tau}\,\mathrm{d}\tau \right| \le \alpha^{-1}|\omega_1|^{-1+\alpha} M_1^P \|u(0)\|_{\mathcal{D}(A)}(t-s)^\alpha. \end{aligned}$$

(c) For negative times $t \le 0$, the relation

$$\begin{aligned} \|y(t)\|_{\mathcal{D}(A)} &\le \left| \int_0^t \big\|Ae^{(t-\tau)A}P\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \|g(\tau) - g(t)\|_{\mathcal{X}}\,\mathrm{d}\tau \right| + \big\|e^{tA}P - P\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \|g(t)\|_{\mathcal{X}} \\ &\le M_1^P[g]_{\mathcal{C}^\alpha(\mathcal{X})} \left| \int_0^t e^{-\omega_1(t-\tau)}(\tau - t)^\alpha\,\mathrm{d}\tau \right| + \big(M_0^P e^{-\omega_1 t} + M_0^P\big)\|g\|_{\mathcal{B}(\mathcal{X})} \\ &\le |\omega_1|^{-1-\alpha}\Gamma(1+\alpha)M_1^P[g]_{\mathcal{C}^\alpha(\mathcal{X})} + 2M_0^P\|g\|_{\mathcal{B}(\mathcal{X})} \end{aligned}$$

holds.

(d) From the identity

$$\begin{aligned} A\big(y(s) - y(t)\big) = &\int_0^t \int_{t-\tau}^{s-\tau} A^2 e^{\sigma A} P\big(g(\tau) - g(t)\big)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &+ \int_t^s Ae^{\tau A}Pg(s)\,\mathrm{d}\tau + \int_t^s Ae^{(s-\tau)A}P\big(g(\tau) - g(s)\big)\,\mathrm{d}\tau \\ &+ \big(e^{tA}P - e^{sA}P + e^{(s-t)A}P - P\big)\big(g(s) - g(t)\big), \end{aligned}$$

for all $s < t \leq 0$ the estimate

$$\begin{aligned}
\|y(s) - y(t)\|_{\mathcal{D}(A)} &\leq M_2^P [g]_{\mathcal{C}^\alpha(\mathcal{X})} \left| \int_0^t \int_{t-\tau}^{s-\tau} e^{-\omega_1 \sigma} (\tau - t)^\alpha \, \mathrm{d}\sigma \, \mathrm{d}\tau \right| \\
&\quad + M_1^P \|g\|_{\mathcal{B}(\mathcal{X})} \left| \int_t^s e^{-\omega_1 \tau} \, \mathrm{d}\tau \right| + M_1^P [g]_{\mathcal{C}^\alpha(\mathcal{X})} \left| \int_t^s e^{-\omega_1 (s-\tau)} (\tau - s)^\alpha \, \mathrm{d}\tau \right| \\
&\quad + 4 M_0^P [g]_{\mathcal{C}^\alpha(\mathcal{X})} (t-s)^\alpha \\
&\leq \Big(4 M_0^P + \alpha^{-1} \big(c_0 + |\omega_1|^{-1+\alpha} \big) M_1^P + \alpha^{-1} |\omega_1|^{-2} \Gamma(1+\alpha) M_2^P \Big) \|g\|_{\mathcal{C}^\alpha(\mathcal{X})} (t-s)^\alpha
\end{aligned}$$

follows. Here, c_0 denotes a bound of the function $x e^{-|\omega_1| x}$ on $[0, \infty)$.

(e) Note further that for all $t \leq 0$ it follows

$$\begin{aligned}
\|z(t)\|_{\mathcal{D}(A)} &\leq \int_{-\infty}^t \big\| A e^{(t-\tau)A} Q \big\|_{\mathcal{X} \leftarrow \mathcal{X}} \|g(\tau) - g(t)\|_{\mathcal{X}} \, \mathrm{d}\tau + \|Q\|_{\mathcal{X} \leftarrow \mathcal{X}} \|g(t)\|_{\mathcal{X}} \\
&\leq M_1^Q [g]_{\mathcal{C}^\alpha(\mathcal{X})} \int_{-\infty}^t e^{-\omega_2 (t-\tau)} (t-\tau)^{-1+\alpha} \, \mathrm{d}\tau + M_0^Q \|g\|_{\mathcal{B}(\mathcal{X})} \\
&\leq \omega_2^{-\alpha} \Gamma(\alpha) M_1^Q [g]_{\mathcal{C}^\alpha(\mathcal{X})} + M_0^Q \|g\|_{\mathcal{B}(\mathcal{X})}.
\end{aligned}$$

(f) Furthermore, the following identity is valid

$$\begin{aligned}
A\big(z(t) - z(s)\big) &= \int_{-\infty}^s \int_{s-\tau}^{t-\tau} A^2 e^{\sigma A} Q \big(g(\tau) - g(s) \big) \, \mathrm{d}\sigma \, \mathrm{d}\tau \\
&\quad + \int_s^t A e^{(t-\tau)A} Q \big(g(\tau) - g(t) \big) \, \mathrm{d}\tau - \big(e^{(t-s)A} Q + Q \big) \big(g(t) - g(s) \big).
\end{aligned}$$

For all times $s < t \leq 0$ we thus receive the estimate

$$\begin{aligned}
\|z(t) - z(s)\|_{\mathcal{D}(A)} &\leq M_2^Q [g]_{\mathcal{C}^\alpha(\mathcal{X})} \int_{-\infty}^s \int_{s-\tau}^{t-\tau} e^{-\omega_2 \sigma} \sigma^{-2} (s-\tau)^\alpha \, \mathrm{d}\sigma \, \mathrm{d}\tau \\
&\quad + M_1^Q [g]_{\mathcal{C}^\alpha(\mathcal{X})} \int_s^t e^{-\omega_2 (t-\tau)} (t-\tau)^{-1+\alpha} \, \mathrm{d}\tau + 2 M_0^Q [g]_{\mathcal{C}^\alpha(\mathcal{X})} (t-s)^\alpha \\
&\leq \Big(2 M_0^Q + \alpha^{-1} M_1^Q + \big(\alpha(1-\alpha)\big)^{-1} M_2^Q \Big) \|g\|_{\mathcal{C}^\alpha(\mathcal{X})} (t-s)^\alpha.
\end{aligned}$$

From this, the desired result for the exact solution follows.

(ii)/(iii) For simplicity, we henceforth suppose that the numerical method (D.2) fulfils

$$R(\infty) = 0.$$

For proving the statements of the lemma for the backward Euler method and a general Runge-Kutta method, we follow part (i) and employ the analogous representation to (D.33). For the Runge-Kutta solution this yields

$$\begin{aligned}
Au_n = R(hA)^n PAu_0 &+ h\sum_{\nu=0}^{n-1} b^T \mathcal{J} AR(hA)^{n-\nu-1} P\big(g(t_\nu+ch) - g(t_{n-1}+ch)\big) \\
&+ b^T\mathcal{J}\big(b^T\mathcal{J}1\!\!1\big)^{-1}\big(R(hA)^n P - P\big) g(t_{n-1}+ch) \\
&\quad + h\sum_{\nu=-\infty}^{n-1} b^T\mathcal{J} AR(hA)^{n-\nu-1} Q\big(g(t_\nu+ch) - g(t_{n-1}+ch)\big) \\
&\quad\quad - b^T\mathcal{J}\big(b^T\mathcal{J}1\!\!1\big)^{-1} Qg(t_{n-1}+ch), \quad n\le 0,
\end{aligned}$$

and especially for the backward Euler solution we obtain the following formula

$$\begin{aligned}
Au_n = (I-hA)^{-n} PAu_0 &+ h\sum_{\nu=0}^{n-1} A(I-hA)^{-(n-\nu)} P\big(g(t_{\nu+1}) - g(t_n)\big) \\
&+ \big((I-hA)^{-n}P - P\big) g(t_n) \\
&\quad + h\sum_{\nu=-\infty}^{n-1} A(I-hA)^{-(n-\nu)} Q\big(g(t_{\nu+1}) - g(t_n)\big) - Qg(t_n), \quad n\le 0.
\end{aligned}$$

Essential for showing the estimates for the α-Hölder norms of the numerical solution are the stability bounds

$$\begin{aligned}
\big\|A^k R(hA)^n P\big\|_{\mathcal{X}\leftarrow\mathcal{X}} &\le \hat{M}_k^P e^{-\omega_1 t_n}, \qquad t_n\le 0, \qquad 0\le k\le 2, \\
\big\|A^k R(hA)^n Q\big\|_{\mathcal{X}\leftarrow\mathcal{X}} &\le \hat{M}_k^Q e^{-\omega_2 t_n} t_n^{-1}, \qquad t_n > 0, \qquad 0\le k\le 2,
\end{aligned}$$

which are valid for a strongly Runge-Kutta method whenever $0 < h \le \overline{h}$ is small enough, see also Section (6.3). In order to prove the desired bound for the Runge-Kutta solution, we slightly modify the above proof for the analytical solution, and we replace the continuous quantities with the discrete ones. The arising Riemann-sums are estimated in a standard way by the corresponding integrals, see also Estimate D.3. We dismiss a detailed derivation and refer to the preceding proofs where similar calculations are performed. □

Lemma D.7 below provides the analogous statements for the positive half line. In this case, the analytical and the numerical solution decay exponentially fast whenever the following conditions

$$Pu(0) = -\int_0^\infty e^{-\tau A} Pg(\tau)\,\mathrm{d}\tau, \tag{D.34a}$$

$$Pu_0 = -h\sum_{\nu=-1}^{\infty} b^T\mathcal{J} R(hA)^{-(\nu+1)} Pg(t_\nu+ch), \tag{D.34b}$$

are satisfied.

Lemma D.7. *Suppose that the sectorial operator A satisfies* (D.30). *Assume further that the initial values $u(0)$ and u_0 fulfil* (D.34) *and that $g \in \mathcal{C}_\alpha^\alpha((0,\infty),\mathcal{X})$ holds for some $0<\alpha<1$. For fixed $h>0$ let $\mathcal{N}=\{t_n = nh : n \in \boldsymbol{N}\}$. Then, the following statements are valid.*

(i) The analytical solution u of equation (D.29) *belongs to $\mathcal{C}_\alpha^\alpha((0,\infty),\mathcal{D}(A))$ and satisfies the estimate*

$$\|u\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}(A))} \le C\big(\|u(0)\|_{\mathcal{D}(A)} + \|g\|_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}\big).$$

The constant C depends on α and further on the bounds for the analytical semigroup.

(ii) For $0<h\le \overline{h}$ sufficiently small, the backward Euler solution $\underline{u}$ of (D.29) *lies in $\mathcal{C}_\alpha^\alpha(\mathcal{N},\mathcal{D}(A))$ and fulfils*

$$\|\underline{u}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}(A))} \le C\big(\|u_0\|_{\mathcal{D}(A)} + \|\underline{g}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{X})}\big)$$

with constant C depending on α and on the bounds for the stability function.

(iii) The Runge-Kutta solution $\underline{u}$ of equation (D.29) *belongs to $\mathcal{C}_\alpha^\alpha(\mathcal{N},\mathcal{D}(A))$ and the stages fulfil $\underline{U} \in \mathcal{C}_\alpha^\alpha(\mathcal{N},\mathcal{D}(A)^s)$. Moreover, the bound*

$$\|\underline{u}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}(A))} + \|\underline{U}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}(A)^s)} \le C\big(\|u_0\|_{\mathcal{D}(A)} + \|\underline{g}\|_{\mathcal{C}_\alpha^\alpha(\mathcal{X}^s)}\big)$$

is valid for $0<h\le\overline{h}$ small enough. In particular, the constant C depends on the choice of α and on the bounds for the stability function of the method.

Proof. The present proof is very close to that of Lemma D.6 wherefore we primarily remark on the differences. Again, we apply the modified variation-of-constants formula (D.9) twice and obtain the following representation for the analytical solution on the positive half line

$$\begin{aligned} Au(t) = e^{tA}QAu(0) &+ \int_0^t Ae^{(t-\tau)A}Q\big(g(\tau)-g(t)\big)\,\mathrm{d}\tau + \big(e^{tA}Q - Q\big)g(t) \\ &- \int_t^\infty Ae^{(t-\tau)A}P\big(g(\tau)-g(t)\big)\,\mathrm{d}\tau - Pg(t), \qquad t \ge 0. \end{aligned}$$

The analogous formula for the Runge-Kutta solution is

$$\begin{aligned} Au_n = R(hA)^nQAu_0 &+ h\sum_{\nu=0}^{n-1} b^T\mathcal{J}AR(hA)^{n-\nu-1}Q\big(g(t_\nu+ch)-g(t_{n-1}+ch)\big) \\ &+ b^T\mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1}\big(R(hA)^nQ - Q\big)g(t_{n-1}+ch) \\ &- h\sum_{\nu=n-1}^{\infty} b^T\mathcal{J}AR(hA)^{n-\nu-1}P\big(g(t_\nu+ch)-g(t_{n-1}+ch)\big) \\ &- b^T\mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1}Pg(t_{n-1}+ch). \end{aligned}$$

We choose constants $\omega_1 < 0$ and $\omega_2 > 0$ such that

$$0 < -\omega_1 < a_1 \qquad \text{and} \qquad 0 < \omega_2 < a_2.$$

Then the bounds from (D.32) and the corresponding relations for the numerical method are valid. In the sequel, we give a proof of the first statement for the analytical solution. The analogue for the backward Euler or a general Runge-Kutta methods then follows by using similar ideas. We omit the details.

(i) As in Lemma D.6 we decompose the exact solution u in the following way

$$\begin{aligned} &Ax(t) = e^{tA}QAu(0), \quad Ay(t) = \int_0^t Ae^{(t-\tau)A}Q\big(g(\tau)-g(t)\big)\,\mathrm{d}\tau + \big(e^{tA}Q - Q\big)g(t), \\ &Az(t) = -\int_t^\infty Ae^{(t-\tau)A}P\big(g(\tau)-g(t)\big)\,\mathrm{d}\tau - Pg(t), \qquad t \geq 0. \end{aligned}$$

Due to the fact that the spectrum of the sectorial operator AQ is contained in the negative complex plane, the desired bounds for the (α,α)-Hölder estimates of x and y are a direct consequence of Lemma D.4. Thus, it remains to consider the map z. The needed auxiliary results are provided by Estimate D.3.

(a) For all positive times $t \geq \varepsilon > 0$ we receive the bound

$$\begin{aligned} \|z(t)\|_{\mathcal{D}(A)} &\leq M_1^P [g]_{\mathcal{C}_\alpha^\alpha(\mathcal{D}(A))}\varepsilon^{-\alpha} \int_t^\infty e^{\omega_1(\tau-t)}(\tau-t)^\alpha\,\mathrm{d}\tau + M_0^P\|g\|_{\mathcal{B}(X)} \\ &\leq \Big(M_0^P + |\omega_1|^{-1-\alpha}\varepsilon^{-\alpha}\Gamma(1+\alpha)M_1^P\Big)\|g\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}(A))}. \end{aligned}$$

(b) Moreover, from the identity

$$A\big(z(t)-z(s)\big) = \int_t^\infty \int_{t-\tau}^{s-\tau} A^2 e^{\sigma A}P\big(g(\tau)-g(t)\big)\,\mathrm{d}\sigma\,\mathrm{d}\tau$$

for all $0 < \varepsilon \leq s < t$ the estimate

$$\begin{aligned} \|z(t)-z(s)\|_{\mathcal{D}(A)} &\leq M_2^P [g]_{\mathcal{C}_\alpha^\alpha(\mathcal{D}(A))}\varepsilon^{-\alpha} \int_t^\infty \int_{\tau-t}^{\tau-s} e^{\omega_1\sigma}(\tau-t)^\alpha\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &\quad + M_1^P [g]_{\mathcal{C}_\alpha^\alpha(\mathcal{D}(A))}\varepsilon^{-\alpha} \int_s^t e^{\omega_1(t-\tau)}(t-\tau)^\alpha\,\mathrm{d}\tau + 2M_0^P [g]_{\mathcal{C}_\alpha^\alpha(\mathcal{D}(A))} \cdot \varepsilon^{-\alpha}(t-s)^\alpha \\ &\leq M_2^P [g]_{\mathcal{C}_\alpha^\alpha(\mathcal{D}(A))}\varepsilon^{-\alpha}\omega^{-1}\big(1-e^{\omega_1(t-s)}\big) \int_t^\infty e^{\omega_1(\tau-t)}(\tau-t)^\alpha\,\mathrm{d}\tau \\ &\quad + c_0 M_1^P [g]_{\mathcal{C}_\alpha^\alpha(\mathcal{D}(A))}\varepsilon^{-\alpha} \int_0^{t-s} \tau^{-1+\alpha}\,\mathrm{d}\tau + 2M_0^P [g]_{\mathcal{C}_\alpha^\alpha(\mathcal{D}(A))} \cdot \varepsilon^{-\alpha}(t-s)^\alpha \\ &\leq \Big(2M_0^P + c_0\alpha^{-1}M_1^P + \alpha^{-1}\omega^{-2}\Gamma(1+\alpha)M_2^P\Big)\|g\|_{\mathcal{C}_\alpha^\alpha(\mathcal{D}(A))} \cdot \varepsilon^{-\alpha}(t-s)^\alpha \end{aligned}$$

follows. □

In the following, we generalize the estimates of the preceding Lemmas D.6 and D.7 to the exponentially weighted Hölder norms. Recall that the constants $a_1 > 0$ and $a_2 > 0$ are specified in (D.30).

Lemma D.8. *Assume that the sectorial operator A satisfies the assumptions from* (D.30).

For some $h > 0$ let $\mathcal{N} = \{t_n = nh : n \in \boldsymbol{Z}, n \leq 0\}$. Whenever the initial values $u(0)$ and u_0 in $\mathcal{D}(A)$ satisfy (D.31) *and the map g belongs to $\mathcal{C}^{\alpha,\omega}((-\infty,0],\mathcal{X})$ for $0 < \alpha < 1$ and some $\omega \leq 0$ such that $-a_2 < \omega \leq -\omega < a_1$, the following statements are valid.*

(i) The backward solution u of equation (D.29) *belongs to $\mathcal{C}^{\alpha,\omega}((-\infty,0],\mathcal{D}(A))$ and satisfies the estimate*

$$\|u\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{D}(A))} \leq C\big(\|u(0)\|_{\mathcal{D}(A)} + \|g\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{X})}\big).$$

The constant C depends on ω and α and further on the bounds for the analytical semigroup.

(ii) For $0 < h \leq \overline{h}$ sufficiently small, the backward Euler solution $\underline{u}$ of (D.29) *lies in $\mathcal{C}^{\alpha,\omega}(\mathcal{N},\mathcal{D}(A))$ and fulfils*

$$\|\underline{u}\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{D}(A))} \leq C\big(\|u_0\|_{\mathcal{D}(A)} + \|\underline{g}\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{X})}\big)$$

with constant C depending on the choices of ω and α and on the bounds for the stability function.

(iii) The Runge-Kutta solution $\underline{u}$ of equation (D.29) *belongs to $\mathcal{C}^{\alpha,\omega}(\mathcal{N},\mathcal{D}(A))$ and the stages fulfil $\underline{U} \in \mathcal{C}^{\alpha,\omega}(\mathcal{N},\mathcal{D}(A)^s)$. Moreover, the bound*

$$\|\underline{u}\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{D}(A))} + \|\underline{U}\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{D}(A)^s)} \leq C\big(\|u_0\|_{\mathcal{D}(A)} + \|\underline{g}\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{X}^s)}\big)$$

is valid for $0 < h \leq \overline{h}$ small enough. In particular, the constant C depends on the choices of ω and α and further on the bounds for the stability function of the method.

Assume that the initial values $u(0)$ and u_0 in $\mathcal{D}(A)$ fulfil Condition (D.34) *and that g belongs to $\mathcal{C}^{\alpha,\omega}_\alpha((0,\infty),\mathcal{X})$ for some $0 < \alpha < 1$ and $\omega \geq 0$ such that $-a_2 < -\omega \leq \omega < a_1$. For fixed $h > 0$ set $\mathcal{N} = \{t_n = nh : n \in \boldsymbol{N}\}$. Then, the following statements hold.*

(i) The analytical solution u of equation (D.29) *belongs to $\mathcal{C}^{\alpha,\omega}_\alpha((0,\infty),\mathcal{D}(A))$ and satisfies the estimate*

$$\|u\|_{\mathcal{C}^{\alpha,\omega}_\alpha(\mathcal{D}(A))} \leq C\big(\|u(0)\|_{\mathcal{D}(A)} + \|g\|_{\mathcal{C}^{\alpha,\omega}_\alpha(\mathcal{X})}\big).$$

The constant C depends on ω and α and further on the bounds for the analytical semigroup.

(ii) *For* $0 < h \le \overline{h}$ *sufficiently small, the backward Euler solution* $\underline{u}$ *of* (D.29) *lies in* $\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{N}, \mathcal{D}(A))$ *and fulfils*

$$\|\underline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D}(A))} \le C\big(\|u_0\|_{\mathcal{D}(A)} + \|\underline{g}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{X})}\big)$$

with constant C *depending on the choices of* ω *and* α *and on the bounds for the stability function.*

(iii) *The Runge-Kutta solution* $\underline{u}$ *of equation* (D.29) *belongs to* $\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{N}, \mathcal{D}(A))$ *and the stages fulfil* $\underline{U} \in \mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{N}, \mathcal{D}(A)^s)$. *Moreover, the bound*

$$\|\underline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D}(A))} + \|\underline{U}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D}(A)^s)} \le C\big(\|u_0\|_{\mathcal{D}(A)} + \|\underline{g}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{X}^s)}\big)$$

is valid for $0 < h \le \overline{h}$ *small enough. In particular, the constant* C *depends on the choices of* ω *and* α *and further on the bounds for the stability function of the method.*

Proof. For proving the above statements we primarily apply the ideas arising in the proofs of the preceding Lemmas D.5, D.6 and D.7. We omit the derivation. □

D.3. Bounds for integrals and Riemann-sums

In the sequel, we state the auxiliary estimates for integrals and Riemann-sums needed in the preceding Section D.2.

Estimate D.1. Fix some $0 < \alpha < 1$ and $0 < \varepsilon \le s < t$. Then, a simple calculation shows

$$\begin{aligned}\int_s^t \tau^{-1}\,\mathrm{d}\tau &= \int_s^t \tau^{-\alpha}\tau^{-1+\alpha}\,\mathrm{d}\tau \le \varepsilon^{-\alpha}\int_s^t \tau^{-1+\alpha}\,\mathrm{d}\tau = \varepsilon^{-\alpha}\int_0^{t-s}(\tau+s)^{-1+\alpha}\,\mathrm{d}\tau \\ &\le \varepsilon^{-\alpha}\int_0^{t-s}\tau^{-1+\alpha}\,\mathrm{d}\tau = \alpha^{-1}\cdot\varepsilon^{-\alpha}(t-s)^\alpha\end{aligned}$$

wherefore the following bound holds

$$\int_s^t \tau^{-1}\,\mathrm{d}\tau \le \alpha^{-1}\cdot\varepsilon^{-\alpha}(t-s)^\alpha. \tag{D.35}$$

For the Beta-function, the common abbreviation

$$B(\alpha, 1-\alpha) = \int_0^1 (1-\xi)^{-1+\alpha}\xi^{-\alpha}\,\mathrm{d}\xi = \int_0^t (t-\tau)^{-1+\alpha}\tau^{-\alpha}\,\mathrm{d}\tau < \infty \tag{D.36}$$

is used. From the relation

$$\int_s^t (t-\tau)^{-1+\alpha}\tau^{-\alpha}\,\mathrm{d}\tau \le \varepsilon^{-\alpha}\int_s^t (t-\tau)^{-1+\alpha}\,\mathrm{d}\tau = \alpha^{-1}\cdot\varepsilon^{-\alpha}(t-s)^\alpha$$

we further receive the estimate

$$\int_s^t (t-\tau)^{-1+\alpha}\tau^{-\alpha}\,\mathrm{d}\tau \le \alpha^{-1}\cdot\varepsilon^{-\alpha}(t-s)^{\alpha}. \tag{D.37}$$

Moreover, for all times $0<\varepsilon\le s<t\le T$ the estimate

$$\int_0^s \int_{s-\tau}^{t-\tau} \sigma^{-2}(s-\tau)^{\alpha}\tau^{-\alpha}\,\mathrm{d}\sigma\,\mathrm{d}\tau \le (\alpha(1-\alpha))^{-1}\cdot\varepsilon^{-\alpha}(t-s)^{\alpha} \tag{D.38}$$

is valid. Due to the fact that the relation

$$\begin{aligned}
\int_0^s \int_{s-\tau}^{t-\tau} \sigma^{-2}(s-\tau)^{\alpha}\tau^{-\alpha}\,\mathrm{d}\sigma\,\mathrm{d}\tau &\le \int_0^s \int_{s-\tau}^{t-\tau} \sigma^{-2+\alpha}\tau^{-\alpha}\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&= (1-\alpha)^{-1}\left(\int_0^s (s-\tau)^{-1+\alpha}\tau^{-\alpha}\,\mathrm{d}\tau - \int_0^s (t-\tau)^{-1+\alpha}\tau^{-\alpha}\,\mathrm{d}\tau\right) \\
&= (1-\alpha)^{-1}\int_s^t (t-\tau)^{-1+\alpha}\tau^{-\alpha}\,\mathrm{d}\tau
\end{aligned}$$

holds, the desired result follows by means of (D.37). In particular, the estimate

$$\int_0^s \int_{s-\tau}^{t-\tau} e^{-\omega\sigma}\sigma^{-2}(s-\tau)^{\alpha}\tau^{-\alpha}\,\mathrm{d}\sigma\,\mathrm{d}\tau \le (\alpha(1-\alpha))^{-1}\cdot\varepsilon^{-\alpha}(t-s)^{\alpha} \tag{D.39}$$

is valid for $\omega>0$ and arbitrary $0<\varepsilon\le s<t$.

The following result provides the corresponding estimates for the discrete case.

Estimate D.2. Choose $0<\alpha<1$ and $n_1>0$. Let further $0<t_{n_1}\le t_m<t_n$ for stepsizes $0<h\le\overline{h}$ chosen sufficiently small. Then, the estimate

$$h\sum_{\nu=m}^{n-1} t_{\nu+1}^{-1} \le \alpha^{-1}\cdot t_{n_1}^{-\alpha}(t_n-t_m)^{\alpha} \tag{D.40}$$

follows from (D.35) by estimating the sum in a standard way by an integral

$$h\sum_{\nu=m}^{n-1} t_{\nu+1}^{-1} \le \int_{t_m}^{t_n} \tau^{-1}\,\mathrm{d}\tau \le \alpha^{-1}\cdot t_{n_1}^{-\alpha}(t_n-t_m)^{\alpha}.$$

In a similar manner, we receive

$$h^2\sum_{\mu=m}^{n-1} t_{\mu+1}^{-2} \le t_{n_1}^{-\alpha}(t_n-t_m)^{\alpha} \tag{D.41}$$

with the help of the relation

$$h^2 \sum_{\mu=m}^{n-1} t_{\mu+1}^{-2} \leq h \int_{t_m}^{t_n} \tau^{-2}\,\mathrm{d}\tau = \frac{h(t_n - t_m)}{t_n t_m} \leq t_{n_1}^{-\alpha}(t_n - t_m)^{\alpha}.$$

Further, we have the following bound

$$h \sum_{\nu=0}^{n-1} (t_n - t_\nu)^{-1}(t_n - t_{\nu+1})^{\alpha} t_{\nu+1}^{-\alpha} \leq B(\alpha, 1-\alpha). \tag{D.42}$$

In order to derive this bound, we consider the function $\tau \mapsto (t_n - \tau)^{-1+\alpha}\tau^{-\alpha}$, and we denote by t^* its minimum in $(0, t_n)$. We choose $p \in \boldsymbol{N}$ such that $t_{p-1} \leq t^* < t_p$. Then, we get

$$\begin{aligned}
h \sum_{\nu=0}^{n-1} (t_n - t_\nu)^{-1}(t_n - t_{\nu+1})^{\alpha} t_{\nu+1}^{-\alpha} &\leq h \sum_{\nu=0}^{p-1} (t_n - t_{\nu+1})^{-1+\alpha} t_{\nu+1}^{-\alpha} + h \sum_{\nu=p}^{n-1} (t_n - t_\nu)^{-1+\alpha} t_\nu^{-\alpha} \\
&\leq \int_0^{t_{p-1}} (t_n - \tau)^{-1+\alpha}\tau^{-\alpha}\,\mathrm{d}\tau + \int_{t_p}^{t_n} (t_n - \tau)^{-1+\alpha}\tau^{-\alpha}\,\mathrm{d}\tau \leq B(\alpha, 1-\alpha),
\end{aligned}$$

which gives the desired result, see also Figure D.2. Besides, Relation (D.37) implies

$$h \sum_{\nu=m}^{n-1} (t_n - t_\nu)^{-1}(t_n - t_{\nu+1})^{\alpha} t_{\nu+1}^{-\alpha} \leq \alpha^{-1} \cdot t_{n_1}^{-\alpha}(t_n - t_m)^{\alpha}. \tag{D.43}$$

For deriving the following bound

$$h^2 \sum_{\nu=0}^{m-1} \sum_{\mu=m-\nu}^{n-\nu-1} t_{\mu+1}^{-2}(t_m - t_{\nu+1})^{\alpha} t_{\nu+1}^{-\alpha} \leq (\alpha(1-\alpha))^{-1} \cdot t_{n_1}^{-\alpha}(t_n - t_m)^{\alpha}, \tag{D.44}$$

we essentially apply the ideas arising in the proof of Relation (D.38). Remark that

$$\begin{aligned}
h^2 \sum_{\nu=0}^{m-1} \sum_{\mu=m-\nu}^{n-\nu-1} t_{\mu+1}^{-2}(t_m - t_{\nu+1})^{\alpha} t_{\nu+1}^{-\alpha} &\leq h \sum_{\nu=0}^{m-1} \int_{t_m - t_\nu}^{t_n - t_\nu} \sigma^{-2}(t_m - t_{\nu+1})^{\alpha} t_{\nu+1}^{-\alpha}\,\mathrm{d}\sigma \\
&\leq h \sum_{\nu=0}^{m-1} \int_{t_m - t_\nu}^{t_n - t_\nu} \sigma^{-2+\alpha} t_{\nu+1}^{-\alpha}\,\mathrm{d}\sigma \\
&= (1-\alpha)^{-1}\left(h \sum_{\nu=0}^{m-1} (t_m - t_\nu)^{-1+\alpha} t_{\nu+1}^{-\alpha} - h \sum_{\nu=0}^{n-1} (t_n - t_\nu)^{-1+\alpha} t_{\nu+1}^{-\alpha} \right) \\
&\quad + (1-\alpha)^{-1} h \sum_{\nu=m}^{n-1} (t_n - t_\nu)^{-1+\alpha} t_{\nu+1}^{-\alpha} \\
&\leq (1-\alpha)^{-1} \int_{t_m}^{t_n} (t_n - \tau)^{-1+\alpha}\tau^{-\alpha}\,\mathrm{d}\tau.
\end{aligned}$$

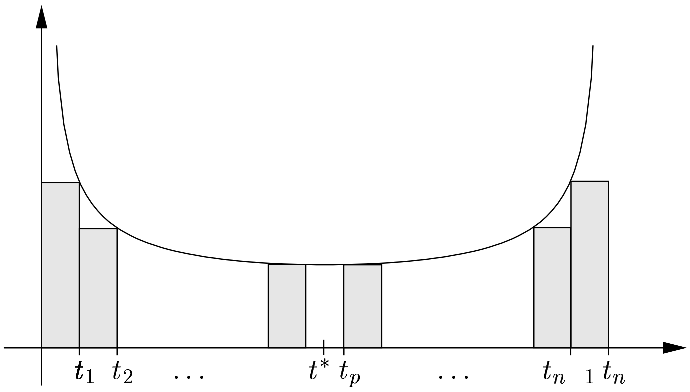

Figure D.2.: Integral and associated Riemann-sum

Now, the previous result (D.43) yields the desired estimate.

The following estimates are useful for proving the results stated in Section D.2.3.

Estimate D.3. Fix some $0 < \alpha < 1$ and $\omega > 0$. Then, for all $s < t \leq 0$ the following estimates hold. On the one hand, we have

$$\left| \int_t^s e^{\omega\tau} \, \mathrm{d}\tau \right| \leq \alpha^{-1} \omega^{-1+\alpha} (t-s)^\alpha$$

since the relation $e^{-\omega x} \leq (\omega x)^{-1+\alpha}$ for $x \geq 0$ implies

$$\begin{aligned} \left| \int_t^s e^{\omega\tau} \, \mathrm{d}\tau \right| &= \int_{-t}^{-s} e^{-\omega\tau} \, \mathrm{d}\tau = \int_{|t|}^{|s|} e^{-\omega\tau} \, \mathrm{d}\tau \\ &\leq \omega^{-1+\alpha} \int_0^{|s|-|t|} \tau^{-1+\alpha} \, \mathrm{d}\tau = \alpha^{-1} \omega^{-1+\alpha} (t-s)^\alpha. \end{aligned}$$

For $\beta > 0$ the Gamma-function is defined through

$$\Gamma(\beta) = \int_0^\infty e^{-\xi} \xi^{-1+\beta} \, \mathrm{d}\xi < \infty.$$

Due to the fact that

$$\left| \int_0^t e^{\omega(t-\tau)} (\tau - t)^\alpha \, \mathrm{d}\tau \right| = \int_0^{|t|} e^{-\omega\tau} \tau^\alpha \, \mathrm{d}\tau \leq \int_0^\infty e^{-\omega\tau} \tau^\alpha \, \mathrm{d}\tau = \omega^{-1-\alpha} \Gamma(1+\alpha)$$

we therefore obtain

$$\left| \int_0^t e^{\omega(t-\tau)} (\tau - t)^\alpha \, \mathrm{d}\tau \right| \leq \omega^{-1-\alpha} \Gamma(1+\alpha).$$

Furthermore, we get

$$\left| \int_0^t \int_{t-\tau}^{s-\tau} e^{\omega\sigma} (\tau - t)^\alpha \, \mathrm{d}\sigma \, \mathrm{d}\tau \right| \leq \alpha^{-1} \omega^{-2} \Gamma(1+\alpha)(t-s)^\alpha$$

by means of the estimate

$$\begin{aligned} \left| \int_0^t \int_{t-\tau}^{s-\tau} e^{\omega\sigma} (\tau - t)^\alpha \, \mathrm{d}\sigma \, \mathrm{d}\tau \right| &= \omega^{-1} \big(1 - e^{-\omega(t-s)}\big) \int_0^t e^{-\omega(\tau - t)} (\tau - t)^\alpha \, \mathrm{d}\tau \\ &\leq \alpha^{-1} \omega^{-1+\alpha} (t-s)^\alpha \int_0^\infty e^{-\omega\tau} \tau^\alpha \, \mathrm{d}\tau = \alpha^{-1} \omega^{-2} \Gamma(1+\alpha)(t-s)^\alpha. \end{aligned}$$

Let c_0 denote a bound of the function $xe^{-\omega x}$ on $[0, \infty)$. From the relation

$$\begin{aligned} \left| \int_t^s e^{\omega(s-\tau)} (\tau - s)^\alpha \, \mathrm{d}\tau \right| &= \int_s^t e^{-\omega(\tau - s)} (\tau - s)^\alpha \, \mathrm{d}\tau \leq \int_0^{t-s} \tau e^{-\omega\tau} \tau^{-1+\alpha} \, \mathrm{d}\tau \\ &\leq c_0 \int_0^{t-s} \tau^{-1+\alpha} \, \mathrm{d}\tau = \alpha^{-1} c_0 (t-s)^\alpha \end{aligned}$$

we consequently receive the following estimate

$$\left| \int_t^s e^{\omega(s-\tau)} (\tau - s)^\alpha \, \mathrm{d}\tau \right| \leq \alpha^{-1} c_0 (t-s)^\alpha.$$

Applying the substitution $\xi = \omega(t - \tau)$ immediately shows

$$\int_{-\infty}^t e^{-\omega(t-\tau)} (t - \tau)^{-1+\alpha} \, \mathrm{d}\tau = \omega^{-\alpha} \Gamma(\alpha).$$

Besides, the following estimates holds

$$\int_{-\infty}^s \int_{s-\tau}^{t-\tau} e^{-\omega\sigma} \sigma^{-2} (s - \tau)^\alpha \, \mathrm{d}\sigma \, \mathrm{d}\tau \leq (\alpha(1-\alpha))^{-1} (t-s)^\alpha.$$

For deriving this relation, we estimate the above integral by

$$\int_{-\infty}^s \int_{s-\tau}^{t-\tau} e^{-\omega\sigma} \sigma^{-2} (s - \tau)^\alpha \, \mathrm{d}\sigma \, \mathrm{d}\tau \leq \int_{-\infty}^s \int_{s-\tau}^{t-\tau} e^{-\omega\sigma} \sigma^{-2+\alpha} \, \mathrm{d}\sigma \, \mathrm{d}\tau,$$

and we further perform the transformation $\tau \to s - \tau$ and interchange the order of integration. This yields

$$\begin{aligned} \int_{-\infty}^s \int_{s-\tau}^{t-\tau} e^{-\omega\sigma} \sigma^{-2+\alpha} \, \mathrm{d}\sigma \, \mathrm{d}\tau &= \int_0^\infty \int_\tau^{\tau+t-s} e^{-\omega\sigma} \sigma^{-2+\alpha} \, \mathrm{d}\sigma \, \mathrm{d}\tau \\ &= \int_0^{t-s} \int_0^\sigma e^{-\omega\sigma} \sigma^{-2+\alpha} \, \mathrm{d}\tau \, \mathrm{d}\sigma + \int_{t-s}^\infty \int_{\sigma-(t-s)}^\sigma e^{-\omega\sigma} \sigma^{-2+\alpha} \, \mathrm{d}\tau \, \mathrm{d}\sigma \\ &= \int_0^{t-s} e^{-\omega\sigma} \sigma^{-1+\alpha} \, \mathrm{d}\sigma + (t-s) \int_{t-s}^\infty e^{-\omega\sigma} \sigma^{-2+\alpha} \, \mathrm{d}\sigma. \end{aligned}$$

Due to the fact that

$$\begin{aligned}\int_0^{t-s} e^{-\omega\sigma}\sigma^{-1+\alpha}\,\mathrm{d}\sigma + (t-s)\int_{t-s}^{\infty} e^{-\omega\sigma}\sigma^{-2+\alpha}\,\mathrm{d}\sigma&\\ \leq \int_0^{t-s}\sigma^{-1+\alpha}\,\mathrm{d}\sigma + (t-s)\int_{t-s}^{\infty}\sigma^{-2+\alpha}\,\mathrm{d}\sigma &= (\alpha(1-\alpha))^{-1}(t-s)^{\alpha},\end{aligned}$$

the desired result follows.

E. The nonlinear problem in semilinear form

In order to study the following nonlinear initial value problem of parabolic type

$$u'(t) = F(t, u(t)), \quad t > 0, \qquad u(0) \text{ given}, \tag{E.1}$$

it is advantageous to linearize the right-hand side of the equation around a point (t^*, v^*). This yields a formally semilinear problem on a Banach space

$$u'(t) = Au(t) + G(t, u(t)), \quad t > 0, \qquad u(0) \text{ given}, \tag{E.2a}$$

where the sectorial operator A and the map G are defined through

$$A = D_v F(t^*, v^*) \qquad \text{and} \qquad G(t, v) = F(t, v) - Av. \tag{E.2b}$$

The center of linearization (t^*, v^*) is chosen accordingly to the actual needs. Useful choices for v^* are for example the initial value or an equilibrium point of (E.1), see Section 5.2 and Chapter 7.

For our analysis of Runge-Kutta time discretizations for nonlinear parabolic problems of the form (E.1), it is crucial that Runge-Kutta methods are invariant under linearization, i.e. the Runge-Kutta solution of (E.2) is equivalent to the Runge-Kutta solution of the original problem (E.1). Due to this, techniques that are used for semilinear equations in order to establish existence and convergence results for Runge-Kutta methods can be applied to the fully nonlinear situation as well.

One of our main tools for proving for example the existence results Proposition 4.1 and Theorem 7.4 is a fixed point iteration in (E.2a). Therefore, we consider in the preceding Chapter D linear problems of the form

$$u'(t) = Au(t) + g(t), \quad t > 0, \qquad u(0) \text{ given}, \tag{E.3}$$

and we investigate the regularity of the exact and the numerical solution of (E.3) under different smoothness assumptions on g. The present chapter is devoted to the study of the map G from (E.2) and organized as follows.

At the beginning of Section E.1, we review the precise requirements on the nonlinearity F arising on the right-hand side of the nonlinear parabolic problem (E.1). Then, bounds for the α-Hölder and Lipschitz constant of the map G defined through (E.2b) are derived, see Section E.1.1. In Sections E.1.2 and E.1.3 we show that the (α,α)-Hölder continuity and the exponentially weighted Hölder continuity of a map v go over to the composition $G(\cdot,v)$, and we further give estimates for the difference between $G(\cdot,v)$ and $G(\cdot,w)$ in the associated norms. As already mentioned before, these results are repeatedly needed in Parts II and III for proving existence results for Runge-Kutta time discretizations of (E.1).

E.1. Properties of the nonlinearity on the right-hand side

Let $\mathcal{X}$ be a Banach space and $\mathcal{O} \subset \mathcal{D}$ an open subset of the densely embedded Banach space $\mathcal{D} \subset \mathcal{X}$. Henceforth, we always suppose that the map

$$F : [0,T] \times \mathcal{O} \longrightarrow \mathcal{X} : (t,v) \longmapsto F(t,v)$$

satisfies the requirements from Section 1.1.1, i.e. F and its Fréchet derivative $D_vF(t,v)$ are α-Hölder continuous in time and Lipschitz continuous in v. More precisely, we assume that for some $0 < \alpha < 1$ and $v^* \in \mathcal{O}$ there are positive constants $R > 0$, $K > 0$ and $L > 0$ such that the relations

$$\begin{aligned} \|F(t,v) - F(s,v)\|_{\mathcal{X}} + \|D_vF(t,v) - D_vF(s,v)\|_{\mathcal{X}\leftarrow\mathcal{D}} &\le K(t-s)^\alpha \\ \|D_vF(t,v) - D_vF(t,w)\|_{\mathcal{X}\leftarrow\mathcal{D}} &\le L\|v-w\|_{\mathcal{D}} \end{aligned} \tag{E.4}$$

hold for all $0 \le s < t \le T$ and $v,w \in \mathcal{O}$ with $\|v - v^*\|_{\mathcal{D}} \le R$ and $\|w - v^*\|_{\mathcal{D}} \le R$. For some element $(t^*,v^*) \in [0,T] \times \mathcal{O}$ we define the map G through

$$G : [0,T] \times \mathcal{O} \longrightarrow \mathcal{X} : (t,v) \longmapsto G(t,v) = F(t,v) - Av \quad \text{with } A = D_vF(t^*,v^*). \tag{E.5}$$

The properties of G derived in Section E.1.1 are a direct consequence of the smoothness assumptions (E.4) on F.

E.1.1. Local α-Hölder and Lipschitz continuity

The first lemma states that $G(t,v)$ is α-Hölder continuous in time and locally Lipschitz-continuous in v with a small Lipschitz constant for all arguments lying sufficiently close to the center of linearization.

Lemma E.1. *For every $R > 0$ there exist constants $K > 0$ and $L > 0$ such that*

$$\|G(t,v) - G(s,v)\|_{\mathcal{X}} \le K(t-s)^\alpha$$

for all $0 \le s < t \le T$ and $v \in \mathcal{O}$. Moreover, for $t \in [0,T]$ and $v,w \in \mathcal{O}$ the estimate

$$\|G(t,v) - G(t,w)\|_{\mathcal{X}} \le \big(K\delta^\alpha + L\varrho\big)\|v-w\|_{\mathcal{D}}$$

is valid whenever $|t-t^*| \le \delta$ as well as $\|v - v^*\|_{\mathcal{D}} \le \varrho \le R$ and $\|w - v^*\|_{\mathcal{D}} \le \varrho \le R$.

Proof. The first bound follows at once from the α-Hölder continuity of F in time

$$\|G(t,v)-G(s,v)\|_{\mathcal{X}} = \|F(t,v)-F(s,v)\|_{\mathcal{X}} \le K(t-s)^{\alpha}.$$

The bound (E.4) also implies that there are constants $K>0$ and $L>0$ such that

$$\begin{aligned}
\|G(t,v)-G(t,w)\|_{\mathcal{X}} &\le \int_0^1 \big\|D_vG(t,\tau v+(1-\tau)w)\big\|_{\mathcal{X}\leftarrow\mathcal{D}}\|v-w\|_{\mathcal{D}}\,\mathrm{d}\tau\\
&= \int_0^1 \big\|D_vF(t,\tau v+(1-\tau)w)-D_vF(t^*,v^*)\big\|_{\mathcal{X}\leftarrow\mathcal{D}}\|v-w\|_{\mathcal{D}}\,\mathrm{d}\tau\\
&\le \int_0^1 \Big(K|t-t^*|^{\alpha}+L\big(\tau\|v-v^*\|_{\mathcal{D}}+(1-\tau)\|w-v^*\|_{\mathcal{D}}\big)\Big)\|v-w\|_{\mathcal{D}}\,\mathrm{d}\tau\\
&\le \big(K\delta^{\alpha}+L\varrho\big)\|v-w\|_{\mathcal{D}}
\end{aligned}$$

for all $t\in[0,T]$ with $|t-t^*|\le\delta$ and for $v,w\in\mathcal{O}$ with $\|v-v^*\|_{\mathcal{D}}\le\varrho\le R$ and $\|w-v^*\|_{\mathcal{D}}\le\varrho\le R$. $\square$

E.1.2. (α,α)-Hölder continuity on bounded time intervals

In the present subsection, we study the composition $G(\cdot,v)$ under the assumption that the map v is (α,α)-Hölder continuous on $(0,T]$ with values in $\mathcal{O}$. We prove (α,α)-Hölder continuity of $G(\cdot,v)$ in $\mathcal{X}$, and we further show that the Lipschitz constant of $G(\cdot,v)$ in the space $\mathcal{C}^{\alpha}_{\alpha}((0,T],\mathcal{X})$ is arbitrarily small, for T small enough and functions v lying sufficiently close to the element v^*. From the continuous case the corresponding results for sequences

$$G(\underline{t},\underline{v}) = \big(G(t_n,v_n)\big)_{n\in\mathcal{N}}, \qquad \underline{v}=(v_n)_{n\in\mathcal{N}},$$

follow easily. For the remainder of this section we fix $h>0$, and we define $N\in\boldsymbol{N}$ through $t_N\le T<t_{N+1}$. Let $\mathcal{N}=\{t_n=nh : n\in\boldsymbol{N},\ 0<n\le N\}$.

Lemma E.2. *(i) Whenever $v\in\mathcal{C}^{\alpha}_{\alpha}((0,T],\mathcal{D})$ for some $0<\alpha<1$ and $v(t)\in\mathcal{O}$ for all $0\le t\le T$, then the map $G(\cdot,v)$ belongs to $\mathcal{C}^{\alpha}_{\alpha}((0,T],\mathcal{X})$.*

(ii) Fix some $0<\alpha<1$. Then, for every $\underline{v}\in\mathcal{O}^N$ the sequence $G(\underline{t},\underline{v})$ lies in $\mathcal{C}^{\alpha}_{\alpha}(\mathcal{N},\mathcal{X})$.

Proof. (i)(a) For every $0\le t\le T$ the element $G(t,v(t))$ is bounded in $\mathcal{X}$ independently of the time t

$$\begin{aligned}
\|G(t,v(t))\|_{\mathcal{X}} &\le \|F(t,v(t))-F(t^*,v(t))\|_{\mathcal{X}}+\|F(t^*,v(t))-F(t^*,v^*)\|_{\mathcal{X}}\\
&\qquad+\|F(t^*,v^*)\|_{\mathcal{X}}+\|A\|_{\mathcal{X}\leftarrow\mathcal{D}}\|v(t)\|_{\mathcal{D}}\\
&\le K|t-t^*|^{\alpha}+L\|v(t)-v^*\|_{\mathcal{D}}+\|F(t^*,v^*)\|_{\mathcal{X}}+\|A\|_{\mathcal{X}\leftarrow\mathcal{D}}\|v(t)\|_{\mathcal{D}}\\
&\le KT^{\alpha}+L\|v-v^*\|_{\mathcal{B}(\mathcal{D})}+\|F(t^*,v^*)\|_{\mathcal{X}}+\|A\|_{\mathcal{X}\leftarrow\mathcal{D}}\|v\|_{\mathcal{B}(\mathcal{D})}.
\end{aligned}$$

(b) Choose $0 < \varepsilon \le s < t \le T$. A calculation, similar to the one in the proof of Lemma E.1, shows

$$\begin{aligned}
\|G(t,v(t)) - G(s,v(s))\|_{\mathcal{X}} &\le \|G(t,v(t)) - G(s,v(t))\|_{\mathcal{X}} + \|G(s,v(t)) - G(s,v(s))\|_{\mathcal{X}} \\
&\le K(t-s)^{\alpha} + \int_0^1 \Big(K|t-t^*|^{\alpha} + L\big(\tau\|v(t)-v^*\|_{\mathcal{D}} + (1-\tau)\|v(s)-v^*\|_{\mathcal{D}}\big)\Big)\,\mathrm{d}\tau \\
&\qquad \cdot \|v(t)-v(s)\|_{\mathcal{D}} \\
&\le \Big(KT^{\alpha} + \big(KT^{\alpha} + L\|v-v^*\|_{\mathcal{B}(\mathcal{D})}\big)[v]_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{D})}\Big)\varepsilon^{-\alpha}(t-s)^{\alpha}.
\end{aligned}$$

From these considerations the first statement of the lemma follows.

(ii) Every sequence $\underline{v} \in \mathcal{O}^N$ belongs to $\mathcal{C}^{\alpha}_{\alpha}(\mathcal{N},\mathcal{D})$, and thus the corresponding result is also true for $G(\underline{t},\underline{v})$. $\square$

The following lemma provides estimates for the quantities

$$\|G(\cdot,v) - G(\cdot,w)\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{X})} \qquad \text{and} \qquad \|G(\underline{t},\underline{v}) - G(\underline{t},\underline{w})\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{X})}$$

in terms of $\|v-w\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{D})}$ and $\|\underline{v}-\underline{w}\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{D})}$, respectively. Notice that our requirements for the continuous and the discrete case are somehow different. In the first part of Lemma E.3 we assume that the (α,α)-Hölder norms relatively to v^* of both arguments v and w are small. For sequences $\underline{v}$ and $\underline{w}$, in view of the convergence results stated in Section 5.1, it is more natural to require only that the quantities $\|\underline{v}-\underline{v}^*\|_{\mathcal{B}(\mathcal{D})}, \|\underline{w}-\underline{v}^*\|_{\mathcal{B}(\mathcal{D})}$ and $[w-\underline{v}^*]_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{D})}$ are small.

Lemma E.3. *For some $R>0$ let $K,L>0$ denote the constants from relation* (E.4)*, and choose $0<\varrho\le R$.*

(i) Then, the following estimate is valid

$$\|G(\cdot,v) - G(\cdot,w)\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{X})} \le 2(KT^{\alpha} + L\varrho)\|v-w\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{D})}$$

for all $v,w \in \mathcal{C}^{\alpha}_{\alpha}((0,T],\mathcal{D})$ with $\|v-v^*\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{D})} \le \varrho$ and $\|w-v^*\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{D})} \le \varrho$ provided that $v(t),w(t) \in \mathcal{O}$ for $0\le t\le T$.

(ii) Whenever $\underline{v},\underline{w} \in \mathcal{O}^N$ are such that $\|\underline{v}-\underline{v}^*\|_{\mathcal{B}(\mathcal{D})} \le \varrho$ and $\|\underline{w}-\underline{v}^*\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{D})} \le \varrho$, then it holds

$$\|G(\underline{t},\underline{v}) - G(\underline{t},\underline{w})\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{X})} \le 2(KT^{\alpha} + L\varrho)\|\underline{v}-\underline{w}\|_{\mathcal{C}^{\alpha}_{\alpha}(\mathcal{D})}.$$

Proof. (i) We know from Lemma E.2 that $G(\cdot,v)$ belongs to $\mathcal{C}^{\alpha}_{\alpha}((0,T],\mathcal{X})$ provided that $v \in \mathcal{C}^{\alpha}_{\alpha}((0,T],\mathcal{D})$. Thus the map

$$H = G(\cdot,v) - G(\cdot,w)$$

is (α, α)-Hölder continuous on $(0, T]$ with values in $\mathcal{X}$ if $v, w \in \mathcal{C}_\alpha^\alpha((0,T], \mathcal{D})$. For the following we introduce the map

$$z : t \longmapsto z(\tau, t) = \tau v(t) + (1-\tau) w(t),$$

for some $0 \leq \tau \leq 1$. With this notation we have

$$\begin{aligned} H(t) &= \int_0^1 D_v G(t, z(\tau, t))\big(v(t) - w(t)\big)\,\mathrm{d}\tau \\ &= \int_0^1 \big(D_v F(t, z(\tau, t)) - D_v F(t^*, v^*)\big)\big(v(t) - w(t)\big)\,\mathrm{d}\tau. \end{aligned}$$

(a) With the help of the bound

$$\|z(\tau, t) - v^*\|_{\mathcal{D}} \leq \tau \|v(t) - v^*\|_{\mathcal{D}} + (1-\tau)\|w(t) - v^*\|_{\mathcal{D}} \leq \varrho$$

we consequently get for $0 \leq t \leq T$

$$\begin{aligned} \|H(t)\|_{\mathcal{X}} &\leq \int_0^1 \big(K|t_n - t^*|^\alpha + L\|z(\tau, t) - v^*\|_{\mathcal{D}}\big)\|v(t) - w(t)\|_{\mathcal{D}}\,\mathrm{d}\tau \\ &\leq \big(KT^\alpha + L\varrho\big)\|v - w\|_{\mathcal{B}(\mathcal{D})}. \end{aligned}$$

(b) Assume $0 < \varepsilon \leq s < t \leq T$. Then, we have

$$\begin{aligned} H(t) - H(s) &= \int_0^1 D_v G(t, z(\tau, t))\big(v(t) - w(t)\big) + D_v G(t, z(\tau, s))\big(v(s) - w(s)\big)\,\mathrm{d}\tau \\ &= \int_0^1 \big(D_v G(t, z(\tau, t)) - D_v G(s, z(\tau, s))\big)\big(v(t) - w(t)\big)\,\mathrm{d}\tau \\ &\quad + \int_0^1 D_v G(s, z(\tau, s))\big(v(t) - w(t) - v(s) + w(s)\big)\,\mathrm{d}\tau \\ &= \int_0^1 \big(D_v F(t, z(\tau, t)) - D_v F(s, z(\tau, s))\big)\big(v(t) - w(t)\big)\,\mathrm{d}\tau \\ &\quad + \int_0^1 \big(D_v F(s, z(\tau, s)) - D_v F(t^*, v^*)\big)\big(v(t) - w(t) - v(s) + w(s)\big)\,\mathrm{d}\tau \end{aligned}$$

and further

$$\begin{aligned} \|H(t) - H(s)\|_{\mathcal{X}} &\leq \int_0^1 \big(K(t-s)^\alpha + L\|z(\tau, t) - z(\tau, s)\|_{\mathcal{D}}\big)\|v(t) - w(t)\|_{\mathcal{D}}\,\mathrm{d}\tau \\ &\quad + \int_0^1 \big(K|s - t^*|^\alpha + L\|z(\tau, s) - v^*\|_{\mathcal{D}}\big)\|(v(t) - w(t)) - (v(s) - w(s))\|_{\mathcal{D}}\,\mathrm{d}\tau. \end{aligned}$$

Note that from $\|v(t)-v(s)\|_{\mathcal{D}} = \|(v(t)-v^*)-(v(s)-v^*)\|_{\mathcal{D}} \le \varrho\,(t-s)^\alpha \varepsilon^{-\alpha}$ the bound

$$\|z(\tau,t)-z(\tau,s)\|_{\mathcal{D}} \le \varrho\,(t-s)^\alpha \varepsilon^{-\alpha}$$

follows. Therefore we receive

$$\begin{aligned}\|H(t)-H(s)\|_{\mathcal{X}} &\le \big(KT^\alpha + L\varrho\big)\big(\|v-w\|_{\mathcal{B}(\mathcal{D})} + [v-w]_{\mathcal{C}^\alpha_\alpha(\mathcal{D})}\big)\varepsilon^{-\alpha}(t-s)^\alpha \\ &= \big(KT^\alpha + L\varrho\big)\|v-w\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})}\varepsilon^{-\alpha}(t-s)^\alpha.\end{aligned}$$

(ii) Remark that the following considerations partially differ from the continuous case. For $\underline{v}, \underline{w} \in \mathcal{O}^N$ the sequence $\underline{H} = (H_n)_{n=1}^N$, defined through

$$\underline{H} = G(\underline{t}, \underline{v}) - G(\underline{t}, \underline{w}),$$

belongs to $\mathcal{C}^\alpha_\alpha(\mathcal{N}, \mathcal{X})$. Following the ideas from part (i), we let

$$\underline{z} = (z_n)_{n=1}^N \qquad \text{with } z_n = \tau v_n + (1-\tau) w_n.$$

With this, the components of $\underline{H}$ read as

$$H_n = \int_0^1 D_v G(t_n, z_n)\big(v_n - w_n\big)\,\mathrm{d}\tau = \int_0^1 \big(D_v F(t_n, z_n) - D_v F(t^*, v^*)\big)\big(v_n - w_n\big)\,\mathrm{d}\tau.$$

(a) Using that $\|z_n - v^*\|_{\mathcal{D}} \le \varrho$ we therefore receive for $0 \le nh \le T$

$$\|H_n\|_{\mathcal{X}} \le \big(KT^\alpha + L\varrho\big)\|\underline{v} - \underline{w}\|_{\mathcal{B}(\mathcal{D})}.$$

(b) Similar to before we have for $0 < n_1 h \le mh < nh \le T$

$$\begin{aligned}\|H_n - H_m\|_{\mathcal{X}} \le \int_0^1 &\big(K(t_n - t_m)^\alpha + L\|z_n - z_m\|_{\mathcal{D}}\big)\|v_n - w_n\|_{\mathcal{D}}\,\mathrm{d}\tau \\ + \int_0^1 &\big(K|t_m - t^*|^\alpha + L\|z_n - v^*\|_{\mathcal{D}}\big)\|(v_n - w_n) - (v_m - w_m)\|_{\mathcal{D}}\,\mathrm{d}\tau.\end{aligned}$$

Note that we use in this case that

$$\begin{aligned}\|z_n - z_m\|_{\mathcal{D}} &\le \|w_n - w_m\|_{\mathcal{D}} + \tau\|(v_n - w_n) - (v_m - w_m)\|_{\mathcal{D}} \\ &\le \big(\varrho + \tau[v-w]_{\mathcal{C}^\alpha_\alpha(\mathcal{D})}\big)t_{n_1}^{-\alpha}(t_n - t_m)^\alpha.\end{aligned}$$

Together with the estimate $\|v-w\|_{\mathcal{B}(\mathcal{D})} \le \|v - v^*\|_{\mathcal{B}(\mathcal{D})} + \|w - v^*\|_{\mathcal{B}(\mathcal{D})} \le 2\varrho$ the bound

$$\begin{aligned}\|H_n - H_m\|_{\mathcal{X}} &\le \Big(KT^\alpha + L\big(\varrho + 1/2[v-w]_{\mathcal{C}^\alpha_\alpha(\mathcal{D})}\big)\Big)\|v-w\|_{\mathcal{B}(\mathcal{D})}t_{n_1}^{-\alpha}(t_n - t_m)^\alpha \\ &\quad + \big(KT^\alpha + L\varrho\big)[v-w]_{\mathcal{C}^\alpha_\alpha(\mathcal{D})}t_{n_1}^{-\alpha}(t_n - t_m)^\alpha \\ &\le \Big(\big(KT^\alpha + L\varrho\big)\|v-w\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D})} + L\varrho[v-w]_{\mathcal{C}^\alpha_\alpha(\mathcal{D})}\Big)t_{n_1}^{-\alpha}(t_n - t_m)^\alpha\end{aligned}$$

follows. This gives the desired result. $\square$

E.1.3. Exponentially weighted Hölder continuity on half lines

In the present section, we consider problem (E.1) near an equilibrium $\overline{u} \in \mathcal{O}$, and we assume that F does not depend explicitly on t. In this situation, it is convenient to linearize the right-hand side of equation (E.1) around the equilibrium point. Thus, the definition of the map G now simplifies to

$$G(v) = F(v) - Av \qquad \text{with} \quad A = F'(\overline{u}),$$

see (E.5). Further, the local Lipschitz condition (E.4) specializes to

$$\|F'(v) - F'(w)\|_{\mathcal{X}\leftarrow\mathcal{D}} \le L\|v - w\|_{\mathcal{D}} \qquad \text{for} \quad \|v - \overline{u}\|_{\mathcal{D}}, \|w - \overline{u}\|_{\mathcal{D}} \le R. \tag{E.6}$$

Hence, the quantity $0 < \alpha < 1$ arising in the Lemmas E.4 and E.5 below, is no longer determined by the properties of F.

In the following auxiliary results we study the composition

$$t \longmapsto e^{\omega t} G(v(t))$$

either on the positive half line for $\omega \ge 0$, or on the negative half line for $\omega \le 0$. Choose $h > 0$. For sequences

$$\bigl(e^{\omega t_n} G(v_n)\bigr)_{n\in\mathcal{N}}, \qquad \underline{v} = (v_n)_{n\in\mathcal{N}},$$

corresponding to the first case where $\omega \ge 0$, we henceforth set $\mathcal{N} = \{t_n = nh : n \in \boldsymbol{N}\}$ and likewise $\mathcal{N} = \{t_n = nh : n \in \boldsymbol{Z} : n \le 0\}$ for the second case. Notice that in the present situation the Hölder continuity of sequences does no longer follow at once. We further employ the notation

$$\underline{v} - \underline{\overline{u}} = (v_n - \overline{u})_{n\in\mathcal{N}} \qquad \text{and} \qquad G(\underline{v}) = \bigl(G(v_n)\bigr)_{n\in\mathcal{N}}.$$

We show next that the exponentially weighted Hölder continuity of v and $\underline{v}$ go over to $G(v)$ and $G(\underline{v})$.

Lemma E.4. *Fix some $0 < \alpha < 1$.*

(i) Let v and $v - \overline{u}$ belong to $\mathcal{C}_\alpha^{\alpha,\omega}((0,\infty),\mathcal{D})$ for some $\omega \ge 0$, and assume $v(t) \in \mathcal{O}$ for all $t \ge 0$. Then, the composition $G(v)$ lies in $\mathcal{C}_\alpha^{\alpha,\omega}((0,\infty),\mathcal{X})$.

If $v, v - \overline{u} \in \mathcal{C}^{\alpha,\omega}((-\infty,0],\mathcal{D})$ for some $\omega \le 0$ and $v(t) \in \mathcal{O}$ for $t \le 0$, then $G(v) \in \mathcal{C}^{\alpha,\omega}((-\infty,0],\mathcal{X})$.

(ii) Suppose that $\underline{v}$ and $\underline{v} - \underline{\overline{u}}$ belong to $\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{N},\mathcal{D})$ for some $\omega \ge 0$ and that $v_n \in \mathcal{O}$ holds for all $n \in \boldsymbol{N}$ such that $t_n \in \mathcal{N} = \{t_n = nh : n \in \boldsymbol{N}\}$. Then, the sequence $G(\underline{v})$ lies in $\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{N},\mathcal{X})$.

Whenever $\underline{v}, \underline{v} - \underline{\overline{u}} \in \mathcal{C}^{\alpha,\omega}(\mathcal{N},\mathcal{D})$ for some $\omega \le 0$ and $v_n \in \mathcal{O}$ for all integers n such that $t_n \in \mathcal{N} = \{t_n = nh : n \in \boldsymbol{Z}, n \le 0\}$, then $G(\underline{v}) \in \mathcal{C}^{\alpha,\omega}(\mathcal{N},\mathcal{X})$.

Proof. (i) Let L_0 denote the Lipschitz constant of F, i.e. it holds

$$\|F(v)-F(w)\|_{\mathcal{X}} \le L_0\|v-w\|_{\mathcal{D}} \qquad \text{if} \quad \|v-\overline{u}\|_{\mathcal{D}} \le R \quad \text{and} \quad \|w-\overline{u}\|_{\mathcal{D}} \le R.$$

We first consider the positive half line $(0,\infty)$.

(a) By using the relation $F(\overline{u})=0$ we receive for every $t \ge 0$

$$\begin{aligned}\big\|e^{\omega t}G(v(t))\big\|_{\mathcal{X}} &\le \|e^{\omega t}\big(F(v(t))-F(\overline{u})\big)\|_{\mathcal{X}} + \big\|Ae^{\omega t}v(t)\big\|_{\mathcal{X}}\\ &\le L_0\big\|e^{\omega t}\big(v(t)-\overline{u}\big)\big\|_{\mathcal{D}} + \big\|A\big\|_{\mathcal{X}\leftarrow\mathcal{D}}\big\|e^{\omega t}v(t)\big\|_{\mathcal{D}}\\ &\le L_0\|v-\overline{u}\|_{\mathcal{B}^\omega(\mathcal{D})} + \|A\|_{\mathcal{X}\leftarrow\mathcal{D}}\|v\|_{\mathcal{B}^\omega(\mathcal{D})}.\end{aligned}$$

(b) For all times $0<\varepsilon\le s<t$ the identity

$$\begin{aligned}e^{\omega t}G(v(t))-e^{\omega s}G(v(s)) = \big(e^{\omega t}-e^{\omega s}\big)F(v(t)) + e^{\omega s}\big(F(v(t))-F(v(s))\big)\\ - A\big(e^{\omega t}v(t)-e^{\omega s}v(s)\big)\end{aligned}$$

is valid. From this, we receive by adding $F(\overline{u})$

$$\begin{aligned}\big\|e^{\omega t}G(v(t))-e^{\omega s}G(v(s))\big\|_{\mathcal{X}} &\le \big|e^{\omega t}-e^{\omega s}\big|\|F(v(t))-F(\overline{u})\|_{\mathcal{X}}\\ &\quad + \|e^{\omega s}\big(F(v(t))-F(v(s))\big)\|_{\mathcal{X}} + \|A\|_{\mathcal{X}\leftarrow\mathcal{D}}\big\|e^{\omega t}v(t)-e^{\omega s}v(s)\big\|_{\mathcal{D}}\\ &\le L_0\big(1-e^{-\omega(t-s)}\big)\big\|e^{\omega t}(v(t)-\overline{u})\big\|_{\mathcal{D}} + L_0\big\|e^{\omega s}\big(v(t)-v(s)\big)\big\|_{\mathcal{D}}\\ &\quad + \|A\|_{\mathcal{X}\leftarrow\mathcal{D}}[v]_{\mathcal{C}^{\alpha,\omega}(\mathcal{D})}\varepsilon^{-\alpha}(t-s)^\alpha.\end{aligned}$$

Due to the validity of $e^{-x}\le x^{-1+\alpha}$ for $x\ge 0$ we have

$$1-e^{-x} = \int_0^x e^{-\xi}\,\mathrm{d}\xi \le \int_0^x \xi^{-1+\alpha}\,\mathrm{d}\xi = \alpha^{-1}x^\alpha, \qquad x\ge 0.$$

This relation implies the bound

$$1-e^{-\omega(t-s)} \le \alpha^{-1}\omega^{-\alpha}(t-s)^\alpha.$$

From this we further receive

$$\begin{aligned}\big\|e^{\omega s}\big(v(t)-v(s)\big)\big\|_{\mathcal{D}} &\le \big\|e^{\omega t}v(t)-e^{\omega s}v(s)\big\|_{\mathcal{D}} + \big|1-e^{-\omega(t-s)}\big|\big\|e^{\omega t}v(t)\big\|_{\mathcal{D}}\\ &\le [v]_{\mathcal{C}^{\alpha,\omega}_\alpha(\mathcal{D})}\varepsilon^{-\alpha}(t-s)^\alpha + \alpha^{-1}\omega^\alpha\|v\|_{\mathcal{B}^\omega(\mathcal{D})}(t-s)^\alpha.\end{aligned}$$

Therefore we finally get the estimate

$$\begin{aligned}\big\|e^{\omega t}G(v(t))-e^{\omega s}G(v(s))\big\|_{\mathcal{X}} \le \alpha^{-1}\omega^\alpha L_0(\|v-\overline{u}\|_{\mathcal{B}^\omega(\mathcal{D})} + \|v\|_{\mathcal{B}^\omega(\mathcal{D})}\big)(t-s)^\alpha\\ + \big(L_0+\|A\|_{\mathcal{X}\leftarrow\mathcal{D}}\big)[v]_{\mathcal{C}^{\alpha,\omega}(\mathcal{D})}\varepsilon^{-\alpha}(t-s)^\alpha\end{aligned}$$

which yields the first statement.

For proving the second result for the negative half line we essentially follow the ideas from before. Notice that the validity of $\omega(t-s)\ge 0$ is crucial for the above arguments.

(ii) These statements follow literally from the first part of the proof by replacing the continuous quantities $t,s,v(t),v(s)$ and v with t_n,t_m,v_n,v_m and $\underline{v}$. ☐

The following auxiliary result is needed in order to prove Lemma E.5.

Remark E.1. Assume that either $I = (0, \infty)$ and $\omega \geq 0$, or $I = (-\infty, 0]$ and $\omega \leq 0$.

(i) Then, the map $e^{-(\cdot)\omega}$ belongs to $\mathcal{C}^\alpha(I, \boldsymbol{R})$ and satisfies

$$\big[e^{-(\cdot)\omega}\big]_{\mathcal{C}^\alpha(\boldsymbol{R})} \leq \alpha^{-1}|\omega|^\alpha.$$

(ii) For $v - \overline{u} \in \mathcal{B}^\omega(I, \mathcal{D})$, the following relation holds

$$\|v - \overline{u}\|_{\mathcal{B}(\mathcal{D})} \leq \|v - \overline{u}\|_{\mathcal{B}^\omega(\mathcal{D})}.$$

Provided that $v - \overline{u} \in \mathcal{C}_\alpha^{\alpha,\omega}((0, \infty), \mathcal{D})$ is fulfiled, the estimate

$$[v]_{\mathcal{C}_\alpha^\alpha(\mathcal{D})} \leq \big(1 + \alpha^{-1}|\omega|^\alpha\big)\|v - \overline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})}$$

is valid, and for maps v satisfying $v - \overline{u} \in \mathcal{C}^{\alpha,\omega}((-\infty, 0], \mathcal{D})$ we further have

$$[v]_{\mathcal{C}^\alpha(\mathcal{D})} \leq \big(1 + \alpha^{-1}|\omega|^\alpha\big)\|v - \overline{u}\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{D})}.$$

Proof. (i) Clearly $\left|e^{-\omega t}\right| \leq 1$ holds whenever $\omega t \geq 0$. By using the relation $e^{-x} \leq x^{-1}$ which is valid for $x \geq 0$ we thus get for all $s, t \in I$ with $s < t$

$$\begin{aligned}\left|e^{-\omega t} - e^{-\omega s}\right| &= \int_{\omega s}^{\omega t} e^{-\tau}\,\mathrm{d}\tau \leq \int_{\omega s}^{\omega t} e^{-(1-\alpha)\tau}\,\mathrm{d}\tau \leq \int_{\omega s}^{\omega t} \tau^{-1+\alpha}\,\mathrm{d}\tau \\ &\leq \int_0^{|\omega|(t-s)} \tau^{-1+\alpha}\,\mathrm{d}\tau = \alpha^{-1}|\omega|^\alpha(t-s)^\alpha.\end{aligned}$$

(ii) We only derive the statement for the positive half line $I = (0, \infty)$ and $\omega \geq 0$. The desired result for the negative half line then follows by applying similar arguments.

As long as the relation $\omega t \geq 0$ holds, the estimate

$$\|v(t) - \overline{u}\|_{\mathcal{D}} = e^{-\omega t}\big\|e^{\omega t}\big(v(t) - \overline{u}\big)\big\|_{\mathcal{D}} \leq \|v - \overline{u}\|_{\mathcal{B}^\omega(\mathcal{D})}$$

is valid. For elements $0 < s < t < \infty$, by means of the identity

$$v(t) - v(s) = \big(e^{-\omega t} - e^{-\omega s}\big)e^{\omega t}\big(v(t) - \overline{u}\big) + e^{-\omega s}\Big(e^{\omega t}\big(v(t) - \overline{u}\big) - e^{\omega s}\big(v(s) - \overline{u}\big)\Big),$$

we further receive

$$\begin{aligned}\|v(t) - v(s)\|_{\mathcal{D}} &\leq \big[e^{-(\cdot)\omega}\big]_{\mathcal{C}^\alpha(I, \boldsymbol{R})}\|v - \overline{u}\|_{\mathcal{B}^\omega(\mathcal{D})}(t-s)^\alpha + [v - \overline{u}]_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})}\varepsilon^{-\alpha}(t-s)^\alpha \\ &\leq \big(1 + \alpha^{-1}|\omega|^\alpha\big)\|v - \overline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})}\varepsilon^{-\alpha}(t-s)^\alpha.\end{aligned}$$

This yields the desired result. ∎

The following result is the analogue of Lemma E.3 for autonomous problems and unbounded time intervals. For Hölder continuous maps v, w and sequences $\underline{v}, \underline{w}$ it provides bounds for the differences

$$G(v) - G(w) \qquad \text{and} \qquad G(\underline{v}) - G(\underline{w})$$

in the exponentially weighted Hölder norm.

Lemma E.5. *For some $R > 0$ let $L > 0$ denote the quantity from* (E.6)*. Fix $0 < \alpha < 1$ and $0 < \varrho \leq R$.*

(i) Then, for $v, w \in \mathcal{C}_\alpha^{\alpha,\omega}((0,\infty), \mathcal{D})$ with $\omega \geq 0$ satisfying $\|v - \overline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} \leq \varrho$ and $\|w - \overline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} \leq \varrho$ it holds

$$\|G(v) - G(w)\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{X})} \leq \big(2 + \alpha^{-1}\omega^\alpha\big) L\varrho \|v - w\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})}$$

provided that $v(t), w(t) \in \mathcal{O}$ for $t \geq 0$.

Whenever $\omega \leq 0$ and $v, w \in \mathcal{C}^{\alpha,\omega}((-\infty,0], \mathcal{D})$ are such that $\|v - \overline{u}\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{D})} \leq \varrho$ and $\|w - \overline{u}\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{D})} \leq \varrho$ then the estimate

$$\|G(v) - G(w)\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{X})} \leq \big(2 + \alpha^{-1}|\omega^\alpha|\big) L\varrho \|v - w\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{D})}$$

is valid.

(ii) If $\omega \geq 0$ for all sequences $\underline{v}, \underline{w} \in \mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{N}, \mathcal{D})$ with the property $\|\underline{v} - \overline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} \leq \varrho$ and $\|\underline{w} - \overline{u}\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})} \leq \varrho$ the bound

$$\|G(\underline{v}) - G(\underline{w})\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{X})} \leq \big(2 + \alpha^{-1}\omega^\alpha\big) L\varrho \|\underline{v} - \underline{w}\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{D})}$$

is valid, whenever $v_n, w_n \in \mathcal{O}$ for $t_n \in \mathcal{N} = \{t_n = nh : n \in \boldsymbol{N}\}$.

Suppose $\omega \leq 0$. Then, the bound

$$\|G(\underline{v}) - G(\underline{w})\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{X})} \leq \big(2 + \alpha^{-1}\omega^\alpha\big) L\varrho \|\underline{v} - \underline{w}\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{D})}$$

holds for all $\underline{v}, \underline{w} \in \mathcal{C}^{\alpha,\omega}(\mathcal{N}, \mathcal{D})$ satisfying $\|\underline{v} - \overline{u}\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{D})} \leq \varrho$ and $\|\underline{w} - \overline{u}\|_{\mathcal{C}^{\alpha,\omega}(\mathcal{D})} \leq \varrho$, if $v_n, w_n \in \mathcal{O}$ for $t_n \in \mathcal{N} = \{t_n = nh : n \in \boldsymbol{Z}, n \leq 0\}$.

Proof. (i) We only derive the first statement for the positive half line. An analogous proof yields the corresponding result for the interval $(-\infty, 0]$.

Similarly to the proof of Lemma E.3, we define the map H through

$$H(t) = G(v(t)) - G(w(t)) = \int_0^1 \big(F'(z(\tau,t)) - F'(\overline{u})\big)\big(v(t) - w(t)\big)\,\mathrm{d}\tau$$

where $z(\tau,t) = \tau v(t) + (1-\tau)w(t)$ for some $0 \le \tau \le 1$. Then, by Lemma E.4, the compositions $G(v)$ and $G(w)$ and thus H belong to $\mathcal{C}_\alpha^{\alpha,\omega}((0,\infty),\mathcal{X})$ if $v,w \in \mathcal{C}_\alpha^{\alpha,\omega}((0,\infty),\mathcal{D})$.

(a) With the help of Remark E.1 and the assumption $\|v-\overline{u}\|_{\mathcal{B}^\omega(\mathcal{D})}, \|w-\overline{u}\|_{\mathcal{B}^\omega(\mathcal{D})} \le \varrho$ we get the following estimate

$$\begin{aligned}\|e^{\omega t}H(t)\|_{\mathcal{X}} &\le \int_0^1 L\big(\tau\|v(t)-\overline{u}\|_{\mathcal{D}} + (1-\tau)\|w(t)-\overline{u}\|_{\mathcal{D}}\big)\|v-w\|_{\mathcal{B}^\omega(\mathcal{D})}\,\mathrm{d}\tau \\ &\le \int_0^1 L\big(\tau\|v-\overline{u}\|_{\mathcal{B}^\omega(\mathcal{D})} + (1-\tau)\|w-\overline{u}\|_{\mathcal{B}^\omega(\mathcal{D})}\big)\|v-w\|_{\mathcal{B}^\omega(\mathcal{D})}\,\mathrm{d}\tau \\ &\le L\varrho\|v-w\|_{\mathcal{B}^\omega(\mathcal{D})}.\end{aligned}$$

(b) Choose $0 < \varepsilon \le s < t$. The identity

$$\begin{aligned}e^{\omega t}H(t) - e^{\omega s}H(s) &= \int_0^1 \big(F'(z(\tau,t) - F'(z(\tau,s))\big)e^{\omega t}\big(v(t)-w(t)\big)\,\mathrm{d}\tau \\ &\quad + \int_0^1 \big(F'(z(\tau,s)) - F'(\overline{u})\big)\Big(e^{\omega t}\big(v(t)-w(t)\big) - e^{\omega s}\big(v(s)-w(s)\big)\Big)\,\mathrm{d}\tau\end{aligned}$$

implies the bound

$$\begin{aligned}\|e^{\omega t}H(t) - e^{\omega s}H(s)\|_{\mathcal{X}} &\le \int_0^1 L\big(\tau\|v(t)-v(s)\|_{\mathcal{D}} + (1-\tau)\|w(t)-w(s)\|_{\mathcal{D}}\big) \\ &\qquad\qquad \cdot \|v-w\|_{\mathcal{B}^\omega(\mathcal{D})}\,\mathrm{d}\tau \\ &\quad + \int_0^1 \big(\tau\|v(s)-\overline{u}\|_{\mathcal{D}} + (1-\tau)\|w(s)-\overline{u}\|_{\mathcal{D}}\big)[v-w]_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})}\varepsilon^{-\alpha}(t-s)^\alpha\,\mathrm{d}\tau.\end{aligned}$$

Together with Remark E.1 this further yields

$$\begin{aligned}\|e^{\omega t}H(t) - e^{\omega s}H(s)\|_{\mathcal{X}} &\le L\varrho\Big(\big(1+\alpha^{-1}\omega^\alpha\big)\|v-w\|_{\mathcal{B}^\omega(\mathcal{D})} + [v-w]_{\mathcal{C}^{\alpha,\omega_i}(\mathcal{D})}\Big)\varepsilon^{-\alpha}(t-s)^\alpha \\ &\le \big(1+\alpha^{-1}\omega^\alpha\big)L\varrho\|v-w\|_{\mathcal{C}_\alpha^{\alpha,\omega}(\mathcal{D})}\varepsilon^{-\alpha}(t-s)^\alpha.\end{aligned}$$

(ii) Again, the results for sequences follow from statement (i) by replacing the continuous quantities with the corresponding discrete ones. $\square$

F. A special initial value problem

In the present chapter, we study the question how the analytical and the Runge-Kutta solution of the abstract initial value problem

$$z'(t) = Az(t) + f(t), \quad t > 0, \qquad z(0) = 0, \tag{F.1}$$

evolve for small times. We henceforth suppose that $A : \mathcal{D}(A) \to \mathcal{X}$ is a sectorial operator on a Banach space $\mathcal{X}$ and that the map f is α-Hölder continuous on the interval $[0, T]$ with values in $\mathcal{X}$.

In order to represent the exact and the numerical solution of (F.1), we make use of the modified variation-of constants formulas derived in Section D.1. For the subsequent considerations, it is useful to split the analytical solution of (F.1) into

$$z(t) = z_1(t) + z_2(t), \qquad 0 \le t \le T,$$

where the maps z_1 and z_2 are defined through

$$\begin{aligned} z_1(t) &= A^{-1}(e^{tA} - I)f(t) \qquad \text{and} \\ z_2(t) &= \int_0^t e^{(t-\tau)A}\big(f(\tau) - f(t)\big)\,\mathrm{d}\tau, \end{aligned} \tag{F.2a}$$

and to consider each part separately. The corresponding decomposition for the backward Euler solution of (F.1) with initial value $z_0 = 0$ is the following

$$\underline{z} = \underline{z}^{(1)} + \underline{z}^{(2)} = \Big(z_n^{(1)} + z_n^{(2)}\Big)_{0 \le t_n \le T}.$$

Here, the sequences $\underline{z}^{(1)}$ and $\underline{z}^{(2)}$ are given by

$$\begin{aligned} z_n^{(1)} &= A^{-1}\big((I - hA)^{-n} - I\big)f(t_n) \qquad \text{and} \\ z_n^{(2)} &= h\sum_{\nu=0}^{n-1}(I - hA)^{-(n-\nu)}\big(f(t_{\nu+1}) - f(t_n)\big). \end{aligned} \tag{F.2b}$$

In compact vector notation, a general Runge-Kutta method with initial value $z_0 = 0$, applied to (F.1), takes the form

$$Z_n' = AZ_n + f(t_n + ch), \qquad Z_n = \mathbb{1}z_n + h\mathcal{O}\!\!\!\!\;\mathcal{A}Z_n', \qquad z_{n+1} = z_n + hb^TZ_n',$$

where $f(t_n + ch) = \big(f(t_n + c_i h)\big)_{i=1}^s$, see Section C.1. As in the preceding chapters, we henceforth employ the abbreviation

$$\mathcal{J} = (\mathcal{I} - hA\mathcal{Q})^{-1},$$

and we denote by $C_{\mathcal{J}}$ a bound of $\mathcal{J}$ on $\mathcal{X}^s$, see Lemma C.1. For the stages of the Runge-Kutta method, analogously to before, we set

$$\underline{Z} = \underline{Z}^{(1)} + \underline{Z}^{(2)} = \Big(Z_n^{(1)} + Z_n^{(2)}\Big)_{0<t_{n+1}\le T},$$

and we define the sequences $\underline{Z}^{(1)}$ and $\underline{Z}^{(2)}$ through the relations

$$\begin{aligned} Z_n^{(1)} &= A^{-1}\mathcal{J}\mathbb{1}b^T\mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1}\big(R(hA)^n - I\big)f(t_{n-1}+ch) \qquad \text{and} \\ Z_n^{(2)} &= h\sum_{\nu=0}^{n-1}\mathcal{J}\mathbb{1}b^T\mathcal{J}R(hA)^{n-\nu-1}\big(f(t_\nu+ch) - f(t_{n-1}+ch)\big) \\ &\qquad + h\mathcal{Q}\mathcal{J}f(t_n+ch). \end{aligned} \tag{F.2c}$$

In Section F.1 we state that the (α,α)-Hölder norms of z and its discrete counterparts $\underline{z}$ and $\underline{Z}$ converge to zero whenever the length of the time interval T tends to zero

$$z \to 0, \quad \underline{z} \to 0 \quad \text{and} \quad \underline{Z} \to 0 \qquad \text{for} \quad T \to 0, \tag{F.3}$$

see Lemmas F.1 and F.2. These statements are needed in the proofs of the local existence results Theorem 4.1 and Proposition 4.1 in order to show that the fixed point iterations for the analytical and the numerical solution map a ball around the initial value into itself. However, as the proofs of the Lemmas F.1 and F.2 use the fact that $\mathcal{D}(A)$ is densely embedded in $\mathcal{X}$, nothing is known about the order of convergence in (F.3).

F.1. Asymptotic behaviour for small times

At first, we consider the map z_1 and its discrete analogues $\underline{z}^{(1)}$ and $\underline{Z}^{(1)}$ defined through Relation (F.2). Fix $h > 0$. For the backward Euler method we set

$$\mathcal{N} = \{t_n = nh : n \in \boldsymbol{N}, 0 < t_n \le T\},$$

and for a general Runge-Kutta method let

$$\mathcal{N} = \{t_n = nh : n \in \boldsymbol{N}, 0 < t_{n+1} \le T\}.$$

The following result shows that the quantities

$$\|z_1\|_{\mathcal{C}^\alpha_\alpha((0,T],\mathcal{D}(A))}, \qquad \|\underline{z}^{(1)}\|_{\mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{D}(A))} \qquad \text{and} \qquad \big\|\underline{Z}^{(1)}\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{N},\mathcal{D}(A)^s)}$$

are arbitrarily small whenever T approximates 0.

Lemma F.1. *Assume $f \in \mathcal{C}^\alpha([0,T],\mathcal{X})$ for some $0<\alpha<1$, and choose $\widetilde{x} \in \mathcal{D}(A)$.*

(i) Then, the following estimate holds with constant C depending on α, on the bounds for the analytical semigroup e^{tA} for $0 \le t \le T$, and on the α-Hölder constant of f, but not on $\widetilde{x}$

$$\|z_1\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} \le C\big(\|f(0)-\widetilde{x}\|_{\mathcal{X}} + T\|\widetilde{x}\|_{\mathcal{D}(A)} + T^\alpha\big).$$

In particular, for $\widetilde{x}$ sufficiently close to $f(0)$, the quantity $\|z_1\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))}$ tends to 0 whenever the length of the time interval T goes to 0.

(ii) For $0<h\le\overline{h}$ sufficiently small the backward Euler solution $\underline{z}^{(1)}$ satisfies

$$\big\|\underline{z}^{(1)}\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} \le C\big(\|f(0)-\widetilde{x}\|_{\mathcal{X}} + T\|\widetilde{x}\|_{\mathcal{D}(A)} + T^\alpha\big)$$

with constant C depending on α, on the bounds for the stability function and on the α-Hölder constant of f, but not on $\widetilde{x}$.

(iii) For a general Runge-Kutta method the following estimate is valid

$$\big\|\underline{Z}^{(1)}\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A)^s)} \le C\big(\|f(0)-\widetilde{x}\|_{\mathcal{X}} + T\|\widetilde{x}\|_{\mathcal{D}(A)} + T^\alpha\big)$$

for $0<h\le\overline{h}$ small enough. The constant C depends on α, on the bounds for the stability function, on the α-Hölder constant of f and further on the bound for $\mathcal{J}$ and on the coefficients of the method, but not on $\widetilde{x}$.

Proof. Let henceforth K denote the α-Hölder constant of f on $[0,T]$. For the constants arising in the bounds for the analytical semigroup and the stability functions, it is useful to introduce some notation. Due to the Results A.1 and C.1, there are constants $M_k>0$ and $\hat{M}_k>0$ such that the estimates

$$\begin{aligned}
\big\|A^k e^{tA}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} &\le M_k t^{-k}, && 0<t\le T, && 0\le k\le 1,\\
\big\|A^k (I-hA)^{-n}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} &\le \hat{M}_k t_n^{-k}, && 0<t_n\le t_N\le T, && 0\le k\le 1,
\end{aligned}$$

hold for $0<h\le\overline{h}$ small enough. For a general Runge-Kutta method we further have

$$\begin{aligned}
\big\|b^T\mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}^s} &\le c_0,\\
\big\|R(hA)^n\big\|_{\mathcal{X}\leftarrow\mathcal{X}} &\le \hat{M}_0, && 0\le t_n\le t_N\le T, && 0\le k\le 1,\\
\big\|b^T\mathcal{J}A^kR(hA)^{n-1}\mathbb{1}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} &\le \hat{M}_k t_n^{-k}, && 0<t_n\le t_N\le T, && 0\le k\le 1,
\end{aligned}$$

provided that the stepsize $0<h\le\overline{h}$ is chosen sufficiently small, see C.1 and also Formula (D.17).

(i)(a) For all $t \in [0,T]$ the following estimate holds if $\widetilde{x} \in \mathcal{D}(A)$

$$\begin{aligned}\|z_1(t)\|_{\mathcal{D}(A)} &\le \big\|(e^{tA}-I)\big(f(t)-f(0)+f(0)-\widetilde{x}\big)\big\|_{\mathcal{X}} + \int_0^t \big\|e^{\tau A}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|\widetilde{x}\|_{\mathcal{D}(A)}\,\mathrm{d}\tau \\ &\le (1+M_0)\Big(KT^\alpha + \big\|f(0)-\widetilde{x}\big\|_{\mathcal{X}}\Big) + M_0T\|\widetilde{x}\|_{\mathcal{D}(A)}.\end{aligned}$$

(b) Further, we have for elements $0<\varepsilon\le s<t\le T$

$$\begin{aligned}\|z_1(t)-z_1(s)\|_{\mathcal{D}(A)} &\le \big\|\big(e^{tA}-e^{sA}\big)f(t)\big\|_{\mathcal{X}} + \big\|\big(e^{sA}-I\big)\big(f(t)-f(s)\big)\big\|_{\mathcal{X}} \\ &\le \int_s^t \big\|Ae^{\tau A}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\,\mathrm{d}\tau\Big(\|f(t)-f(0)\|_{\mathcal{X}} + \big\|f(0)-\widetilde{x}\big\|_{\mathcal{X}}\Big) \\ &\quad + \int_s^t \big\|e^{\tau A}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|\widetilde{x}\|_{\mathcal{D}(A)}\,\mathrm{d}\tau + (1+M_0)K(t-s)^\alpha \\ &\le M_1\big(KT^\alpha + \big\|f(0)-\widetilde{x}\big\|_{\mathcal{X}}\big)\int_s^t \tau^{-1}\,\mathrm{d}\tau + M_0\|\widetilde{x}\|_{\mathcal{D}(A)}(t-s) \\ &\quad + (1+M_0)K(t-s)^\alpha.\end{aligned}$$

With the help of the bound (D.35) from Estimate D.1 this yields

$$[z_1]_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} \le \alpha^{-1}M_1\big(KT^\alpha + \big\|f(0)-\widetilde{x}\big\|_{\mathcal{X}}\big) + M_0\|\widetilde{x}\|_{\mathcal{D}(A)}T + (1+M_0)KT^\alpha.$$

Due to the fact that the domain of A lies dense in $\mathcal{X}$, we can choose $\widetilde{x}$ such that the first statement of the lemma follows.

(ii)(a) Choose $\widetilde{x} \in \mathcal{D}(A)$. Then, we get for $0 \le nh \le Nh \le T < (N+1)h$

$$\begin{aligned}\big\|z_n^{(1)}\big\|_{\mathcal{D}(A)} &\le \big\|\big((I-hA)^{-n}-I\big)\big(f(t_n)-f(0)+f(0)-\widetilde{x}\big)\big\|_{\mathcal{X}} \\ &\quad + h\sum_{\nu=0}^{n-1}\big\|(I-hA)^{-(\nu+1)}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|\widetilde{x}\|_{\mathcal{D}(A)} \\ &\le (1+\hat{M}_0)\Big(Kt_N^\alpha + \big\|f(0)-\widetilde{x}\big\|_{\mathcal{X}}\Big) + \hat{M}_0t_N\|\widetilde{x}\|_{\mathcal{D}(A)}.\end{aligned}$$

(b) Similarly to the continuous case, we have for $0 < n_1h \le mh < nh \le Nh$

$$\begin{aligned}\big\|z_n^{(1)}-z_m^{(1)}\big\|_{\mathcal{D}(A)} &\le \big\|\big((I-hA)^{-n}-(I-hA)^{-m}\big)f(t_n)\big\|_{\mathcal{X}} \\ &\quad + \big\|\big((I-hA)^{-m}-I\big)\big(f(t_n)-f(t_m)\big)\big\|_{\mathcal{X}} \\ &\le h\sum_{\nu=m}^{n-1}\big\|A(I-hA)^{-(\nu+1)}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\Big(\|f(t_n)-f(0)\|_{\mathcal{X}} + \big\|f(0)-\widetilde{x}\big\|_{\mathcal{X}}\Big) \\ &\quad + h\sum_{\nu=m}^{n-1}\big\|(I-hA)^{-(\nu+1)}\big\|_{\mathcal{X}\leftarrow\mathcal{X}}\|\widetilde{x}\|_{\mathcal{D}(A)} + (1+\hat{M}_0)K(t_n-t_m)^\alpha\end{aligned}$$

and furthermore

$$\big\|z_n^{(1)} - z_m^{(1)}\big\|_{\mathcal{D}(A)} \le \hat{M}_1\big(Kt_N^\alpha + \big\|f(0) - \widetilde{x}\big\|_{\mathcal{X}}\big) h \sum_{\nu=m}^{n-1} t_{\nu+1}^{-1} + \hat{M}_0 \|\widetilde{x}\|_{\mathcal{D}(A)}(t_n - t_m) \\ + (1 + \hat{M}_0) K (t_n - t_m)^\alpha.$$

Estimating the Riemann-sum by an integral and applying again (D.35) from Estimate D.1 yields the relation

$$\big[\underline{z}^{(1)}\big]_{\mathcal{C}_\alpha^\alpha(\mathcal{D}(A))} \le \alpha^{-1}\hat{M}_1\big(Kt_N^\alpha + \big\|f(0) - \widetilde{x}\big\|_{\mathcal{X}}\big) + \hat{M}_0\|\widetilde{x}\|_{\mathcal{D}(A)} t_N + (1 + \hat{M}_0)Kt_N^\alpha$$

for $0 < h \le \overline{h}$ sufficiently small.

(iii)(a) Likewise to the previous considerations, by using the identity

$$\big(R(hA) - I\big)^{-1}\big(R(hA)^n - I\big) = \sum_{\nu=0}^{n-1} R(hA)^\nu,$$

we have for $0 \le (n+1)h \le Nh \le T < (N+1)h$

$$\begin{aligned} \big\|Z_n^{(1)}\big\|_{\mathcal{D}(A)^s} &\le \big\|\mathcal{J}\mathbb{1} b^T \mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1}\big(R(hA)^n - I\big) \\ &\qquad \cdot \big(f(t_{n-1} + ch) - \mathbb{1} f(0) + \mathbb{1}\big(f(0) - \widetilde{x}\big)\big)\big\|_{\mathcal{X}^s} \\ &\quad + h \sum_{\nu=0}^{n-1} \big\|\mathcal{J}\mathbb{1} b^T \mathcal{J} R(hA)^\nu \mathbb{1}\big\|_{\mathcal{X}^s \leftarrow \mathcal{X}^s} \|\widetilde{x}\|_{\mathcal{D}(A)} \\ &\le C_{\mathcal{J}} c_0 (1 + \hat{M}_0)\Big(Kt_N^\alpha + \big\|f(0) - \widetilde{x}\big\|_{\mathcal{X}}\Big) + C_{\mathcal{J}}\hat{M}_0 t_N \|\widetilde{x}\|_{\mathcal{D}(A)}. \end{aligned}$$

(b) For $0 < n_1 h \le (m+1)h < (n+1)h \le Nh$ the following bound holds

$$\begin{aligned} \big\|Z_n^{(1)} - Z_m^{(1)}\big\|_{\mathcal{D}(A)^s} &\le \big\|\mathcal{J}\mathbb{1} b^T \mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1}\big(R(hA)^n - R(hA)^m\big) f(t_{n-1} + ch)\big\|_{\mathcal{X}^s} \\ &\quad + \big\|\mathcal{J}\mathbb{1} b^T \mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1}\big(R(hA)^m - I\big)\big(f(t_{n-1} + ch) - f(t_{m-1} + ch)\big)\big\|_{\mathcal{X}^s} \\ &\le h \sum_{\nu=m}^{n-1} \big\|\mathcal{J}\mathbb{1} b^T \mathcal{J} A R(hA)^\nu\big\|_{\mathcal{X}^s \leftarrow \mathcal{X}^s}\Big(\|f(t_{n-1} + ch) - \mathbb{1} f(0)\|_{\mathcal{X}^s} + \big\|f(0) - \widetilde{x}\big\|_{\mathcal{X}}\Big) \\ &\quad + h \sum_{\nu=m}^{n-1} \big\|\mathcal{J}\mathbb{1} b^T \mathcal{J} R(hA)^\nu\big\|_{\mathcal{X}^s \leftarrow \mathcal{X}^s} \|\widetilde{x}\|_{\mathcal{D}(A)} + C_{\mathcal{J}} c_0 (1 + \hat{M}_0) K (t_n - t_m)^\alpha \\ &\le C_{\mathcal{J}} \hat{M}_1 \big(Kt_N^\alpha + \big\|f(0) - \widetilde{x}\big\|_{\mathcal{X}}\big) \cdot h \sum_{\nu=m}^{n-1} t_{\nu+1}^{-1} + C_{\mathcal{J}}\hat{M}_0 \|\widetilde{x}\|_{\mathcal{D}(A)}(t_n - t_m) \\ &\quad + C_{\mathcal{J}} c_0 (1 + \hat{M}_0) K (t_n - t_m)^\alpha. \end{aligned}$$

From this, with the help of (D.35) we further get

$$\begin{aligned}[\underline{Z}^{(1)}]_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A)^s)} \le \alpha^{-1} C_{\mathcal{J}} \hat{M}_1 \big(Kt_N^\alpha + \|f(0) - \widetilde{x}\|_{\mathcal{X}}\big) + C_{\mathcal{J}} \hat{M}_0 \|\widetilde{x}\|_{\mathcal{D}(A)} t_N \\ + C_{\mathcal{J}} c_0 (1 + \hat{M}_0) K t_N^\alpha.\end{aligned}$$

This yields the desired result and concludes the proof. □

In the sequel, we study the behaviour of the map z_2 and the sequences $\underline{z}^{(2)}$ and $\underline{Z}^{(2)}$ given by (F.2) for small times. Notice that only the statement for a general Runge-Kutta method uses the requirement that $\mathcal{D}(A)$ lies dense in $\mathcal{X}$.

Lemma F.2. *Assume that f belongs to $\mathcal{C}^\alpha([0,T],\mathcal{X})$ with $0 < \alpha < 1$.*

(i) Then, the map $z_2 \in \mathcal{C}^\alpha_\alpha((0,T],\mathcal{D}(A))$ satisfies

$$\|z_2\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} \le CT^\alpha$$

where the constant C depends on α, on the bounds for the analytical semigroup e^{tA} for $0 \le t \le T$, and on the α-Hölder constant of f. In particular, the norm of z_2 in $\mathcal{C}^\alpha_\alpha((0,T],\mathcal{D}(A))$ tends to 0 whenever T goes to 0.

(ii) For the backward Euler solution $\underline{z}^{(2)}$ the relation

$$\big\|\underline{z}^{(2)}\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A))} \le CT^\alpha.$$

is valid for stepsizes $0 < h \le \overline{h}$ small enough with constant C depending on α, on the bounds for the stability function and further on the α-Hölder constant of f.

(iii) Choose $\widetilde{x} \in \mathcal{D}(A)$. Then, the following bound holds for a general Runge-Kutta method whenever $0 < h \le \overline{h}$ is sufficiently small

$$\big\|\underline{Z}^{(2)}\big\|_{\mathcal{C}^\alpha_\alpha(\mathcal{D}(A)^s)} \le C\big(T^\alpha + \|f(0) - \widetilde{x}\|_{\mathcal{X}} + h\|\widetilde{x}\|_{\mathcal{D}(A)}\big).$$

The constant C depends on α, on the bounds for the stability function, on the α-Hölder constant of f and further on the bound for $\mathcal{J}$ and on the coefficients of the method.

Proof. We employ the notation introduced in the proof of Lemma F.1. In particular, K denotes the α-Hölder constant of f on $[0,T]$. For $0 < h \le \overline{h}$ sufficiently small the relations

$$\begin{aligned}\big\|A^2 e^{tA}\big\|_{\mathcal{X} \leftarrow \mathcal{X}} &\le M_2 t^{-2}, \qquad 0 < t \le T, \\ \big\|A^2 (I - hA)^{-n}\big\|_{\mathcal{X} \leftarrow \mathcal{X}} &\le \hat{M}_2 t_n^{-2}, \qquad 0 < t_n \le t_N \le T, \qquad n \ge 2,\end{aligned}$$

are valid. For a general Runge-Kutta method we furthermore apply the bound

$$\big\|b^T\mathcal{J}b^T\mathcal{J}\mathbb{1}A^2R(hA)^{n-2}\big\|_{\mathcal{X}\leftarrow\mathcal{X}} \le \hat{M}_2 t_n^{-2}, \qquad 0 < t_n \le t_N \le T, \qquad n \ge 2,$$

for $0 < h \le \overline{h}$ small enough.

(i)(a) For all $0 \le t \le T$ the following estimate holds

$$\|z_2(t)\|_{\mathcal{D}(A)} \le M_1 K \int_0^t (t-\tau)^{-1+\alpha}\,\mathrm{d}\tau \le \alpha^{-1} M_1 K T^\alpha.$$

(b) On the other hand, we have for $0 < \varepsilon \le s < t \le T$

$$\begin{aligned} A\big(z_2(t) - z_2(s)\big) &= \int_0^s \int_{s-\tau}^{t-\tau} A^2 e^{\sigma A}\big(f(\tau) - f(s)\big)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &\quad + \big(e^{tA} - e^{(t-s)A}\big)\big(f(s) - f(t)\big)\,\mathrm{d}\tau + \int_s^t A e^{(t-\tau)A}\big(f(\tau) - f(t)\big)\,\mathrm{d}\tau. \end{aligned}$$

With the help of the bound

$$\begin{aligned} \int_0^s \int_{s-\tau}^{t-\tau} \sigma^{-2}(s-\tau)^\alpha\,\mathrm{d}\sigma\,\mathrm{d}\tau &\le \int_0^s \int_{s-\tau}^{t-\tau} \sigma^{-2+\alpha}\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &= \frac{1}{\alpha(1-\alpha)}\big((t-s)^\alpha + s^\alpha - t^\alpha\big) \le \frac{1}{\alpha(1-\alpha)}(t-s)^\alpha, \end{aligned}$$

we consequently receive

$$\begin{aligned} \|z_2(t) - z_2(s)\|_{\mathcal{D}(A)} &\le M_2 K \int_0^s \int_{s-\tau}^{t-\tau} \sigma^{-2}(s-\tau)^\alpha\,\mathrm{d}\sigma\,\mathrm{d}\tau + 2M_0 K (t-s)^\alpha \\ &\quad + M_1 K \int_s^t (t-\tau)^{-1+\alpha}\,\mathrm{d}\tau \\ &\le \big(2M_0 + \alpha^{-1}M_1 + (\alpha(1-\alpha))^{-1}M_2\big)\varepsilon^{-\alpha}(t-s)^\alpha \end{aligned}$$

which implies the estimate

$$[z_2]_{\mathcal{C}_\alpha^\alpha(\mathcal{D}(A))} \le \big(2M_0 + \alpha^{-1}M_1 + (\alpha(1-\alpha))^{-1}M_2\big)KT^\alpha.$$

(ii)(a) For every $0 \le nh \le Nh \le T < (N+1)h$ we have

$$\begin{aligned} \big\|z_n^{(2)}\big\|_{\mathcal{D}(A)} &\le \hat{M}_1 K h \sum_{\nu=0}^{n-1} (t_n - t_\nu)^{-1+\alpha}(t_n - t_{\nu+1})^\alpha \\ &\le \hat{M}_1 K \int_0^{t_n} (t_n - \tau)^{-1+\alpha}\,\mathrm{d}\tau \le \alpha^{-1}\hat{M}_1 K t_N^\alpha. \end{aligned}$$

(b) Moreover, for all $0 < n_1 h \le mh < nh \le Nh$ the relation

$$\begin{aligned} A\big(z_n^{(2)} - z_m^{(2)}\big) = h^2 \sum_{\nu=0}^{m-1} \sum_{\mu=m-\nu}^{n-\nu-1} & A^2 (I - hA)^{-(\mu+1)} \big(f(t_{\nu+1}) - f(t_m)\big) \\ & + \big((I - hA)^{-n} - (I - hA)^{-(n-m)}\big)\big(f(t_m) - f(t_n)\big) \\ & + h \sum_{\nu=m}^{n-1} A (I - hA)^{-(n-\nu)} \big(f(t_{\nu+1}) - f(t_n)\big) \end{aligned}$$

holds, which implies the estimate

$$\begin{aligned} \big\|z_n^{(2)} - z_m^{(2)}\big\|_{\mathcal{D}(A)} &\le \hat{M}_2 K h^2 \sum_{\nu=0}^{m-1} \sum_{\mu=m-\nu}^{n-\nu-1} t_{\mu+1}^{-2} (t_m - t_{\nu+1})^\alpha + 2\hat{M}_0 K (t_n - t_m)^\alpha \\ &\quad + \hat{M}_1 K h \sum_{\nu=m}^{n-1} (t_n - t_\nu)^{-1+\alpha} \\ &\le \big(2\hat{M}_0 + \alpha^{-1}\hat{M}_1 + (\alpha(1-\alpha))^{-1}\hat{M}_2\big) K t_N^\alpha \cdot t_{n_1}^{-\alpha} (t_n - t_m)^\alpha . \end{aligned}$$

Note that for example the following bound holds

$$\begin{aligned} h^2 \sum_{\nu=0}^{m-1} \sum_{\mu=m-\nu}^{n-\nu-1} t_{\mu+1}^{-2} (t_m - t_{\nu+1})^\alpha &\le h^2 \sum_{\nu=0}^{m-1} \sum_{\mu=m-\nu}^{n-\nu-1} t_{\mu+1}^{-2+\alpha} \le h \sum_{\nu=0}^{m-1} \int_{t_m - t_\nu}^{t_n - t_\nu} \sigma^{-2+\alpha} \,\mathrm{d}\sigma \\ &= (1-\alpha)^{-1} h \sum_{\nu=0}^{m-1} (t_m - t_\nu)^{-1+\alpha} - (t_n - t_\nu)^{-1+\alpha} \\ &\le (1-\alpha)^{-1} \int_0^{t_m} (t_m - \tau)^{-1+\alpha} - (t_n - \tau)^{-1+\alpha} \,\mathrm{d}\tau \le \big(\alpha(1-\alpha)\big)^{-1} (t_n - t_m)^\alpha . \end{aligned}$$

(iii)(a) For all $0 \le (n+1)h \le t_N$ we receive

$$\begin{aligned} \big\|Z_n^{(2)}\big\|_{\mathcal{D}(A)^s} &\le h \sum_{\nu=0}^{n-1} \big\|\mathcal{J}\mathbb{1} b^T \mathcal{J} A R(hA)^{n-\nu-1}\big\|_{\mathcal{X}^s \leftarrow \mathcal{X}^s} \|f(t_\nu + ch) - f(t_{n-1} + ch)\|_{\mathcal{X}^s} \\ &\quad + \big\|hA\mathcal{Q}\mathcal{J}\big((f(t_n + ch) - f(0)) + \mathbb{1}\big(f(0) - \widetilde{x}\big)\big)\|_{\mathcal{X}^s} + h\|\mathcal{Q}\mathcal{J}\|_{\mathcal{X}^s \leftarrow \mathcal{X}^s} \big\|\widetilde{x}\big\|_{\mathcal{D}(A)} \\ &\le C_{\mathcal{J}} \hat{M}_1 K h \sum_{\nu=0}^{n-1} (t_n - t_\nu)^{-1+\alpha} (t_{n+1} - t_\nu)^\alpha + (1 + C_{\mathcal{J}})\big(K t_N^\alpha + \big\|f(0) - \widetilde{x}\big\|_{\mathcal{X}}\big) \\ &\quad + h\|\mathcal{Q}\mathcal{J}\|_{\mathcal{X}^s \leftarrow \mathcal{X}^s} \big\|\widetilde{x}\big\|_{\mathcal{D}(A)} \\ &\le \alpha^{-1} \hat{M}_1 K t_N^\alpha + (1 + C_{\mathcal{J}})\big(K t_N^\alpha + \big\|f(0) - \widetilde{x}\big\|_{\mathcal{X}}\big) + h\|\mathcal{Q}\mathcal{J}\|_{\mathcal{X}^s \leftarrow \mathcal{X}^s} \big\|\widetilde{x}\big\|_{\mathcal{D}(A)} . \end{aligned}$$

(b) In order to derive a bound for the (α,α)-Hölder seminorm of $\underline{Z}^{(2)}$, it is convenient to use the decomposition $Z_n^{(2)} = S_n + h\mathcal{Q}\mathcal{J}f(t_n+ch)$ where

$$AS_n = h\sum_{\nu=0}^{n-1} \mathcal{J}\mathbb{1}b^T\mathcal{J}AR(hA)^{n-\nu-1}\big(f(t_\nu+ch)-f(t_{n-1}+ch)\big).$$

For the second term we immediately get the bound

$$\big\|hA\mathcal{Q}\mathcal{J}\big(f(t_{n-1}+ch)-f(t_{m-1}+ch)\big)\big\|_{\mathcal{X}^s} \le (1+C_{\mathcal{J}})Kt_N^\alpha\cdot t_{n_1}^\alpha(t_n-t_m)^\alpha.$$

For the difference $S_n - S_m$ the following relation holds

$$\begin{aligned}
A(S_n-S_m) = h\sum_{\nu=0}^{m-1} &\mathcal{J}\mathbb{1}b^T\mathcal{J}A\big(R(hA)^{n-\nu-1}-R(hA)^{m-\nu-1}\big)\\
&\cdot\big(f(t_\nu+ch)-f(t_{m-1}+ch)\big)\\
&+h\sum_{\nu=0}^{m-1}\mathcal{J}\mathbb{1}b^T\mathcal{J}AR(hA)^{n-\nu-1}\big(f(t_{m-1}+ch)-f(t_{n-1}+ch)\big)\\
&\quad+h\sum_{\nu=m}^{n-1}\mathcal{J}\mathbb{1}b^T\mathcal{J}AR(hA)^{n-\nu-1}\big(f(t_{\nu+1}+ch)-f(t_{n-1}+ch)\big)
\end{aligned}$$

and furthermore

$$\begin{aligned}
A(S_n-S_m) &= h^2\sum_{\nu=0}^{m-1}\sum_{\nu=m-\nu}^{n-\nu-1}\mathcal{J}\mathbb{1}b^T\mathcal{J}b^T\mathcal{J}\mathbb{1}A^2R(hA)^{\mu-1}\big(f(t_\nu+ch)-f(t_{m-1}+ch)\big)\\
&+\mathcal{J}\mathbb{1}b^T\mathcal{J}\big(b^T\mathcal{J}\mathbb{1}\big)^{-1}\big(R(hA)^n-R(hA)^{n-m}\big)\big(f(t_{m-1}+ch)-f(t_{n-1}+ch)\big)\\
&\quad+h\sum_{\nu=m}^{n-1}\mathcal{J}\mathbb{1}b^T\mathcal{J}AR(hA)^{n-\nu-1}\big(f(t_{\nu+1}+ch)-f(t_{n-1}+ch)\big).
\end{aligned}$$

For all $0 < n_1h \le (m+1)h < (n+1)h \le Nh$ the above identity implies essentially the same bound as for the backward Euler solution, namely

$$\begin{aligned}
\big\|S_n-S_m\big\|_{\mathcal{D}(A)} &\le C_{\mathcal{J}}\hat{M}_2Kh^2\sum_{\nu=0}^{m-1}\sum_{\mu=m-\nu}^{n-\nu-1}t_{\mu+1}^{-2}(t_m-t_{\nu+1})^\alpha + 2C_{\mathcal{J}}c_0\hat{M}_0K(t_n-t_m)^\alpha\\
&\quad+C_{\mathcal{J}}\hat{M}_1Kh\sum_{\nu=m}^{n-1}(t_n-t_\nu)^{-1+\alpha}\\
&\le C_{\mathcal{J}}\big(2c_0\hat{M}_0+\alpha^{-1}\hat{M}_1+(\alpha(1-\alpha))^{-1}\hat{M}_2\big)Kt_N^\alpha\cdot t_{n_1}^{-\alpha}(t_n-t_m)^\alpha.
\end{aligned}$$

which concludes the proof of the lemma. $\square$

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Symbol index

Sets and intervals

$\boldsymbol{N} = \{1, 2, 3, \dots\}$	Positive integers
$\boldsymbol{N}_0 = \boldsymbol{N} \cup \{0\}$	Non-negative integers
$\boldsymbol{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$	All integers
$\boldsymbol{R}$	Real numbers
$\boldsymbol{R}_{>0} = \{x \in \boldsymbol{R} : x > 0\}$	Positive real numbers
$\boldsymbol{R}_{\geq 0} = \boldsymbol{R}_{>0} \cup \{0\}$	Non-negative real numbers
$\boldsymbol{C}$	Complex numbers
$(a, b) = \{x \in \boldsymbol{R} : a < x < b\}$	Open interval
$[a, b] = \{x \in \boldsymbol{R} : a \leq x \leq b\}$	Closed interval
$[a, b), (a, b]$	Half-open intervals
$X \times Y = \{(x, y) : x \in X, y \in Y\}$	Cartesian product of arbitrary sets X and Y
$X^s = \underbrace{X \times \cdots \times X}_{s \text{ times}}$	s-fold product of X

Common abbreviations

$\max, \min$	Maximum and minimum
$\sup, \inf$	Supremum and infimum
ess sup	Essential supremum
$\lim_{x \to x_0}$	Limit
diag	Diagonal matrix
$\Re(\lambda), \Im(\lambda)$	Real and imaginary part of $\lambda \in \boldsymbol{C}$
$\lvert\lambda\rvert, \arg(\lambda)$	Modulus and argument of $\lambda \in \boldsymbol{C}$
$\alpha = (\alpha_1, \dots, \alpha_n) \in \boldsymbol{N}_0^n$	Multi-index
$\lvert\alpha\rvert = \alpha_1 + \cdots + \alpha_n$	Definition of $\lvert\alpha\rvert$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \boldsymbol{N}_0^n$
$\partial = (\partial_{x_1}, \dots, \partial_{x_n})$	Nabla operator
$\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$	Definition of ∂^α for $\alpha = (\alpha_1, \dots, \alpha_n) \in \boldsymbol{N}_0^n$

Normed spaces and operators

$(\boldsymbol{R}, \lvert\cdot\rvert)$	Real line
$(\boldsymbol{C}, \lvert\cdot\rvert)$	Complex plane
$(\boldsymbol{R}^n, \lVert\cdot\rVert_{\boldsymbol{R}^n})$	n-dimensional Euclidian space
$(\mathcal{X}, \lVert\cdot\rVert_{\mathcal{X}})$	Banach space
$\lVert x\rVert_{\mathcal{X}^s} = \max\{\lVert x_i\rVert_{\mathcal{X}} : 1 \le i \le s\}$	Definition of $\lVert\cdot\rVert_{\mathcal{X}^s}$
$(\mathcal{X}^s, \lVert\cdot\rVert_{\mathcal{X}^s})$	Product space
$I : \mathcal{X} \to \mathcal{X} : x \mapsto x$	Identity operator on Banach space $\mathcal{X}$
$A : \mathcal{D}(A) \subset \mathcal{X} \to \mathcal{X} : x \mapsto Ax$	Linear operator in $\mathcal{X}$ with domain $\mathcal{D}(A)$
$\lVert B\rVert_{\mathcal{Y}\leftarrow\mathcal{X}}$	Operator norm of $B : \mathcal{X} \to \mathcal{Y}$ linear

Derivatives

$f', f'', f^{(k)}$	Derivatives of order $1, 2$ and k for $f : \boldsymbol{R} \to \boldsymbol{R}$
$\partial_{x_i} f, \partial_{x_i x_j} f$	Partial derivatives for $f : \boldsymbol{R}^n \to \boldsymbol{R} : (x_1, \ldots, x_n) \to f(x_1, \ldots, x_n)$
$D_v f$	Fréchet derivative of $f(t, v)$ with respect to v

Matrix and vector notation

$\mathcal{A} = (a_{ij})_{1 \le i,j \le s} \in \boldsymbol{R}^{s \times s}$	Runge-Kutta matrix
$\mathcal{I} = \operatorname{diag}(1, \ldots, 1) \in \boldsymbol{R}^{s \times s}$	Identity matrix
$b = (b_1, \ldots, b_s)^T \in \boldsymbol{R}^s$	Vector of weights
$c = (c_1, \ldots, c_s)^T \in [0, 1]^s$	Vector of nodes
$\mathbb{1} = (1, \ldots, 1)^T \in \boldsymbol{R}^s$	Vector of ones
$U_n = (U_{n1}, \ldots, U_{ns})^T$, $U_n' = (U_{n1}', \ldots, U_{ns}')^T$	Stages of the method

Other notation

$B_r(a) = \{\lambda \in \boldsymbol{C} : \lvert\lambda - a\rvert < r\}$	Open ball
$M_\vartheta = \{\lambda \in \boldsymbol{C} : \arg(-\lambda) \le \vartheta\}$	Stability region
$S_\varphi(a) = \{\lambda \in \boldsymbol{C} : \lvert\arg(\lambda - a)\rvert \le \varphi\}$	Complement of a cone

Curriculum vitae

Personal data

Name	Thalhammer
First name	Mechthild Maria Franziska
Gender	Female
Dates of birth	April 22, 1974 in Innsbruck (Austria)
Citizenship	Austrian
Civil status	Single
Position	Assistant, Universität Innsbruck

Education

1980 – 1984	Primary school, Völs in Tirol
1984 – 1992	Bundesgymnasium und Bundesrealgymnasium Sillgasse, Innsbruck
1992 – 1997	Study of Mathematics, Universität Innsbruck
March 1997	Mag. rer. nat. Universität Innsbruck

Research projects

1. *The long-time behaviour of discretizations of parabolic evolution equations*
   Supported by Acciones Integradas (Austria – Spain 1998/99)
   January 1998 - December 1999
   Conjoint research project with Prof. C. Palencia (Universidad de Valladolid, Spain) and A. Univ. Prof. Dr. A. Ostermann (Universität Innsbruck)
2. *Zeitdiskretisierung parabolischer Evolutionsgleichungen*
   Supported by the Austrian Science Fund FWF
   January 1999 - November 2001
   Conjoint research project with A. Univ. Prof. Dr. A. Ostermann (Universität Innsbruck)

Preprints and publications

1. M. Thalhammer, *Die numerische Behandlung freier Randwertprobleme.* Diploma thesis, 1997. Supervisor: Alexander Ostermann
2. A. Ostermann and M. Thalhammer, *Non-smooth data error estimates for linearly implicit Runge-Kutta methods.* IMA J. Numer. Anal. 20, 167-184 (2000)
3. C. González, A. Ostermann, C. Palencia and M. Thalhammer, *Backward Euler discretization of fully nonlinear parabolic problems.* To appear in Math. Comp.
4. A. Ostermann and M. Thalhammer, *Convergence of Runge-Kutta methods for nonlinear parabolic problems.* Submitted to Applied Numerical Math.

Research stays abroad

May 1998	Universidad de Valladolid, Spain, with C. González und C. Palencia (supported by Acciones Integradas)
October 1999	Universidad de Valladolid, Spain, with C. González und C. Palencia (supported by Acciones Integradas)
February 2000	Université de Genève, Switzerland, with A. Ostermann (supported by the Swiss National Science Foundation)

Talks

1. ICM 1998 (Berlin, Germany): *Non-smooth data error estimates for linearly-implicit Runge-Kutta methods*
2. 18th Biennial Conference on Numerical Analysis 1999 (Dundee, Scotland): *Non-smooth data error estimates for linearly-implicit Runge-Kutta methods and applications to long-term behaviour*
3. Euroconference on Numerical Methods for Evolution Partial Differential Equations 2000 (Anogia, Crete): *Runge-Kutta time discretization of nonlinear parabolic problems*

Colloquia

1. Universidad de Valladolid (October 1999): *Time discretization of nonlinear parabolic problems*
2. Université de Genève (February 2000): *On the convergence of Rosenbrock and W-methods for semilinear parabolic problems*

Summer schools

Summer School on Numerical Methods for ODEs 2000 (Toblach, Italy)

Prizes

December 1997	Würdigungspreis des Bundesministers für Wissenschaft, Verkehr und Kunst für Absolventen von Diplomstudien für das Jahr 1997