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Stability of linear multistep methods and applications to nonlinear parabolic problems

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Abstract

In the present paper, stability and convergence properties of linear multistep methods are investigated. The attention is focused on parabolic problems and variable stepsizes. Under weak assumptions on the method and the stepsize sequence an asymptotic stability result is shown. Further, stability bounds for linear nonautonomous parabolic problems with Hölder continuous operator are given. With the help of these results, convergence estimates for semilinear and fully nonlinear parabolic problems are derived.

Keywords: Linear multistep methods; Stability; Variable stepsizes; Semilinear parabolic problems; Fully nonlinear parabolic problems; Convergence; Geometric properties

1. Introduction

In this paper, we study the stability and convergence properties of linear multistep methods applied to nonlinear parabolic problems. Our analysis admits variable stepsizes and is based on an abstract framework of sectorial operators and analytic semigroups in Banach spaces.

Stability results for variable stepsize multistep discretizations generally require that the ratios of two subsequent steps are bounded from below and above, i.e., $\Omega_1 \leqslant h_n/h_{n-1} \leqslant \Omega_2$ with appropriate Ω_1 and Ω_2. For ordinary differential equations, two conceptually different types of stability estimates are found in literature, see [9, Section III.5].

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The first approach ensures stability *independently* of the chosen stepsize sequence. To our knowledge, Grigorieff [8] was the first who analyzed BDF discretizations of ordinary differential equations in this way. Recently, stability bounds that depend only weakly on the stepsize sequence have been derived for BDF discretizations of parabolic problems in a Hilbert space setting by Becker [2], Calvo and Grigorieff [3], and Emmrich [5]. Notwithstanding the merits of this approach, it has its shortcomings as well, since it gives, in general, quite disappointing values for Ω_1 and Ω_2. For BDF5, e.g., one obtains the stringent condition $0.997 \leqslant h_n/h_{n-1} \leqslant 1.003$ in order to ensure zero-stability without further restrictions on the stepsize sequence.

The second approach allows the stability factor to *depend* on the stepsize sequence. To obtain (practical) stability, however, it must be guaranteed that this factor remains bounded by a (reasonable) constant. For ordinary differential equations, this approach was used by Gear and Tu [6]. Under the assumption that the stepsize sequence depends smoothly on the local errors, they obtained favourable convergence results. More recently, this direction has been further exploited for linear parabolic problems in a series of papers by Palencia [14,15] and Palencia and García-Archilla [16]. The results of Palencia, however, are not sufficient to obtain convergence for nonlinear problems. Our motivation for the present paper was to derive the missing stability estimates and to develop a convergence theory of multistep methods for nonlinear parabolic problems.

The present paper is structured as follows: In Section 2, we first introduce the analytical framework, based on the theory of sectorial operators in Banach spaces, and we specify the requirements on the numerical method. We then derive our main stability results for asymptotically stable analytic semigroups. This is the key for proving asymptotic stability of multistep discretizations. In Section 3, we extend the stability results of Section 2 to arbitrary sectorial operators, and then in Section 4 to linear nonautonomous parabolic problems with Hölder continuous operator. In Section 5, we apply the stability results to semilinear parabolic problems, and we derive a convergence result for finite times. In Section 6, we give applications to fully nonlinear parabolic problems. We study the long-term behaviour of multistep discretizations nearby an asymptotically stable equilibrium, and we state a convergence result for smooth solutions on compact time intervals. Corresponding results for Runge–Kutta methods are found in our papers [7,13,18].

Throughout this paper, we employ the following notation. For normed spaces Y and Z, the space $L(Y,Z)$ comprises all linear operators from Y to Z. It is endowed with the usual operator norm denoted by $\|\cdot\|_{Z\leftarrow Y}$. For an integer $k \geqslant 1$, the norm on the product space Y^k is defined by $\|y\|_{Y^k} = \max\{\|y_i\|_Y\colon\ 1 \leqslant i \leqslant k\}$ for $y = (y_1,\dots,y_k)^{\mathrm{T}} \in Y^k$. In order to simplify the notation, we usually dismiss the dimensions in the operator norm and write $\|\cdot\|_{Z\leftarrow Y}$ instead of $\|\cdot\|_{Z^k\leftarrow Y^k}$ for short. We recall that for an arbitrary matrix B with coefficients b_{ij} and a linear operator A, the (i,j)th component of the Kronecker product $B \otimes A$ equals $b_{ij}A$. We further distinguish between the identity operator I and the identity matrix $\mathcal{I}$ on $\mathbb{R}^k$.

Henceforth, C denotes a generic constant with possibly different values at different occurrences.

2. Asymptotic stability for time-independent operators

In this section, we derive fundamental stability estimates for linear multistep methods with variable stepsizes. Our results substantially rely on the papers [15] and [16].

We study the abstract initial value problem on a Banach space $(X, \|\cdot\|_X)$

$$u'(t) = Au(t), \quad t > 0, \qquad u(0) \text{ given}, \tag{1}$$

where A is a densely defined and closed linear operator on X. The domain D of A is endowed with the graph norm $\|\cdot\|_D$. Our main assumption on A is the following, cf. [12] or [10].

HA1. *We suppose that $A \in L(D, X)$ is sectorial on X, i.e., for some constants $a \in \mathbb{R}$, $M \geqslant 1$ and $\varphi \in (0, \pi/2)$, the resolvent of A fulfills the condition*

$$\left\|(\lambda I - A)^{-1}\right\|_{X \leftarrow X} \leqslant \frac{M}{|\lambda - a|}, \quad \lambda \in \mathbb{C} \setminus S_\varphi(a), \tag{2}$$

on the complement of the sector $S_\varphi(a) = \{\lambda \in \mathbb{C}\colon |\arg(a - \lambda)| \leqslant \varphi\} \cup \{a\}$.

Let $(h_n)_{n\geqslant 0}$ denote the sequence of positive time steps with corresponding ratios $\omega_n = h_n/h_{n-1}$, $n \geqslant 1$, and set $\boldsymbol{\omega}_n = (\omega_n, \ldots, \omega_{n+k-2})$. The associated grid points are denoted by $t_n = h_0 + h_1 + \cdots + h_{n-1}$. Throughout the paper, we use the following assumption on the stepsize sequence.

HS1. *We assume that there exists $\Omega > 1$ such that the stepsize ratios satisfy $\Omega^{-1} \leqslant \omega_n \leqslant \Omega$ for all $n \geqslant 1$.*

We first draw some conclusions from this hypothesis that are all-important for our stability results. Let $(h_n)_{n\geqslant 0}$ be a stepsize sequence satisfying HS1. For the subsequence $h_{k-1}, h_k, \ldots, h_{k+j-2}$ of length j, consider the associated sequence of ordered stepsizes $h_{\pi(1)} \leqslant h_{\pi(2)} \leqslant \cdots \leqslant h_{\pi(j)}$ and set

$$\tau_\varkappa^{(j)} = h_{\pi(1)} + \cdots + h_{\pi(\varkappa)} \quad \text{for } 0 \leqslant \varkappa \leqslant j. \tag{3a}$$

From the identity

$$t_{n+k-1} = h_{n+k-2} + \cdots + h_{j+k-1} + h_{\pi(j)} + \cdots + h_{\pi(\varkappa+1)} + \tau_\varkappa^{(j)} + t_{k-1},$$

with the help of HS1 and the obvious estimates $h_{j+k-1} \leqslant \Omega h_{\pi(j)}$ and $h_{\pi(\varkappa+1)} \leqslant \Omega \tau_\varkappa^{(j)}$, we get the useful relation

$$t_{n+k-1} - t_{k-1} \leqslant C\Omega^{n-\varkappa} \tau_\varkappa^{(j)} \quad \text{for } 1 \leqslant \varkappa \leqslant j \leqslant n. \tag{3b}$$

We further note for later use that

$$\begin{aligned} &C t_{n+k-1} \leqslant t_{n+k-1} - t_{k-1} \leqslant t_{n+k-1}, \quad n \geqslant 1, \\ &t_{n+k-1} - t_{k-1} \leqslant C\Omega^n h_{k-1}. \end{aligned} \tag{3c}$$

The numerical approximation u_{n+k} to the solution of (1) at time t_{n+k} by a linear multistep method is given recursively by

$$\sum_{i=0}^{k} \alpha_{ni} u_{n+i} = h_{n+k-1} A \sum_{i=0}^{k} \beta_{ni} u_{n+i}, \quad n \geqslant 0. \tag{4}$$

This relation involves the coefficients α_{ni} and β_{ni}, $0 \leqslant i \leqslant k$, that may depend on $\boldsymbol{\omega}_{n+1}$, and on the starting values $u_0, u_1, \ldots, u_{k-1}$. For more information on variable stepsize linear multistep methods, we refer to the monograph [9].

In order to write the numerical scheme (4) in compact vector form, we denote

$$U_n = (u_n, u_{n+1}, \ldots, u_{n+k-1})^{\mathrm{T}},$$

for $n \geqslant 0$. Further, we introduce the functions

$$\begin{aligned} J_{n+1}(z) &= (\alpha_{nk} - \beta_{nk} z)^{-1}, \\ s_{n+1,i}(z) &= -J_{n+1}(z) \cdot (\alpha_{ni} - \beta_{ni} z), \quad 0 \leqslant i \leqslant k-1. \end{aligned} \tag{5}$$

Here, the first index indicates the dependence on $\boldsymbol{\omega}_{n+1}$. Then, the companion matrix of the method is given by

$$r_{n+1}(z) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ s_{n+1,0}(z) & s_{n+1,1}(z) & \dots & \dots & s_{n+1,k-1}(z) \end{pmatrix}.$$

For constant time steps, we denote the companion matrix by $r(z)$ for short.

The above notation allows us to rewrite (4) as

$$U_n = \prod_{j=1}^{n} r_j(h_{j+k-2}A)\, U_0, \quad n \geqslant 0. \tag{6}$$

We note that the factors arising in the product do not commute, in general.

Throughout the paper, we require the following stability assumption for constant stepsizes.

HM1. *We assume that the linear multistep method* (4) *is $A(\varphi)$-stable and strictly stable at* 0 *and infinity. Thus, $\lambda = 1$ is the only eigenvalue of the companion matrix at* 0 *with modulus one, and the spectral radius of the companion matrix at infinity, $\varrho = \varrho(r(\infty))$, is less than one.*

The following hypothesis is needed for variable stepsizes.

HM2. *We assume that the coefficients α_{ni} and β_{ni} in* (4) *are bounded for all stepsize sequences satisfying* HS1. *We further require that the rational functions $J_n(z)$ and $s_{ni}(z)$ in* (5) *remain bounded for $z \in S_\varphi(0)$.*

If the k-step method is consistent of order p, the principal eigenvalue $\lambda_1(z)$ of $r(z)$ fulfills

$$\lambda_1(z) = \mathrm{e}^z + \mathcal{O}\big(z^{p+1}\big), \quad z \to 0,$$

see [11]. Note that this relation implies

$$\lambda_1(z) = \mathrm{e}^{z+\eta(z)} \quad \text{with } \eta(z) = \mathcal{O}\big(z^{p+1}\big) \text{ for } z \to 0,$$

which is important for the results we have in mind.

HM3. *We assume that the linear multistep method* (4) *has order $p \geqslant 1$.*

Example. The k-step BDF methods, for $1 \leqslant k \leqslant 6$, satisfy HM1–2 with $\varrho = 0$ for any $\Omega > 0$, and HM3 with $p = k$, see [9].

Now we are ready to state the stability result. The corresponding results for Runge–Kutta methods are given in [1]. In order to simplify the notation we introduce the abbreviation

$$C^{\Delta}_{\omega,n} = C \prod_{j=1}^{n+k-2} \bigl(1 + \Delta|\omega_j - 1|\bigr)^2 \leqslant C\mathrm{e}^{2\Delta \sum_{j=1}^{n+k-2} |\omega_j - 1|}. \tag{7}$$

Theorem 1. *Consider the linear multistep discretization* (4) *of Eq.* (1)*, and assume that* HA1 *with* $a < 0$, HS1*, and* HM1–3 *hold. If* $\mu \geqslant 1$ *satisfies* $\varrho\Omega^{\mu} < 1$*, then the following bound is valid for all* $n \geqslant 1$

$$\Biggl\| \prod_{j=1}^{n} r_j(h_{j+k-2}A) \Biggr\|_{D\leftarrow D} \leqslant \frac{C^{\Delta}_{\omega,n}}{1 + t^{\mu}_{n+k-1}}.$$

If in addition $\mu \geqslant 2$*, we have for all* $n \geqslant 1$

$$\Biggl\| \prod_{j=1}^{n} r_j(h_{j+k-2}A) - \prod_{j=1}^{n} r_j(\infty) \Biggr\|_{D\leftarrow X} \leqslant \frac{C^{\Delta}_{\omega,n}}{t_{n+k-1} + t^{\mu}_{n+k-1}}.$$

Recall that under hypothesis HA1 with $a < 0$, the semigroup e^{tA} decays exponentially fast to 0, since for any $\tilde{a} > a$

$$\bigl\|\mathrm{e}^{tA}\bigr\|_{D\leftarrow D} + t\bigl\|\mathrm{e}^{tA}\bigr\|_{D\leftarrow X} \leqslant C\mathrm{e}^{\tilde{a}t}, \quad t \geqslant 0.$$

Let the stepsize sequence be such that $t_n \to \infty$ for $n \to \infty$. If the quotient

$$\frac{C^{\Delta}_{\omega,n}}{1 + t^{\mu}_{n+k-1}} \to 0, \quad \text{for } n \to \infty, \tag{8}$$

then the numerical method is asymptotically stable. For *practical* purposes, however, it is essential that the quotient in (8) remains bounded by a *reasonable* constant for all n. This is achieved, for example, in the following situation.

Example. Suppose that the size of the gridpoints grows exponentially fast $t_n \geqslant Cq^n$ with some $1 < q \leqslant \Omega$ and that for some $\mu \geqslant 2$

$$\bigl(1 + \Delta(\Omega - 1)\bigr)^2 < q^{\mu} \quad \text{and} \quad \varrho\Omega^{\mu} < 1. \tag{9}$$

Then, the numerical method is asymptotically stable, and the constant in (8) is reasonable, if $|a|h_0$ is not too small. Note that (9) holds for μ sufficiently large, if $\varrho = 0$.

Proof of Theorem 1. Our proof is strongly based on the work of Palencia. An application of a matrix version of [15, Lemma 1 and Theorem 2] to the shifted operator $A - aI$ shows that the following bound holds

$$\bigl\|g(A)\bigr\| \leqslant C\|g\|_{\varphi,a} + C\|g\|_{\varphi,a} \log^{+}\biggl(\frac{N_{\varphi,a}(g)}{\|g\|_{\varphi,a}}\biggr),$$

for any holomorphic mapping g defined on some neighbourhood of $S_{\varphi}(a)$ taking values in the space of complex $k \times k$ matrices. We here denote

$$\|g\|_{\varphi,a} = \sup\bigl\{\bigl\|g(\lambda)\bigr\|: \lambda \in S_{\varphi}(a)\bigr\},$$
$$N_{\varphi,a}(g) = \bigl\|g(a)\bigr\| + \bigl\|g(\infty)\bigr\| + \sqrt{Z_{\varphi,a}(g)I_{\varphi,a}(g)},$$

with

$$Z_{\varphi,a}(g) = \sup\bigl\{ \bigl\| (\lambda - a)^{-1}\bigl(g(\lambda) - g(a)\bigr)\bigr\| \colon \lambda \in S_\varphi(a)\bigr\},$$
$$I_{\varphi,a}(g) = \sup\bigl\{ \bigl\| (\lambda - a)\bigl(g(\lambda) - g(\infty)\bigr)\bigr\| \colon \lambda \in S_\varphi(a)\bigr\}.$$

The first part of the theorem follows by applying this bound to the function

$$g(\lambda) = \prod_{j=1}^{n} r_j(h_{j+k-2}\lambda), \tag{10}$$

and Lemma 4 below. More precisely, we use the inequality $x\log^+(y/x) \leqslant \log^+(y/b) + b/\mathrm{e}$ which holds for $y \geqslant 0$ and $x, b > 0$. We recall here that $\log^+ x = \max(0, \log x)$. Setting $x = \|g\|_{\varphi,a}$, $y = N_{\varphi,a}(g)$ and

$$b = \frac{1}{1 + t_{n+k-1}^{\mu}} \prod_{j=1}^{n+k-2} \bigl(1 + \Delta|\omega_j - 1|\bigr)^2,$$

then yields the first estimate of the theorem with the additional factor $1 + \log^+ t_{n+k-1}$. This factor, however, can be omitted by slightly increasing μ. The second part of the theorem follows in the same way by using the function

$$G(\lambda) = \lambda\left(\prod_{j=1}^{n} r_j(h_{j+k-2}\lambda) - \prod_{j=1}^{n} r_j(\infty)\right), \tag{11}$$

and Lemma 5. □

The auxiliary results that are needed in the above proof are collected in the remainder of this section. First, we study the behaviour of the companion matrix for constant time steps. We remark that $r(z)$ satisfies an estimate of the form

$$\bigl\|r(z_1) - r(z_2)\bigr\| \leqslant C \min\bigl(|z_1 - z_2|, \bigl|z_1^{-1} - z_2^{-1}\bigr|\bigr), \quad z_1, z_2 \in S_\varphi(0).$$

The following lemma is an extension of [16, Theorem A.1]. For a related decomposition of the companion matrix, see [4].

Lemma 1. *Let $r(z)$ be the companion matrix of a linear multistep method satisfying* HM1 *and* HM3. *Then there exists a map T defined on $S_\varphi(0)$ with values in the space of complex $k \times k$ matrices with the following properties: For any $0 < \varrho < \delta < 1$, there exist a neighbourhood $B = \{z \in \mathbb{C}\colon |z| \leqslant \sigma\}$ of the origin and a constant $0 < c < 1$ such that the following estimates hold*

$$\begin{aligned} &\bigl\|T(z)r(z)T(z)^{-1}\bigr\| \leqslant \mathrm{e}^{c\,\mathrm{Re}\,z}, && z \in \Sigma_0 = B \cap S_\varphi(0),\\ &\bigl\|T(z)r(z)T(z)^{-1}\bigr\| \leqslant \delta, && z \in \Sigma_\infty = S_\varphi(0) \setminus \Sigma_0. \end{aligned} \tag{12a}$$

Furthermore, we have for all $z, z_1, z_2 \in S_\varphi(0)$

$$\begin{aligned} &\bigl\|T(z)\bigr\| \leqslant C, \qquad \bigl\|T(z)^{-1}\bigr\| \leqslant C,\\ &\bigl\|T(z_1) - T(z_2)\bigr\| \leqslant C \min\bigl(|z_1 - z_2|, \bigl|z_1^{-1} - z_2^{-1}\bigr|\bigr). \end{aligned} \tag{12b}$$

Proof. The lemma is a consequence of [16, Theorem A.1]. In order to show the additional estimates (12a), we choose a Lipschitz-continuous map ψ that coincides with the exponential $\psi(z) = \mathrm{e}^{cz}$ near the origin and satisfies $\psi(z) = \delta$ in a neighborhood of ∞ in such a way that $\varrho\big(\psi(z)^{-1} r(z)\big) < 1$ holds on $S_\varphi(0) \setminus \{0\}$. Then, the result follows from an application of Theorem A.1 in *loc.cit.* □

In order to study the product $g(\lambda)$ it is useful to introduce the map

$$\Phi : S_\varphi(0) \to \mathbb{C} : z \mapsto \Phi(z) = \begin{cases} \mathrm{e}^{c\,\mathrm{Re}\,z} & \text{if } z \in \Sigma_0, \\ \delta & \text{if } z \in \Sigma_\infty, \end{cases}$$

which essentially captures the behaviour of $r(z)$, see (12a).

Lemma 2. *Under the assumptions of Lemma* 1 *and* HM2, *it holds*

$$\left\| \prod_{j=1}^{n} r_j(h_{j+k-2}\lambda) \right\| \leqslant C^{\Delta}_{\omega,n} \cdot \prod_{j=1}^{n} \Phi(h_j \lambda), \qquad \lambda \in S_\varphi(a).$$

Proof. The proof is very close to that of [16, Lemma 3.2]. Replacing relation (32) of *loc.cit.* with (12a) and tracing its effects, yields the result. In particular, the very form of $C^{\Delta}_{\omega,n}$ in (7) as a product is obtained. □

Lemma 3. *Let the stepsize sequence satisfy* HS1, *and let* $\gamma > 0$ *and* $\mu > 0$ *be such that* $\gamma\,\Omega^\mu \leqslant 1$. *Then, for* $c > 0$, *there exists a constant* C *such that*

$$\gamma^{n-m} \mathrm{e}^{-c\tau_m^{(j)}} \leqslant \frac{C}{1 + t^{\mu}_{n+k-1}} \quad \textit{for all } \ 0 \leqslant m \leqslant j \leqslant n$$

with $\tau_m^{(j)}$ *given by* (3a).

Proof. From (3) we obtain $t^{\mu}_{n+k-1} \gamma^{n-m} \leqslant C(\gamma\,\Omega^\mu)^{n-m} (\tau_m^{(j)})^\mu$, and the assertion follows at once from the uniform boundedness of $s^\mu \mathrm{e}^{-cs}$ for positive s. □

We are now ready to derive the desired estimates for the function $g(\lambda)$, defined in (10).

Lemma 4. *Under the assumptions of the theorem, it holds*

$$\sup_{\lambda \in S_\varphi(a)} \big\| g(\lambda) \big\| \leqslant \frac{C^{\Delta}_{\omega,n}}{1 + t^{\mu}_{n+k-1}},$$

$$\sup_{\lambda \in S_\varphi(a)} \big\| (\lambda - a)^{-1} \big(g(\lambda) - g(a)\big) \big\| \leqslant \frac{C^{\Delta}_{\omega,n}}{1 + t^{\mu}_{n+k-1}}\, t_{n+k-1},$$

$$\sup_{\lambda \in S_\varphi(a)} \big\| (\lambda - a) \big(g(\lambda) - g(\infty)\big) \big\| \leqslant \frac{C^{\Delta}_{\omega,n}}{1 + t^{\mu-1}_{n+k-1}}\, t^{-1}_{n+k-1}.$$

Proof. It is convenient to employ the following abbreviations

$$r_j = r_j(h_{j+k-2}\lambda), \qquad \varrho_j = r_j(\infty). \tag{13}$$

We choose $\varrho < \delta < 1$ such that $\delta\Omega^{\mu} < 1$. Let $\varkappa$ denote the number of indices $k-1 \leqslant m \leqslant n-1$ such that $|h_m\lambda| < \sigma$. From Lemma 2 and (3b), we get

$$\|g(\lambda)\| \leqslant C^{\Delta}_{\omega,n}\delta^{n-\varkappa}\mathrm{e}^{c\tau^{(n)}_{\varkappa}\operatorname{Re}\lambda}.$$

Since $\operatorname{Re}\lambda \leqslant a < 0$, the first assertion of the lemma follows at once from (3) and Lemma 3. For the second bound, we use the telescopic identity

$$g(\lambda) - g(a) = \sum_{j=1}^{n}\prod_{l=j+1}^{n} r_l(h_{l+k-2}a)\big(r_j - r_j(h_{j+k-2}a)\big)\prod_{i=1}^{j-1} r_i,$$

and the estimate $\|r_j - r_j(h_{j+k-2}a)\| \leqslant Ch_{j+k-2}|\lambda - a|$. This yields

$$\big\|(\lambda-a)^{-1}\big(g(\lambda)-g(a)\big)\big\| \leqslant C^{\Delta}_{\omega,n}\sum_{j=1}^{n} h_{j+k-2}\delta^{n-\varkappa_j-1}\mathrm{e}^{c\tau^{(j-1)}_{\varkappa_j}a},$$

with $0 \leqslant \varkappa_j \leqslant n-1$, and the same arguments as before yield the desired bound. The last estimate follows in a similar way from

$$g(\lambda) - g(\infty) = \sum_{j=1}^{n}\prod_{l=j+1}^{n} \varrho_l(r_j - \varrho_j)\prod_{i=1}^{j-1} r_i.$$

For fixed $j \geqslant 2$, let $\varkappa = \varkappa(j)$ denote the number of indices $k-1 \leqslant m \leqslant j+k-3$ such that $|h_m\lambda| < \sigma$. Using $\|(\lambda-a)(r_j - \varrho_j)\| \leqslant Ch^{-1}_{j+k-2}$ for $\varkappa(j) = 0$ and $|\lambda - a|\tau_{\varkappa}\mathrm{e}^{c\tau_{\varkappa}\operatorname{Re}\lambda} \leqslant C$ for $\varkappa(j) \geqslant 1$ yields the desired result. □

We next study the behaviour of $G(\lambda)$, defined in (11).

Lemma 5. *Under the assumptions of the theorem, it holds*

$$\sup_{\lambda\in S_{\varphi}(a)}\|G(\lambda)\| \leqslant \frac{C^{\Delta}_{\omega,n}}{1+t^{\mu-1}_{n+k-1}}\, t^{-1}_{n+k-1},$$

$$\sup_{\lambda\in S_{\varphi}(a)}\big\|(\lambda-a)^{-1}\big(G(\lambda)-G(a)\big)\big\| \leqslant \frac{C^{\Delta}_{\omega,n}}{1+t^{\mu}_{n+k-1}}(1+t_{n+k-1}),$$

$$\sup_{\lambda\in S_{\varphi}(a)}\big\|(\lambda-a)\big(G(\lambda)-G(\infty)\big)\big\| \leqslant \frac{C^{\Delta}_{\omega,n}}{1+t^{\mu-2}_{n+k-1}}\, t^{-2}_{n+k-1}.$$

Proof. Since $G(\lambda) = \lambda(g(\lambda) - g(\infty))$, the first estimate follows in the same way as that in the previous lemma. For the second estimate, we use the identity

$$G(\lambda) - G(a) = (\lambda - a)\big(g(\lambda) - g(a)\big) + a\big(g(\lambda) - g(a)\big),$$

and the previous lemma. In order to show the last relation, we define with (13) the analytic function $\psi_j(\lambda) = \lambda(r_j - \varrho_j)$ which is bounded at infinity by Ch^{-1}_{j+k-2}. Using

$$G(\infty) = \sum_{j=1}^{n}\prod_{l=j+1}^{n}\varrho_l\psi_j(\infty)\prod_{i=1}^{j-1}\varrho_i,$$

and the telescopic identity, we get

$$G(\lambda) - G(\infty) = \sum_{j=1}^{n} \prod_{l=j+1}^{n} \varrho_l \big(\psi_j(\lambda) - \psi_j(\infty)\big) \prod_{i=1}^{j-1} \varrho_i + \sum_{j=1}^{n} \prod_{l=j+1}^{n} \varrho_l (r_j - \varrho_j)\lambda \left(\prod_{i=1}^{j-1} r_i - \prod_{i=1}^{j-1} \varrho_i \right).$$

Expanding the last term again with the telescopic identity, the desired bound now follows as in the previous lemma. □

3. Stability on compact time intervals

In this section, we derive stability estimates for (4) on compact time intervals $[0, T]$. We first give an extension of Theorem 1 to nonnegative a. We make use of the following hypothesis which is familiar from the convergence analysis of linear multistep methods for ODEs, see [9, Theorem III.5.7].

HS2. *We assume that the stability factors $C^{\Delta}_{\omega,n}$ in* (7) *are uniformly bounded by a constant for all $n \geqslant 0$.*

We emphasize that the size of the constant in HS2 may depend on the length of the considered time interval.

Theorem 2. *Consider the linear multistep discretization* (4) *of Eq.* (1) *on the interval* $[0, T]$, *and assume that* HA1, HS1–2, HM1–3 *hold and that* $\varrho\Omega^2 < 1$. *Then, there exist positive constants H and C such that for $0 < h_j \leqslant H$ the following bounds are valid for all $n \geqslant 1$ with $t_n \leqslant T$*

$$\left\| \prod_{j=1}^{n} r_j(h_{j+k-2}A) \right\|_{D \leftarrow D} \leqslant C, \qquad \left\| \prod_{j=1}^{n} r_j(h_{j+k-2}A) - \prod_{j=1}^{n} r_j(\infty) \right\|_{D \leftarrow X} \leqslant \frac{C}{t_{n+k-1}}.$$

The constant C depends on the constants that appear in our assumptions and on T, but it is independent of n.

Proof. Our proof relies on a smart idea of Palencia [14, Section 3]. Since our assumptions on the stepsize sequence here are different, we shortly comment on the necessary modifications. For $b > a \geqslant 0$, let $f_j = (1 + \Delta|\omega_j - 1|)^{-2}$ and

$$\tilde{r}_j = f_j \cdot r_j\big(h_{j+k-2}(A - bI)\big) \quad \text{and} \quad \hat{r}_j = f_j \cdot r_j(h_{j+k-2}A).$$

Note that the (k, m)-entry of $\hat{r}_j - \tilde{r}_j$ is given by

$$h_{j+k-2} b f_j \cdot (\beta_{j-1,k}\alpha_{j-1,m} - \alpha_{j-1,k}\beta_{j-1,m})\, J_j(h_{j+k-2}A) J_j\big(h_{j+k-2}(A - bI)\big). \tag{14}$$

Therefore, there exists a constant C such that

$$\|\tilde{r}_j - \hat{r}_j\| \leqslant C \cdot h_{j+k-2}.$$

The first assertion of the theorem now follows at once from the telescopic identity

$$\prod_{j=1}^{n} \hat{r}_j = \sum_{j=1}^{n} \prod_{l=j+1}^{n} \tilde{r}_l \big(\hat{r}_j - \tilde{r}_j\big) \prod_{i=1}^{j-1} \hat{r}_i + \prod_{j=1}^{n} \tilde{r}_j,$$

and a discrete Gronwall lemma. To obtain the second bound, we write

$$(\mathcal{I}\otimes A)\left(\prod_{j=1}^{n}\hat r_j-\prod_{j=1}^{n}\tilde r_j\right)=\sum_{j=1}^{n}\prod_{l=j+1}^{n}\tilde r_l\big(\hat r_j-\tilde r_j\big)(\mathcal{I}\otimes A)\left(\prod_{i=1}^{j-1}\hat r_i-\prod_{i=1}^{j-1}\tilde r_i\right)$$
$$+(\mathcal{I}\otimes A)\sum_{j=1}^{n}\prod_{l=j+1}^{n}\tilde r_l\big(\hat r_j-\tilde r_j\big)\prod_{i=1}^{j-1}\tilde r_i.$$

To bound the inhomogeneity, we use again (14). Due to

$$\left\|\mathcal{I}\otimes(A-bI)^{\theta}\prod_{j=2}^{n}\hat r_j\cdot\big(\mathcal{I}\otimes J_1\big(h_{k-1}(A-bI)\big)\big)\right\|_{X\leftarrow X}\leqslant\frac{C}{t_{n+k-1}^{\theta}},\quad 0\leqslant\theta\leqslant 1,$$

which follows from Theorem 1 by interpolation, the inhomogeneity is seen to be bounded by

$$\sum_{j=1}^{n}h_{j+k-2}\,(t_{n+k-1}-t_{j+k-2})^{-1/2}\,t_{j+k-2}^{-1/2}\leqslant C.$$

The desired result now follows again with a discrete Gronwall lemma. □

The following lemma is a discrete version of the well-known identity

$$A\int_0^t \mathrm{e}^{\tau A}\,\mathrm{d}\tau=\mathrm{e}^{tA}-I.$$

We denote again $r_j=r_j(h_{j+2-k}A)$, $J_j=J_j(h_{j+2-k}A)$, and further

$$e_k=(0,\dots,0,1)^{\mathrm{T}}\quad\text{and}\quad \mathbb{1}=(1,\dots,1)^{\mathrm{T}}\in\mathbb{R}^k. \tag{15}$$

Recall that a linear multistep method is consistent of order 0, if $\alpha_{j0}+\cdots+\alpha_{j,k-1}=0$ for all j.

Lemma 6. *Assume that* HA1, HS1, HM1–2 *hold, and that the multistep method is consistent of order* 0. *We then have*

$$h_{j+k-1}(\beta_{j0}+\cdots+\beta_{j,k-1})(e_k\otimes AJ_{j+1})=(r_{j+1}-I)\mathbb{1},$$

and in particular

$$\sum_{j=1}^{n}h_{j+k-2}\left(\prod_{l=j+1}^{n}r_l\right)\big(e_k\otimes(\beta_{j-1,0}+\cdots+\beta_{j-1,k-1})AJ_j\big)=\left(\prod_{j=1}^{n}r_j-I\right)\mathbb{1}.$$

The proof is straightforward and therefore omitted.

4. Stability for time-dependent operators

In this section, we consider the time-dependent problem

$$u'(t)=A(t)u(t),\quad 0<t\leqslant T,\qquad u(0)\ \text{given}, \tag{16}$$

where $A:[0,T]\to L(D,X)$ for some $T>0$. Our basic assumptions on the operator $A(t)$ rely on [12].

HA2. *We assume that the operator $A(t)$ satisfies* HA1 *uniformly in t. In particular, $A(t)$ is supposed to have a fixed domain D.*

The following assumption concerning the Hölder continuity of A is motivated by the framework considered in [13].

HA3. *We suppose that $A\in\mathcal{C}^\alpha([0,T],L(D,X))$ for some $0<\alpha\leqslant 1$, i.e., there exists a constant $L>0$ such that*

$$\left\|A(t)-A(s)\right\|_{X\leftarrow D}\leqslant L(t-s)^\alpha,\quad \textit{for all } 0\leqslant s<t\leqslant T.$$

A linear k-step method, applied to (16) takes the form

$$\sum_{i=0}^{k}\alpha_{ni}u_{n+i}=h_{n+k-1}\sum_{i=0}^{k}\beta_{ni}A(t_{n+i})u_{n+i},\quad n\geqslant 0. \tag{17}$$

Again, it is convenient to work with $U_n=(u_n,u_{n+1},\dots,u_{n+k-1})^{\mathrm{T}}$. For this purpose, we denote

$$s_{n+1,i}(z,w)=-(\alpha_{nk}-\beta_{nk}z)^{-1}(\alpha_{ni}-\beta_{ni}w),$$

and we define the companion matrix of the method through

$$r_{n+1}(z_0,z_1,\dots,z_k)=\begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ \vdots & 0 & 1 & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \dots & 0 & 1\\ s_{n+1,0}(z_k,z_0) & s_{n+1,1}(z_k,z_1) & \dots & \dots & s_{n+1,k-1}(z_k,z_{k-1})\end{pmatrix}.$$

Further, we set

$$s_{ni}(z)=s_{ni}(z,z)\quad\text{and}\quad r_n(z)=r_n(z,\dots,z),\quad n\geqslant 1,$$

which makes our new notation compatible with that of the previous sections. Besides, for integers $j\geqslant 0$, we set $A_j=A(t_j)$, and we write for short

$$R_{n+1}=r_{n+1}(h_{n+k-1}A_n,\dots,h_{n+k-1}A_{n+k}),\quad n\geqslant 0.$$

This allows us to rewrite the numerical method (17) as

$$U_n=R_nR_{n-1}\cdots R_1U_0,\quad n\geqslant 0. \tag{18}$$

We are now in a position to give the stability result for (17). Henceforth, we denote $h_{\max}=\max\{h_j\colon 0\leqslant t_j\leqslant T\}$.

Theorem 3. *Consider the linear multistep discretization* (17) *of Eq.* (16) *on the interval* $[0,T]$, *and assume that* HA2–3, HS1–2, HM1–3 *hold and that $\varrho\Omega^2<1$. Then, there exist positive constants H and C such that for $0<h_j\leqslant H$ the following bounds are valid for all $n\geqslant 1$ with $t_n\leqslant T$*

$$\Big\|\prod_{j=1}^{n} R_j\Big\|_{X\leftarrow X} + \Big\|\prod_{j=1}^{n} R_j\Big\|_{D\leftarrow D} \leqslant C, \tag{19a}$$

$$\Big\|\prod_{j=2}^{n} R_j \cdot \big(\mathcal{I}\otimes J_1(h_{k-1}A_k)\big)\Big\|_{D\leftarrow X} \leqslant C\left(\frac{1}{t_{n+k-1}} + \frac{|\log h_{\max}|}{t_{n+k-1}^{1-\alpha}}\right). \tag{19b}$$

The constant C depends on the constants that appear in our assumptions and on T, but it is independent of n.

Proof. The main idea for proving the theorem is to compare R_n with the frozen operator $r_n = r_n(h_{n+k-2}A)$. To show the first estimate in the norm of D, we choose $A = A_{n+k-1}$ and use the telescopic identity and the bounds of Theorem 2 to get

$$\Big\|\prod_{j=1}^{n} R_j\Big\|_{D\leftarrow D} \leqslant C\sum_{j=1}^{n}(t_{n+k-1}-t_{j+k-2})^{-1}\|R_j - r_j\|_{X\leftarrow D}\Big\|\prod_{i=1}^{j-1} R_i\Big\|_{D\leftarrow D} + C.$$

Due to HA3, we have

$$\|R_j - r_j\|_{X\leftarrow D} \leqslant C\cdot h_{j+k-2}(t_{n+k-1}-t_{j+k-2})^{\alpha},$$

and the application of a discrete Gronwall lemma yields the desired result. The corresponding estimate in the norm of X is obtained in a similar way from

$$\prod_{j=m}^{n} R_j = \sum_{j=m}^{n}\prod_{l=j+1}^{n} R_l(R_j - r_j)\prod_{i=m}^{j-1} r_i + \prod_{j=m}^{n} r_j, \quad 1\leqslant m\leqslant n,$$

by choosing $A = A_{m+k-1}$. A preliminary estimate for (19b) is obtained with the same choice of A from the identity

$$\prod_{j=m+1}^{n} R_j - \prod_{j=m+1}^{n} r_j = \sum_{j=m+1}^{n}\left(\prod_{l=j+1}^{n} R_l - \prod_{l=j+1}^{n} r_l\right)(R_j - r_j)\prod_{i=m+1}^{j-1} r_i + \sum_{j=m+1}^{n}\prod_{l=j+1}^{n} r_l(R_j - r_j)\prod_{i=m+1}^{j-1} r_i.$$

Multiplying this relation from the right with $\mathcal{I}\otimes J_m(h_{m+k-2}A_{m+k-1})$ shows that the inhomogeneity, as an operator from X to D, is bounded by

$$\sum_{j=m+1}^{n} h_{j+k-2}\,(t_{n+k-1}-t_{j+k-2})^{-1}\,(t_{j+k-2}-t_{m+k-2})^{\alpha-1} \leqslant C\cdot\big(1 + |\log h_{n+k-2}|\big).$$

With the help of a discrete Gronwall lemma, we thus get a preliminary bound for (19b) with $\log h_{n+k-2}$ in place of $\log h_{\max}$. In a similar way, we get a bound with $\log h_{k-1}$ instead.

It remains to show the sharper estimate with $\log h_{\max}$. For this, let $k-1 \leqslant m \leqslant n+k-2$ be an index with $h_m = h_{\max}$. Depending on the size of t_m, we distinguish two cases. If $2t_m \geqslant t_{n+k-1} - t_{k-1}$, we write with $J_1 = J_1(h_{k-1}A_k)$

$$\left\| \prod_{j=2}^{n} R_j \cdot (\mathcal{I} \otimes J_1) \right\|_{D \leftarrow X} \leqslant \|R_n \cdots R_{m+1}\|_{D \leftarrow D} \left\| R_m \cdots R_2(\mathcal{I} \otimes J_1) \right\|_{D \leftarrow X}$$

and use (19a) and the preliminary bound from above to obtain the desired estimate. If $2t_m \leqslant t_{n+k-1} - t_{k-1}$, we use the identity

$$\left\| \prod_{j=2}^{n} R_j \cdot (\mathcal{I} \otimes J_1) \right\|_{D \leftarrow X} \leqslant \left\| R_n \cdots R_{m+1}(\mathcal{I} \otimes J_m) \right\|_{D \leftarrow X} \times \left\| \big(\mathcal{I} \otimes (\alpha_{m-1,k} - \beta_{m-1,k} h_{m+k-2} A_{m+k-1})\big) R_m \cdots R_2(\mathcal{I} \otimes J_1) \right\|_{X \leftarrow X}.$$

Expressing $R_m \cdots R_2 - \varrho_m \cdots \varrho_2$ through the telescopic identity then yields as before the desired result. □

5. Applications to semilinear parabolic problems

As a first application of our stability results, we study the behaviour of time discretizations for semilinear parabolic problems by linear multistep methods with variable stepsizes. In the following, we briefly sketch a convergence result for finite times.

For our purposes, it is useful to employ an abstract formulation of the parabolic initial-boundary value problem as an initial value problem on a Banach space $(X, \|\cdot\|_X)$

$$u'(t) = Au(t) + f\big(u(t)\big), \quad t > 0, \qquad u(0) \text{ given}. \tag{20}$$

Here, the linear operator $A : D \to X$ is assumed to be sectorial. We further suppose that the map $f : \mathcal{O} \subset X_\theta \to X : v \mapsto f(v)$ defined on some open subset of an interpolation space $X_\theta = [X, D]_\theta$, $0 \leqslant \theta < 1$, is Fréchet differentiable and that its Fréchet derivative $Df(v)$ satisfies a local Lipschitz condition. Reaction–diffusion equations and the incompressible Navier–Stokes equations fit into this analytical framework, see [10,12,17].

As linear multistep methods are invariant under linearization, we may assume without loss of generality that Eq. (20) is already linearized around $u(0)$. Consequently, f satisfies

$$\left\| f(v) - f(w) \right\|_X \leqslant L\varrho \|v - w\|_{X_\theta} \tag{21}$$

for all $v, w \in X_\theta$ with $\|v - u(0)\|_{X_\theta} \leqslant \varrho$ and $\|w - u(0)\|_{X_\theta} \leqslant \varrho$.

Applying a linear k-step method (4) of order p to Eq. (20) yields

$$U_{n+1} = r_{n+1} U_n + h_{n+k-1}(\mathcal{I} \otimes J_{n+1}) f(U_{n+1}), \quad n \geqslant 0. \tag{22}$$

Here, we make use of the notation introduced in Section 2. In particular, we have $U_n = (u_n, u_{n+1}, \ldots, u_{n+k-1})^{\mathrm{T}}$, $r_{n+1} = r_{n+1}(h_{n+k-1}A)$ and $J_{n+1} = (\alpha_{nk} - h_{n+k-1}\beta_{nk}A)^{-1}$. Furthermore, with $e_k = (0, \ldots, 0, 1)^{\mathrm{T}} \in \mathbb{R}^k$, we denote

$$f(U_{n+1}) = \sum_{i=0}^{k} \beta_{ni} e_k \otimes f(u_{n+i}).$$

Solving (22), we receive the discrete variation-of-constants formula

$$U_n = \prod_{j=1}^{n} r_j U_0 + \sum_{m=1}^{n} h_{m+k-2} \prod_{j=m+1}^{n} r_j (\mathcal{I} \otimes J_m) f(U_m), \quad n \geqslant 0. \tag{23}$$

We now carry out a fixed point iteration based on this relation. That is, for finite sequences $V = (V_n)_{n=0}^N$ belonging to a ball around the constant sequence $U(0)$ with components $U(0) = \mathbb{1} \otimes u(0)$

$$\mathcal{V} = \Big\{ V = (V_n)_{n=0}^N \colon \ \big\| V - U(0) \big\|_{X_\theta,\infty} = \max_{0 \leqslant n \leqslant N} \big\| e^{-\gamma t_{n+k-1}} \big(V_n - U(0)\big) \big\|_{X_\theta} \leqslant \varrho \Big\},$$

we define a map $\Psi : \mathcal{V} \to \mathcal{V}$ through

$$\Psi(V)_n = \prod_{j=1}^{n} r_j U_0 + \sum_{m=1}^{n} h_{m+k-2} \prod_{j=m+1}^{n} r_j (\mathcal{I} \otimes J_m) f(V_m), \quad n \geqslant 0.$$

We remark that under the requirements of Theorem 2 the estimate

$$\Big\| \prod_{j=1}^{n} r_j \Big\|_{X_\theta \leftarrow X_\theta} + (t_{n+k-1} - t_{m+k-2})^\theta \Big\| \prod_{j=m+1}^{n} r_j (\mathcal{I} \otimes J_m) \Big\|_{X_\theta \leftarrow X} \leqslant C, \tag{24}$$

follows easily by interpolation. Using moreover (21), it is straightforward to show that Ψ is a contraction with contraction factor $\kappa < 1$ for stepsizes sufficiently small and exponent $\gamma > 0$ large enough, since

$$\kappa = C L_\varrho \max_{0 \leqslant n \leqslant N} \sum_{m=1}^{n} h_{m+k-2} \frac{\mathrm{e}^{-\gamma (t_{n+k-1} - t_{m+k-1})}}{(t_{n+k-1} - t_{m+k-2})^\theta},$$

with the constant C from (24). Moreover, Ψ maps $\mathcal{V}$ to $\mathcal{V}$ if U_0 lies sufficiently close to $U(0)$. Hence, an application of Banach's fixed point theorem proves the existence of the numerical solution. Besides, the vector $\widehat{U}_n = (\hat{u}_n, \hat{u}_{n+1}, \ldots, \hat{u}_{n+k-1})^{\mathrm{T}}$ comprising the exact solution $\hat{u}_n = u(t_n)$ satisfies (23) with additional defects D_m

$$\widehat{U}_n = \prod_{j=1}^{n} r_j \, \widehat{U}_0 + \sum_{m=1}^{n} h_{m+k-2} \prod_{j=m+1}^{n} r_j \, (\mathcal{I} \otimes J_m) \big(f\big(\widehat{U}_m\big) + D_m \big), \quad n \geqslant 0.$$

Provided that the $(p+1)$st order derivative of u remains bounded, we have $\|D_m\|_X \leqslant C h_{m+k-2}^p$. Therefore, due to the fact that

$$\begin{aligned} \big\| U_n - \widehat{U}_n \big\|_{X_\theta} &\leqslant \big\| U - \widehat{U} \big\|_{X_\theta,\infty} \leqslant \frac{1}{1-\kappa} \big\| \Psi\big(\widehat{U}\big) - \widehat{U} \big\|_{X_\theta,\infty} \\ &\leqslant C \big\| U_0 - \widehat{U}_0 \big\|_{X_\theta} + C \sum_{m=1}^{N} \frac{h_{m+k-2}}{(t_{n+k-1} - t_{m+k-2})^\theta} \|D_m\|_X, \end{aligned}$$

the desired convergence estimate follows.

Theorem 4. *In the above situation, apply a linear k-step method* (4) *of order p to Eq.* (20)*. Assume further that the requirements of Theorem 2 are satisfied and that the derivative $u^{(p+1)}(t)$ of the true* *solution remains bounded in X for $t \in [0,T]$. Then, for initial values $u_0, u_1, \ldots, u_{k-1} \in X_\theta$ that lie sufficiently close to $u(0)$ and for stepsize sequences $(h_j)_{j\geqslant 0}$ with $0 < h_j \leqslant h_{\max}$ small enough, the associated numerical solution fulfills the relation*

$$\left\|u_n - u(t_n)\right\|_{X_\theta} \leqslant C \max_{0\leqslant i\leqslant k-1} \left\|u_i - u(t_i)\right\|_{X_\theta} + C \sum_{m=k}^{n} \frac{h_{m-1}^{p+1}}{(t_n - t_{m-1})^\theta},$$

as long as $0 \leqslant t_n \leqslant T$. The constant C depends on the constants that appear in our assumptions and on T, but it is independent of n.

6. Applications to fully nonlinear parabolic problems

In this section, we study variable stepsize linear multistep time discretizations of fully nonlinear parabolic problems. As in the preceding section, we employ an abstract formulation of the partial differential equation and we work within the setting of sectorial operators. Our assumptions on the equation

$$u'(t) = F\big(t, u(t)\big), \quad t > 0, \qquad u(0) \text{ given}, \tag{25}$$

are mainly that of [12]. For nonlinear initial-boundary value problems that can be cast in this analytical framework, see also [7] and [13].

In the following, we specify two illustrations. First, we give a result on the dynamical behaviour nearby a stable equilibrium of the equation, and secondly, a convergence result for finite time intervals.

6.1. Asymptotically stable stationary solutions

We consider an autonomous equation on a Banach space $(X, \|\cdot\|_X)$

$$u'(t) = F\big(u(t)\big), \quad t > 0, \tag{26}$$

with right side $F : \mathcal{D} \subset D \to X$ defined on some open subset $\mathcal{D}$ of another densely embedded Banach space $D \subset X$. Our assumptions on (26) are that of [12], see also [7]. Thus, the Fréchet derivative $DF : \mathcal{D} \to L(D,X)$ satisfies a local Lipschitz condition. Further, for any $v \in \mathcal{D}$, the linear operator $DF(v)$ is sectorial and its graph norm is equivalent to the norm $\|\cdot\|_D$ in D. We suppose that $\bar{u} \in \mathcal{D}$ is an asymptotically stable equilibrium point of Eq. (26), that is, $F(\bar{u}) = 0$, and the sectorial operator $A = DF(\bar{u})$ fulfills the resolvent estimate (2) with $a < 0$.

Linearizing the right side of (26) around the equilibrium point $\bar{u}$ yields a formally semilinear problem

$$u'(t) = Au(t) + G\big(u(t)\big), \quad t > 0, \tag{27}$$

with map G defined through $G(v) = F(v) - Av$ for $v \in \mathcal{D}$. For a linear k-step method (4) applied to (27), in accordance with the notation of Sections 2 and 5, we thus receive the following relation with $G_j = G(u_j)$

$$U_{n+1} = r_{n+1} U_n + h_{n+k-1} e_k \otimes \left(J_{n+1} \sum_{i=0}^{k} \beta_{ni} G_{n+i} \right), \quad n \geqslant 0.$$

We represent the numerical solution by means of a discrete version of a modification of the variation-of-constants formula

$$U_n = \prod_{j=1}^{n} r_j U_0 + \sum_{m=1}^{n} h_{m+k-2} \prod_{j=m+1}^{n} r_j e_k \otimes \left(J_m \sum_{i=0}^{k} \beta_{m-1,i} G_{n+k-1} \right)$$
$$+ \sum_{m=1}^{n} h_{m+k-2} \prod_{j=m+1}^{n} r_j e_k \otimes \left(J_m \sum_{i=0}^{k} \beta_{m-1,i} (G_{m+i-1} - G_{n+k-1}) \right). \tag{28}$$

This relation remains well-defined in a space of weighted α-Hölder continuous sequences for some $0 < \alpha < 1$, that is, the set

$$\mathcal{C}_\alpha^\alpha(D) = \left\{ V = (V_n)_{n\geqslant 0}\colon\ V_n \in \mathcal{D}^k,\ |||V|||_D = \sup_{n\geqslant 0} \|V_n\|_D + \sup_{0<m<n} t_m^\alpha (t_n - t_m)^{-\alpha} \|V_n - V_m\|_D < \infty \right\},$$

endowed with the norm $|||\cdot|||_D$. With the help of Lemma 6, we are able to bound the second term on the right side of formula (28).

For our situation, it is known that if the initial value lies close to the equilibrium point $\bar{u}$, then the true solution decays against $\bar{u}$ exponentially fast. Permitting increasing stepsizes, a similar result holds true for linear multistep methods if stability estimates of the form

$$\left\| \prod_{j=1}^{n} r_j \right\|_{D\leftarrow D} \leqslant \frac{C}{1 + t_{n+k-1}^\eta},$$
$$\left\| \prod_{j=m+1}^{n} r_j (\mathcal{I} \otimes J_m) \right\|_{D\leftarrow X} \leqslant \frac{C}{t_{n+k-1} - t_{m+k-2} + (t_{n+k-1} - t_{m+k-2})^{\eta+1}}, \tag{29}$$

with exponent $\eta > 0$ hold for $n \geqslant 1$, see Theorem 1 and the subsequent discussion.

Theorem 5. *Under the above requirements on F, let $\bar{u}$ be an asymptotically stable equilibrium point of* (26). *Apply a linear k-step method with stepsizes $(h_j)_{j\geqslant 0}$ such that* (29) *is valid. Then, for $0 < \nu < \eta$, there exist constants $\delta > 0$ and $C > 0$ such that for all initial values $u_0, u_1, \ldots, u_{k-1} \in \mathcal{D}$ with $\|u_i - \bar{u}\|_D \leqslant \delta$, $0 \leqslant i \leqslant k-1$, the numerical solution $(u_n)_{n\geqslant k}$ satisfies the estimate*

$$\|u_n - \bar{u}\|_D \leqslant \frac{C}{1 + t_n^\nu} \max_{0\leqslant i\leqslant k-1} \|u_i - \bar{u}\|_D, \qquad n \geqslant 0.$$

Proof. We shortly indicate the proof of Theorem 5. For a precise explanation of the employed techniques, we refer to [7] and [18]. For constructing the numerical solution, we use the ideas of the preceding section. We carry out a fixed point iteration relying on (28) in a subset of $\mathcal{C}_\alpha^\alpha(D)$. In order to capture the decaying behaviour of the numerical solution, we introduce appropriate weights. More precisely, for $0 < \nu < \eta$, we define the norm

$$|||V|||_{\nu,D} = \sup_{n\geqslant 0} \left\| \left(1 + t_n^\nu\right) V_n \right\|_D + \sup_{0<m<n} t_m^\alpha (t_n - t_m)^{-\alpha} \left\| \left(1 + t_n^\nu\right) V_n - \left(1 + t_m^\nu\right) V_m \right\|_D,$$

and set $\mathcal{V} = \{V = (V_n)_{n\geqslant 0}\colon\ |||V - \overline{U}|||_{\nu,D} \leqslant \varrho\}$ where $\overline{U}$ denotes the constant sequence with components equal to $\mathbb{1} \otimes \bar{u}$. Then the iteration based on (28) turns out to be a contraction on $\mathcal{V}$ provided that ϱ and δ are chosen sufficiently small. □

6.2. Convergence for finite times

Another approach that avoids the technicalities which arise in connection with the modified variation-of-constants formula and the consideration of the sequence space $\mathcal{C}_\alpha^\alpha(D)$ is based on a slightly stronger setting. This framework is presented in [13].

We consider an initial value problem of the form

$$u'(t) = F\big(t, u(t)\big), \quad t > 0, \qquad u(0) \text{ given}, \tag{30}$$

where the right-hand side function $F : [0, T] \times \mathcal{D} \to X : (t, v) \mapsto F(t, v)$ is defined on an open subset $\mathcal{D} \subset D$ of a densely embedded Banach space $D \subset X$. We suppose that F is twice continuously Fréchet differentiable and that its Fréchet derivative $D_2F(t, v)$ with respect to the second variable is a sectorial operator in X. Moreover, we assume that the graph-norm of $D_2F(t, v)$ is equivalent to the norm of D for all $0 \leqslant t \leqslant T$ and for all $v \in \mathcal{D}$. In view of our convergence result, we further suppose that the true solution of (30) is differentiable. As a consequence, the hypotheses HA2–3 are satisfied with $\alpha = 1$ on $[0, T]$. Linearizing around $u(t)$ leads to the equation

$$u'(t) = A(t)u(t) + G\big(t, u(t)\big), \quad t > 0,$$

involving the time-dependent sectorial operator $A(t) = D_2F(t, u(t))$. Here, the nonlinearity G is defined by $G(t, v) = F(t, v) - A(t)v$ for $(t, v) \in [0, T] \times \mathcal{D}$. In the present situation, the discrete variation-of-constants formula is still meaningful. For a linear multistep method (17), we receive the following relation

$$U_n = \prod_{j=1}^{n} R_j\, U_0 + \sum_{m=1}^{n} h_{m+k-2} \prod_{j=m+1}^{n} R_j(\mathcal{I} \otimes J_m)\, G(U_m), \quad n \geqslant 0. \tag{31}$$

Here, we use the abbreviations introduced in Section 4. In particular, we let $A_n = A(t_n)$, $R_{n+1} = r_{n+1}(h_{n+k-1}A_n, \ldots, h_{n+k-1}A_{n+k})$ and $J_{n+1} = (\alpha_{nk} - h_{n+k-1}\beta_{nk}A_{n+k})^{-1}$. Besides, we set

$$G(U_{n+1}) = \sum_{i=0}^{k} \beta_{ni}\, e_k \otimes G(t_{n+i}, u_{n+i}).$$

Following [13], we employ as in Section 5 a fixed point iteration based on (31). We define the fixed point operator Ψ on a tube around the true solution $\widehat{U}_n = (u(t_n), \ldots, u(t_{n+k-1}))^{\mathrm{T}}$

$$\mathcal{V} = \Big\{V = (V_n)_{n=0}^{N}\colon\ \big\|V - \widehat{U}\big\|_{D,\infty} = \max_{0\leqslant n\leqslant N} \big\|V_n - \widehat{U}_n\big\|_D \leqslant \varrho h_{\max}^{p/2}\Big\}.$$

By means of the stability estimates from Theorem 3 it follows that Ψ is a contraction and maps $\mathcal{V}$ to $\mathcal{V}$ if for all $n \leqslant N$

$$C\, h_{\max}^{p/2} \sum_{m=k}^{n} \left(\frac{h_{m-1}}{t_n - t_{m-1}} + \frac{h_{m-1}|\log h_{\max}|}{(t_n - t_{m-1})^{1-\alpha}} \right) < 1,$$

with a constant C depending on the stability constant, the Lipschitz constant of F, the bound on the $(p+1)$st-order derivative of the true solution, on the coefficients of the method, and on ϱ. In particular, this bound is satisfied if

$$\big(1+|\log h_{\min}|\big)\, h_{\max}^{p/2} < \gamma, \tag{32}$$

with γ sufficiently small. We remark that this is essentially a condition on the maximal stepsize $h_{\max}$.

We are now prepared to state the convergence result for finite time intervals.

Theorem 6. *In the above situation and under the assumptions of Theorem* 3*, apply a linear k-step method* (17) *of order p to Eq.* (30)*. Suppose further that the derivative $u^{(p+1)}(t)$ of the true solution remains bounded in X for $t \in [0,T]$. Then, provided that the stepsize sequence $(h_j)_{j\geqslant 0}$ satisfies* (32) *with γ sufficiently small, the following bound is valid. For initial values $u_0, u_1, \ldots, u_{k-1}$ in $\mathcal{D}$ with $\|u_i - u(t_i)\|_D$ sufficiently small, the associated numerical solution fulfills the relation*

$$\big\|u_n - u(t_n)\big\|_D \leqslant C \max_{0\leqslant i\leqslant k-1} \big\|u_i - u(t_i)\big\|_D + C \sum_{m=k}^{n} \left(\frac{h_{m-1}^{p+1}}{t_n - t_{m-1}} + \frac{h_{m-1}^{p+1} |\log h_{\max}|}{(t_n - t_{m-1})^{1-\alpha}} \right),$$

for all $0 \leqslant t_n \leqslant T$. The constant C depends on the constants that appear in our assumptions and on T, but it is independent of n.

References

[1] N. Bakaev, A. Ostermann, Long-term stability of variable stepsize approximations of semigroups, Math. Comp. 71 (2002) 1545–1567.

[2] J. Becker, A second order backward difference method with variable steps for a parabolic problem, BIT 38 (1998) 644–662.

[3] M. Calvo, R.D. Grigorieff, Time discretisation of parabolic problems with the variable 3-step BDF, BIT 42 (2002) 689–701.

[4] A. Eder, G. Kirlinger, A normal form for multistep companion matrices, Math. Models Methods Appl. Sci. 11 (2001) 57–70.

[5] E. Emmrich, Stability and error of the variable two-step BDF for semilinear parabolic problems, Extended version of Preprint No. 703, TU Berlin, Berlin, 2001.

[6] C.W. Gear, K.W. Tu, The effect of variable mesh size on the stability of multistep methods, SIAM J. Numer. Anal. 11 (1974) 1025–1043.

[7] C. González, A. Ostermann, C. Palencia, M. Thalhammer, Backward Euler discretization of fully nonlinear parabolic problems, Math. Comp. 71 (2002) 125–145.

[8] R.D. Grigorieff, Stability of multistep-methods on variable grids, Numer. Math. 42 (1983) 359–377.

[9] E. Hairer, S.P. Nørsett, G. Wanner, Solving Ordinary Differential Equations I. Nonstiff Problems, second revised ed., Springer, Berlin, 1993.

[10] D. Henry, Geometric Theory of Semilinear Parabolic Equations, in: Lecture Notes in Math., vol. 840, Springer, Berlin, 1981.

[11] A.T. Hill, E. Süli, Upper semicontinuity of attractors for linear multistep methods approximating sectorial evolution equations, Math. Comp. 64 (1995) 1097–1122.

[12] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, Basel, 1995.

[13] A. Ostermann, M. Thalhammer, Convergence of Runge–Kutta methods for nonlinear parabolic problems, Appl. Numer. Math. (2002) 367–380.

[14] C. Palencia, A stability result for sectorial operators in Banach spaces, SIAM J. Numer. Anal. 30 (1993) 1373–1384.

[15] C. Palencia, On the stability of variable stepsize rational approximations of holomorphic semigroups, Math. Comp. 62 (1994) 93–103.

[16] C. Palencia, B. García-Archilla, Stability of linear multistep methods for sectorial operators in Banach spaces, Appl. Numer. Math. 445 (1993) 503–520.
[17] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, New York, 1983.
[18] M. Thalhammer, Runge–Kutta Time Discretization of Fully Nonlinear Parabolic Problems, Thesis, Universität Innsbruck, Innsbruck, 2000.