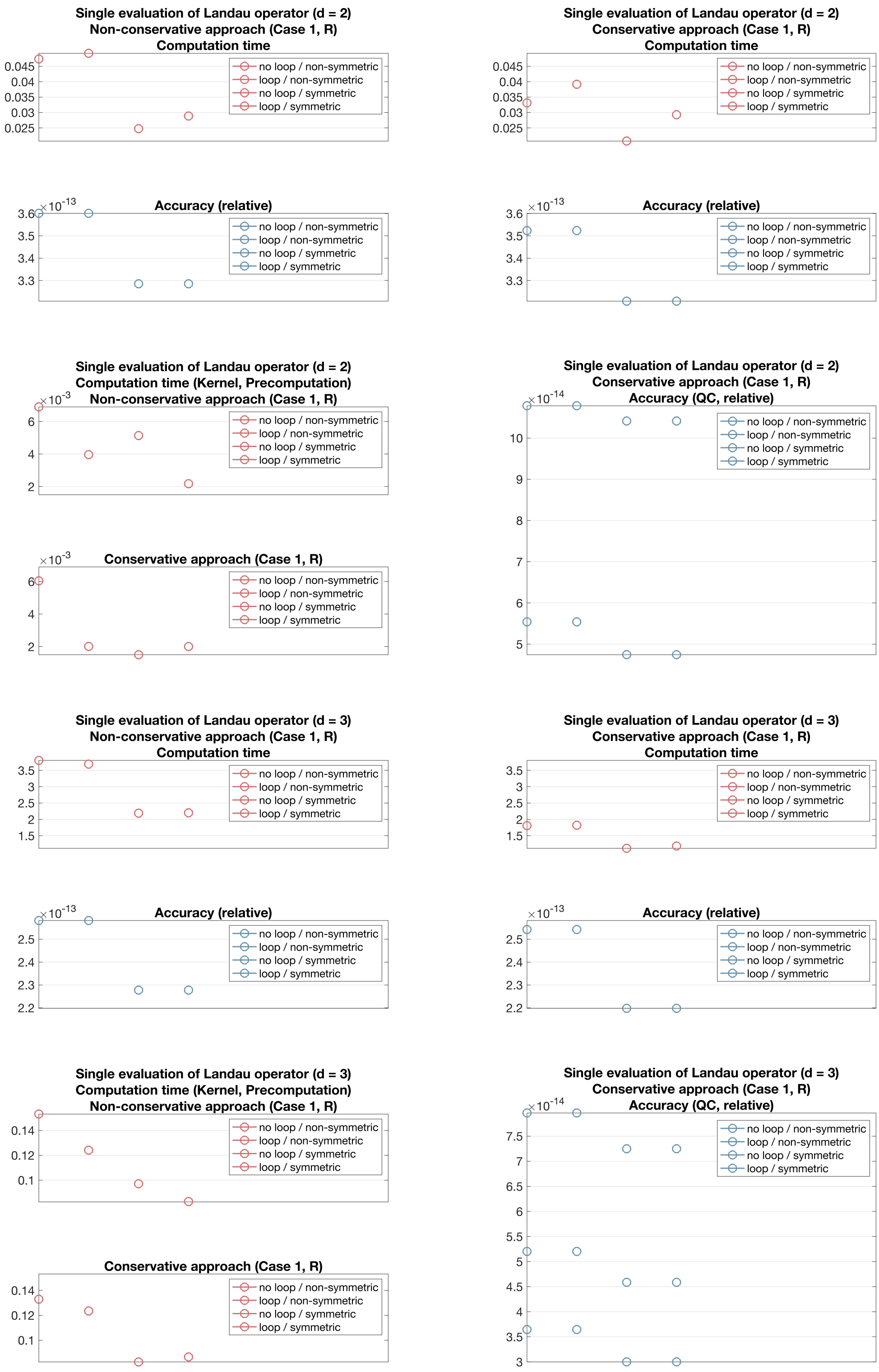

Fig. 1. Maxwellian and two more test problems with regular kernels. Case 2: Loss of accuracy due to numerical differentiation. Case 3: Non-conservative approach yields higher accuracy.

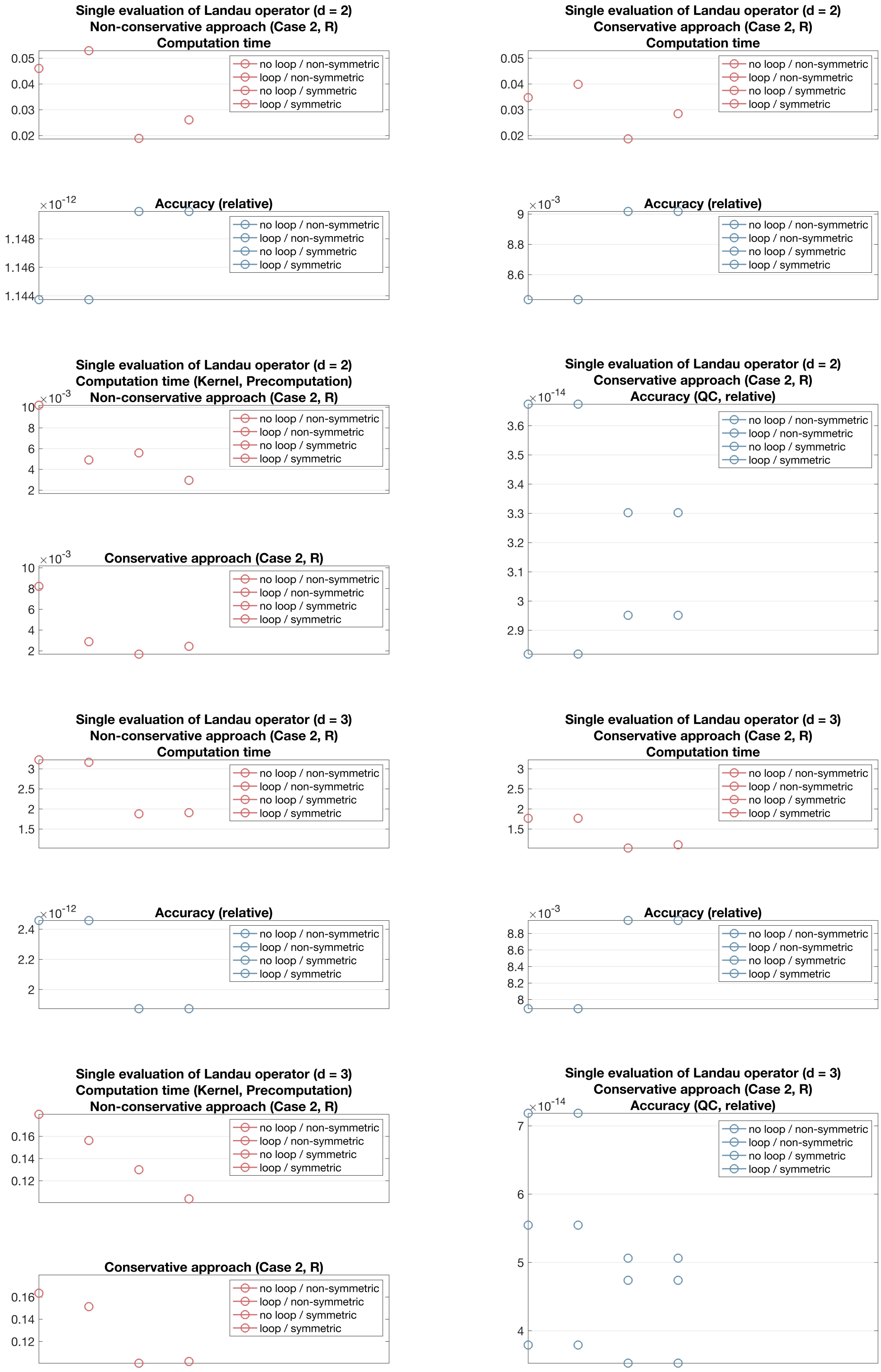
Single evaluation of Landau operator (d = 2)
Non-conservative approach (Case 2, R)
Computation time
Accuracy (relative)
Single evaluation of Landau operator (d = 2)
Conservative approach (Case 2, R)
Computation time
Accuracy (relative)
Single evaluation of Landau operator (d = 2)
Computation time (Kernel, Precomputation)
Non-conservative approach (Case 2, R)
Conservative approach (Case 2, R)
Single evaluation of Landau operator (d = 2)
Conservative approach (Case 2, R)
Accuracy (QC, relative)
Single evaluation of Landau operator (d = 3)
Non-conservative approach (Case 2, R)
Computation time
Accuracy (relative)
Single evaluation of Landau operator (d = 3)
Conservative approach (Case 2, R)
Computation time
Accuracy (relative)
Single evaluation of Landau operator (d = 3)
Computation time (Kernel, Precomputation)
Non-conservative approach (Case 2, R)
Conservative approach (Case 2, R)
Single evaluation of Landau operator (d = 3)
Conservative approach (Case 2, R)
Accuracy (QC, relative)
no loop / non-symmetric
loop / non-symmetric
no loop / symmetric
loop / symmetric

Fig. 2. Case 2

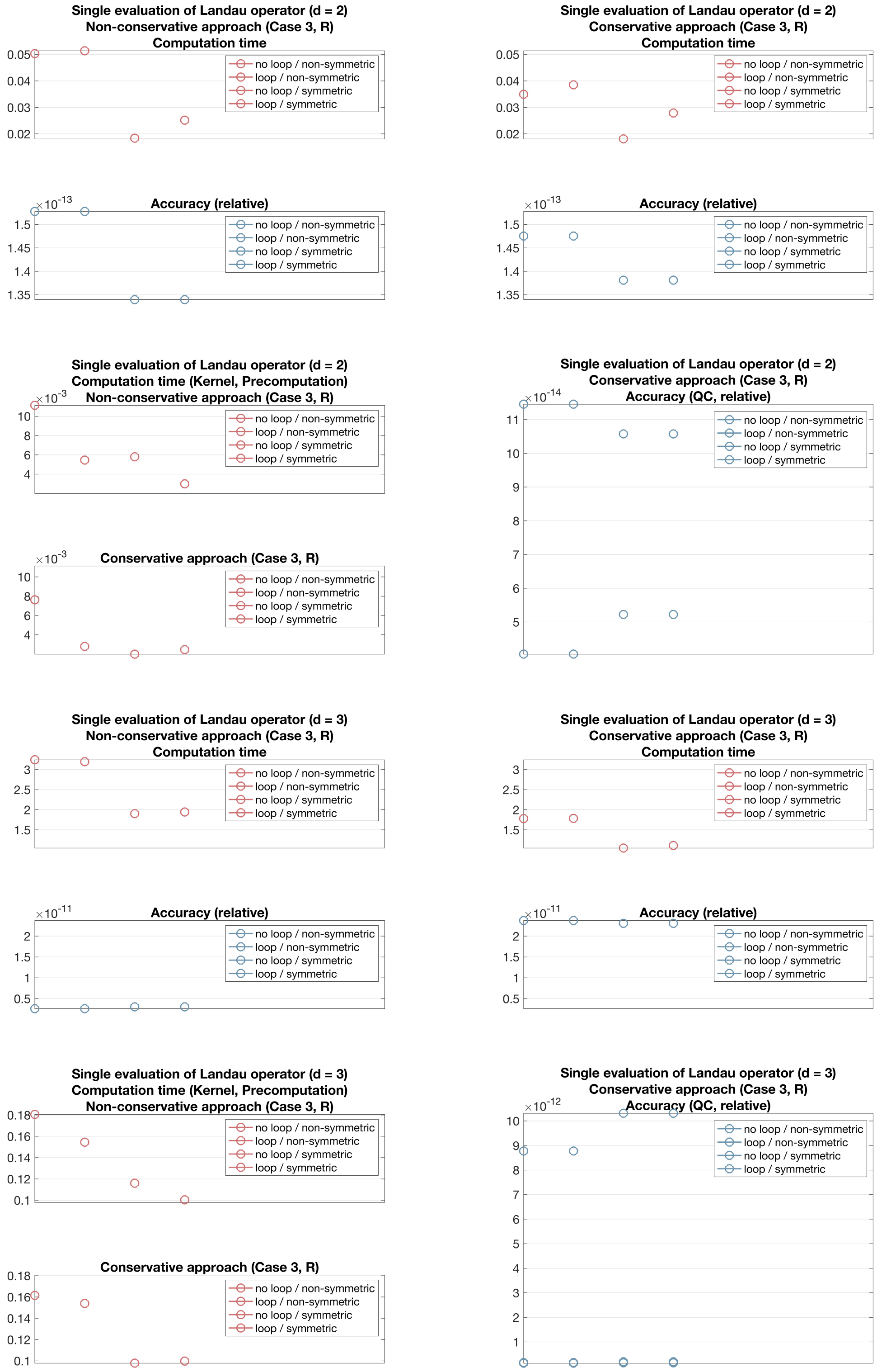
Single evaluation of Landau operator (d = 2)
Non-conservative approach (Case 3, R)
Computation time
Single evaluation of Landau operator (d = 2)
Conservative approach (Case 3, R)
Computation time
Accuracy (relative)
Accuracy (relative)
Single evaluation of Landau operator (d = 2)
Computation time (Kernel, Precomputation)
Non-conservative approach (Case 3, R)
Single evaluation of Landau operator (d = 2)
Conservative approach (Case 3, R)
Accuracy (QC, relative)
Conservative approach (Case 3, R)
Single evaluation of Landau operator (d = 3)
Non-conservative approach (Case 3, R)
Computation time
Single evaluation of Landau operator (d = 3)
Conservative approach (Case 3, R)
Computation time
Accuracy (relative)
Accuracy (relative)
Single evaluation of Landau operator (d = 3)
Computation time (Kernel, Precomputation)
Non-conservative approach (Case 3, R)
Single evaluation of Landau operator (d = 3)
Conservative approach (Case 3, R)
Accuracy (QC, relative)
Conservative approach (Case 3, R)
no loop / non-symmetric
loop / non-symmetric
no loop / symmetric
loop / symmetric

Fig. 3. Case 3

The Landau operator reads as

$$\Big(Q(f,f)\Big)(v) = \partial_v \cdot \Big(Q_C(f,f)\Big)(v)$$

involving the operator

$$\begin{aligned}
& d = 2\,, \\
& \Big(Q_C(f,f)\Big)(v) \\
& \quad = \int_\Omega \varphi(v-w) \begin{pmatrix} (v_2-w_2)^2 & -(v_1-w_1)(v_2-w_2) \\ -(v_1-w_1)(v_2-w_2) & (v_1-w_1)^2 \end{pmatrix} \\
& \qquad \times \begin{pmatrix} \partial_{v_1} f(v)\, f(w) - f(v)\, \partial_{w_1} f(w) \\ \partial_{v_2} f(v)\, f(w) - f(v)\, \partial_{w_2} f(w) \end{pmatrix} \mathrm{d}w\,, \quad v \in \Omega\,,
\end{aligned}$$

$$\begin{aligned}
& d = 3\,, \\
& \Big(Q_C(f,f)\Big)(v) \\
& = \int_\Omega \varphi(v-w) \\
& \times \begin{pmatrix} (v_2-w_2)^2 + (v_3-w_3)^2 & -(v_1-w_1)(v_2-w_2) & -(v_1-w_1)(v_3-w_3) \\ -(v_1-w_1)(v_2-w_2) & (v_1-w_1)^2 + (v_3-w_3)^2 & -(v_2-w_2)(v_3-w_3) \\ -(v_1-w_1)\,(v_3-w_3) & -(v_2-w_2)(v_3-w_3) & (v_1-w_1)^2 + (v_2-w_2)^2 \end{pmatrix} \\
& \times \begin{pmatrix} \partial_{v_1} f(v)\, f(w) - f(v)\, \partial_{w_1} f(w) \\ \partial_{v_2} f(v)\, f(w) - f(v)\, \partial_{w_2} f(w) \\ \partial_{v_3} f(v)\, f(w) - f(v)\, \partial_{w_3} f(w) \end{pmatrix} \mathrm{d}w\,, \quad v \in \Omega\,.
\end{aligned}$$

Our pragmatic approach to deduce a first alternative representation relies on expanding the integrand. In order to specify the resulting integrals, we found it useful to employ the symbolic notation

$$\begin{aligned}
& d = 2\,, \\
& I_{w\, i_1 i_2\, \varphi\, 00\, f\, j_1 j_2}(v) = \int_\Omega w_1^{i_1} w_2^{i_2}\, \varphi(v-w)\, \partial_{w_1}^{j_1} \partial_{w_2}^{j_2} f(w)\, \mathrm{d}w\,, \quad v \in \Omega\,, \\
& d = 3\,, \\
& I_{w\, i_1 i_2 i_3\, \varphi\, 000\, f\, j_1 j_2 j_3}(v) = \int_\Omega w_1^{i_1} w_2^{i_2} w_2^{i_3}\, \varphi(v-w)\, \partial_{w_1}^{j_1} \partial_{w_2}^{j_2} \partial_{w_3}^{j_3} f(w)\, \mathrm{d}w\,, \quad v \in \Omega\,,
\end{aligned}$$

which indicates the degrees of the monomials as well as the orders of the derivatives of the density function.

Representation in two dimensions. In two dimensions, this yields the identity

$$
\begin{aligned}
&d = 2\,,\\
&\Big(Q_C(f,f)\Big)(v) = \begin{pmatrix}\Big(Q_C^{(1)}(f,f)\Big)(v)\\ \Big(Q_C^{(2)}(f,f)\Big)(v)\end{pmatrix}\,,\\
&\Big(Q_C^{(1)}(f,f)\Big)(v)\\
&= \Big(Q_{00}^{(1)}(f)\Big)(v)\, f(v) + \Big(Q_{10}^{(1)}(f)\Big)(v)\,\partial_{v_1} f(v) + \Big(Q_{01}^{(1)}(f)\Big)(v)\,\partial_{v_2} f(v)\,,\\
&\Big(Q_C^{(2)}(f,f)\Big)(v)\\
&= \Big(Q_{00}^{(2)}(f)\Big)(v)\, f(v) + \Big(Q_{10}^{(2)}(f)\Big)(v)\,\partial_{v_1} f(v) + \Big(Q_{01}^{(2)}(f)\Big)(v)\,\partial_{v_2} f(v)\,,
\end{aligned}
$$

$$
\begin{aligned}
&\Big(Q_{00}^{(1)}(f)\Big)(v)\\
&= v_1\, v_2\, I_{w\,00\,\varphi\,00\,f\,01}(v) - v_2^2\, I_{w\,00\,\varphi\,00\,f\,10}(v)\\
&\qquad - v_1\, I_{w\,01\,\varphi\,00\,f\,01}(v) + v_2\,\Big(2\, I_{w\,01\,\varphi\,00\,f\,10}(v) - I_{w\,10\,\varphi\,00\,f\,01}(v)\Big)\\
&\qquad - I_{w\,02\,\varphi\,00\,f\,10}(v) + I_{w\,11\,\varphi\,00\,f\,01}(v)\,,\\
&\Big(Q_{00}^{(2)}(f)\Big)(v)\\
&= -\, v_1^2\, I_{w\,00\,\varphi\,00\,f\,01}(v) + v_1\, v_2\, I_{w\,00\,\varphi\,00\,f\,10}(v)\\
&\qquad v_1\,\Big(-\, I_{w\,01\,\varphi\,00\,f\,10}(v) + 2\, I_{w\,10\,\varphi\,00\,f\,01}(v)\Big) - v_2\, I_{w\,10\,\varphi\,00\,f\,10}(v)\\
&\qquad - I_{w\,20\,\varphi\,00\,f\,01}(v) + I_{w\,11\,\varphi\,00\,f\,10}(v)\,,\\
&\Big(Q_{10}^{(1)}(f)\Big)(v)\\
&= v_2^2\, I_{w\,00\,\varphi\,00\,f\,00}(v) - 2\, v_2\, I_{w\,01\,\varphi\,00\,f\,00}(v) + I_{w\,02\,\varphi\,00\,f\,00}(v)\,,\\
&\Big(Q_{01}^{(1)}(f)\Big)(v)\\
&= -\, v_1\, v_2\, I_{w\,00\,\varphi\,00\,f\,00}(v) + v_1\, I_{w\,01\,\varphi\,00\,f\,00}(v) + v_2\, I_{w\,10\,\varphi\,00\,f\,00}(v)\\
&\qquad + I_{w\,11\,\varphi\,00\,f\,00}(v)\,,\\
&\Big(Q_{10}^{(2)}(f)\Big)(v)\\
&= v_1\, v_2\, I_{w\,00\,\varphi\,00\,f\,00}(v) + v_1\, I_{w\,01\,\varphi\,00\,f\,00}(v) + v_2\, I_{w\,10\,\varphi\,00\,f\,00}(v)\\
&\qquad + I_{w\,11\,\varphi\,00\,f\,00}(v)\,,\\
&\Big(Q_{01}^{(2)}(f)\Big)(v)\\
&= v_1^2\, I_{w\,00\,\varphi\,00\,f\,00}(v) - 2\, v_1\, I_{w\,10\,\varphi\,00\,f\,00}(v) + I_{w\,20\,\varphi\,00\,f\,00}(v)\,.
\end{aligned}
$$

$d = 3\,,$

$$\Big(Q_C(f,f)\Big)(v) = \begin{pmatrix} \Big(Q_C^{(1)}(f,f)\Big)(v) \\ \Big(Q_C^{(2)}(f,f)\Big)(v) \\ \Big(Q_C^{(3)}(f,f)\Big)(v) \end{pmatrix}\,,$$

$$\begin{aligned}
&\Big(Q_C^{(1)}(f,f)\Big)(v) \\
&= \Big(Q_{000}^{(1)}(f)\Big)(v)\, f(v) \\
&\qquad + \Big(Q_{100}^{(1)}(f)\Big)(v)\, \partial_{v_1} f(v) + \Big(Q_{010}^{(1)}(f)\Big)(v)\, \partial_{v_2} f(v) + \Big(Q_{001}^{(1)}(f)\Big)(v)\, \partial_{v_3} f(v)\,, \\
&\Big(Q_C^{(2)}(f,f)\Big)(v) \\
&= \Big(Q_{000}^{(2)}(f)\Big)(v)\, f(v) \\
&\qquad + \Big(Q_{100}^{(2)}(f)\Big)(v)\, \partial_{v_1} f(v) + \Big(Q_{010}^{(2)}(f)\Big)(v)\, \partial_{v_2} f(v) + \Big(Q_{001}^{(2)}(f)\Big)(v)\, \partial_{v_3} f(v)\,, \\
&\Big(Q_C^{(3)}(f,f)\Big)(v) \\
&= \Big(Q_{000}^{(3)}(f)\Big)(v)\, f(v) \\
&\qquad + \Big(Q_{100}^{(3)}(f)\Big)(v)\, \partial_{v_1} f(v) + \Big(Q_{010}^{(3)}(f)\Big)(v)\, \partial_{v_2} f(v) + \Big(Q_{001}^{(3)}(f)\Big)(v)\, \partial_{v_3} f(v)\,,
\end{aligned}$$

$$
\begin{aligned}
&\Big(Q^{(1)}_{000}(f)\Big)(v)\\
&= v_1\, v_2\, I_{w\,000\,\varphi\,000\,f\,010}(v) + v_1\, v_3\, I_{w\,000\,\varphi\,000\,f\,001}(v) - \Big(v_2^2 + v_3^2\Big)\, I_{w\,000\,\varphi\,000\,f\,100}(v)\\
&\quad - v_1\, \Big(I_{w\,010\,\varphi\,000\,f\,010}(v) + I_{w\,001\,\varphi\,000\,f\,001}(v)\Big)\\
&\quad + v_2\, \Big(2\, I_{w\,010\,\varphi\,000\,f\,100}(v) - I_{w\,100\,\varphi\,000\,f\,010}(v)\Big)\\
&\quad + v_3\, \Big(2\, I_{w\,001\,\varphi\,000\,f\,100}(v) - I_{w\,100\,\varphi\,000\,f\,001}(v)\Big)\\
&\quad - I_{w\,020\,\varphi\,000\,f\,100}(v) - I_{w\,002\,\varphi\,000\,f\,100}(v)\\
&\quad + I_{w\,110\,\varphi\,000\,f\,010}(v) + I_{w\,101\,\varphi\,000\,f\,001}(v)\,,\\
&\Big(Q^{(2)}_{000}(f)\Big)(v)\\
&= -\, v_1^2\, I_{w\,000\,\varphi\,000\,f\,010}(v) + v_1\, v_2\, I_{w\,000\,\varphi\,000\,f\,100}(v) + v_2\, v_3\, I_{w\,000\,\varphi\,000\,f\,001}(v)\\
&\quad - v_3^2\, I_{w\,000\,\varphi\,000\,f\,010}(v)\\
&\quad + v_1\, \Big(-\, I_{w\,010\,\varphi\,000\,f\,100}(v) + 2\, I_{w\,100\,\varphi\,000\,f\,010}(v)\Big)\\
&\quad - v_2\, \Big(I_{w\,100\,\varphi\,000\,f\,100}(v) + I_{w\,001\,\varphi\,000\,f\,001}(v)\Big)\\
&\quad + v_3\, \Big(2\, I_{w\,001\,\varphi\,000\,f\,010}(v) - I_{w\,010\,\varphi\,000\,f\,001}(v)\Big)\\
&\quad - I_{w\,200\,\varphi\,000\,f\,010}(v) - I_{w\,002\,\varphi\,000\,f\,010}(v)\\
&\quad + I_{w\,110\,\varphi\,000\,f\,100}(v) + I_{w\,011\,\varphi\,000\,f\,001}(v)\,,\\
&\Big(Q^{(3)}_{000}(f)\Big)(v)\\
&= -\, \Big(v_1^2 + v_3^2\Big)\, I_{w\,000\,\varphi\,000\,f\,001}(v) + v_1\, v_3\, I_{w\,000\,\varphi\,000\,f\,100}(v)\\
&\quad + v_2\, v_3\, I_{w\,000\,\varphi\,000\,f\,010}(v)\\
&\quad + v_1\, \Big(-\, I_{w\,001\,\varphi\,000\,f\,100}(v) + 2\, I_{w\,100\,\varphi\,000\,f\,001}(v)\Big)\\
&\quad + v_2\, \Big(-\, I_{w\,001\,\varphi\,000\,f\,010}(v) + 2\, I_{w\,010\,\varphi\,000\,f\,001}(v)\Big)\\
&\quad - v_3\, \Big(I_{w\,010\,\varphi\,000\,f\,010}(v) + I_{w\,100\,\varphi\,000\,f\,100}(v)\Big)\\
&\quad - I_{w\,200\,\varphi\,000\,f\,001}(v) - I_{w\,020\,\varphi\,000\,f\,001}(v)\\
&\quad + I_{w\,101\,\varphi\,000\,f\,100}(v) + I_{w\,011\,\varphi\,000\,f\,010}(v)\,,
\end{aligned}
$$

$$
\begin{aligned}
&\Big(Q^{(1)}_{100}(f)\Big)(v)\\
&= \Big(v_2^2 + v_3^2\Big)\, I_{w\,000\,\varphi\,000\,f\,000}(v)\\
&\qquad - 2\,v_2\, I_{w\,010\,\varphi\,000\,f\,000}(v) - 2\,v_3\, I_{w\,001\,\varphi\,000\,f\,000}(v)\\
&\qquad + I_{w\,020\,\varphi\,000\,f\,000}(v) + I_{w\,002\,\varphi\,000\,f\,000}(v)\,,\\
&\Big(Q^{(1)}_{010}(f)\Big)(v)\\
&= -\,v_1\,v_2\, I_{w\,000\,\varphi\,000\,f\,000}(v)\\
&\qquad + v_1\, I_{w\,010\,\varphi\,000\,f\,000}(v) + v_2\, I_{w\,100\,\varphi\,000\,f\,000}(v)\\
&\qquad - I_{w\,110\,\varphi\,000\,f\,000}(v)\,,\\
&\Big(Q^{(1)}_{001}(f)\Big)(v)\\
&= -\,v_1\,v_3\, I_{w\,000\,\varphi\,000\,f\,000}(v)\\
&\qquad + v_1\, I_{w\,001\,\varphi\,000\,f\,000}(v) + v_3\, I_{w\,100\,\varphi\,000\,f\,000}(v)\\
&\qquad - I_{w\,101\,\varphi\,000\,f\,000}(v)\,,\\
&\Big(Q^{(2)}_{100}(f)\Big)(v)\\
&= -\,v_1\,v_2\, I_{w\,000\,\varphi\,000\,f\,000}(v)\\
&\qquad + v_1\, I_{w\,010\,\varphi\,000\,f\,000}(v) + v_2\, I_{w\,100\,\varphi\,000\,f\,000}(v)\\
&\qquad - I_{w\,110\,\varphi\,000\,f\,000}(v)\,,\\
&\Big(Q^{(2)}_{010}(f)\Big)(v)\\
&= \Big(v_1^2 + v_3^2\Big)\, I_{w\,000\,\varphi\,000\,f\,000}(v)\\
&\qquad - 2\,v_1\, I_{w\,100\,\varphi\,000\,f\,000}(v) - 2\,v_3\, I_{w\,001\,\varphi\,000\,f\,000}(v)\\
&\qquad + I_{w\,200\,\varphi\,000\,f\,000}(v) + I_{w\,002\,\varphi\,000\,f\,000}(v)\,,\\
&\Big(Q^{(2)}_{001}(f)\Big)(v)\\
&= -\,v_2\,v_3\, I_{w\,000\,\varphi\,000\,f\,000}(v)\\
&\qquad + v_2\, I_{w\,001\,\varphi\,000\,f\,000}(v) + v_3\, I_{w\,010\,\varphi\,000\,f\,000}(v)\\
&\qquad - I_{w\,011\,\varphi\,000\,f\,000}(v)\,,\\
&\Big(Q^{(3)}_{100}(f)\Big)(v)\\
&= -\,v_1\,v_3\, I_{w\,000\,\varphi\,000\,f\,000}(v)\\
&\qquad + v_1\, I_{w\,001\,\varphi\,000\,f\,000}(v) + v_3\, I_{w\,100\,\varphi\,000\,f\,000}(v)\\
&\qquad - I_{w\,101\,\varphi\,000\,f\,000}(v)\,,\\
&\Big(Q^{(3)}_{010}(f)\Big)(v)\\
&= -\,v_2\,v_3\, I_{w\,000\,\varphi\,000\,f\,000}(v)\\
&\qquad + v_2\, I_{w\,001\,\varphi\,000\,f\,000}(v) + v_3\, I_{w\,010\,\varphi\,000\,f\,000}(v)\\
&\qquad - I_{w\,011\,\varphi\,000\,f\,000}(v)\,,\\
&\Big(Q^{(3)}_{001}(f)\Big)(v)\\
&= \Big(v_1^2 + v_2^2\Big)\, I_{w\,000\,\varphi\,000\,f\,000}(v)\\
&\qquad - 2\,v_1\, I_{w\,100\,\varphi\,000\,f\,000}(v) - 2\,v_2\, I_{w\,010\,\varphi\,000\,f\,000}(v)\\
&\qquad + I_{w\,200\,\varphi\,000\,f\,000}(v) + I_{w\,002\,\varphi\,000\,f\,000}(v)\,.
\end{aligned}
$$

Simplification for constant kernel (similar to approach in non-conservative form).