J. Differential Equations 251 (2011) 82–118

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Doubly nonlinear evolution equations of second order: Existence and fully discrete approximation

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ARTICLE INFO

Article history:
Received 7 April 2010
Revised 18 February 2011
Available online 5 March 2011

MSC:
47J35
65M12
65J08
34G20
35G25
35L70
35K55
35R09
47H05
47G20

Keywords:
Evolution equation of second order
Integro-differential equation
Monotone operator
Weak solution
Time discretization
Variable time grid
Internal approximation
Convergence

ABSTRACT

Existence of solutions for a class of doubly nonlinear evolution equations of second order is proven by studying a full discretization. The discretization combines a time stepping on a non-uniform time grid, which generalizes the well-known Störmer–Verlet scheme, with an internal approximation scheme.

The linear operator acting on the zero-order term is supposed to induce an inner product, whereas the nonlinear time-dependent operator acting on the first-order time derivative is assumed to be hemicontinuous, monotone and coercive (up to some additive shift), and to fulfill a certain growth condition. The analysis also extends to the case of additional nonlinear perturbations of both the operators, provided the perturbations satisfy a certain growth and a local Hölder-type continuity condition. A priori estimates are then derived in abstract fractional Sobolev spaces.

Convergence in a weak sense is shown for piecewise polynomial prolongations in time of the fully discrete solutions under suitable requirements on the sequence of time grids.

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.02.014

1. Introduction

A variety of problems arising in mechanics, elasticity theory, molecular dynamics, and quantum mechanics can be described by, in general, nonlinear partial differential equations of second order in time. In these equations also time derivatives of first order may appear as e.g. in the case of damping. Examples are the viscous regularization of the Sine– or Klein–Gordon equation, the equations describing a vibrating membrane or a vibrating nonlocal beam, the equations describing phase transformations in shape-memory alloys, and further the equations in thermo-visco-elasticity.

The functional analytic formulation of the foregoing problems leads to initial value problems of the form

$$u'' + Au' + Bu = f \quad \text{in } (0, T), \qquad u(0) = u_0, \qquad u'(0) = v_0. \tag{1.1}$$

The operators A and B are the Nemytskii operators corresponding to a family of nonlinear operators $A(t)$ and $B(t)$ ($t \in [0, T]$), respectively. In this paper, we consider the following situation.

We suppose $A(t) = A_0(t) + A_1(t)$, where $A_0(t) : V_A \to V_A^*$ is the principle part corresponding, in general, to the highest spatial derivatives and $A_1(t) : V_A \to V_A^*$ is a certain perturbation arising from semi-linearities. Here, $(V_A, \|\cdot\|_{V_A})$ is a real reflexive separable Banach space that is dense and continuously embedded in a Hilbert space $(H, (\cdot\,,\cdot), |\cdot|)$ such that $V_A \subseteq H \subseteq V_A^*$ forms a Gelfand triple. Our main assumption is that the hemicontinuous operator $A_0(t)$ is, up to some additive shift, monotone and coercive (with exponent $p \geqslant 2$), uniformly in t, and that a growth condition (with exponent $p-1$) is satisfied. Moreover, the perturbation $A_1(t)$ is assumed to fulfill a certain lower semi-boundedness such that $A(t)$ remains, up to an additive shift, coercive, a certain growth condition, and a local Hölder-type continuity condition. The operator $A_1(t) : V_A \to V_A^*$ ($t \in [0, T]$) is strongly continuous provided V_A is compactly embedded in H.

Similarly, let $B(t) = B_0 + B_1(t)$ with a time-independent principle part $B_0 : V_B \to V_B^*$ and a (possibly time-dependent) perturbation $B_1(t)$. It is crucial to assume that B_0 is linear, bounded, symmetric, and strongly positive. These assumptions force V_B to be a Hilbert space. We also assume that V_B is separable and that $V_B \subseteq H \subseteq V_B^*$ forms a Gelfand triple. One may also allow a time dependence of B_0 (see [19]). However, for readability, we do not consider this case here. Again, the perturbation $B_1(t)$ is assumed to fulfill a certain growth condition and a local Hölder-type continuity condition. It turns out, however, that we shall require that $B_1(t)$ ($t \in [0, T]$) maps V_B into V_A^* (instead of V_B^*).

Besides the assumptions above, we suppose that $V := V_A \cap V_B$ is dense in both the spaces V_A and V_B, which yields the scale

$$V_A \cap V_B = V \subseteq V_C \subseteq H = H^* \subseteq V_C^* \subseteq V^* = V_A^* + V_B^*, \quad C \in \{A, B\},$$

with dense and continuous embeddings.

A full theory of existence and uniqueness for linear evolution problems of second order is given in [14]. For semi-linear problems (with A_0 being linear), we refer to [27,3]. Results on the existence, uniqueness, and regularity in the nonlinear case as well as on the convergence of the Galerkin method can be found in [16, Kap. 7], [29, Ch. 33], and [22, pp. 296ff., 342ff.] for the rather restrictive case $V_A = V_B$. Results allowing $V_A \neq V_B$ are found in the seminal work [19] of Lions and Strauss, see also [4, Ch. V] as well as [15] for a special class of problems of the form (1.1) and [2,17] for particular examples. In contrast to the aforementioned work, we allow more involved problems including also perturbations of the principle parts.

Recently, in [13], we could prove the convergence of a semi-discretization in time, and thereby also existence of a solution, in the case that V_A is dense and continuously embedded in V_B. In the present work, one of our aims is to avoid the restriction $V_A \subseteq V_B$. This is achieved by considering a full discretization that combines a temporal discretization with an internal approximation of V and employing inverse inequalities. This allows to show existence of a solution for a wider class of problems and thus generalizing the results known from the references cited above. Indeed, proving that a sequence of numerical solutions obtained from the full discretization converges in a weak sense, which is of interest on its own, also proves existence. It is notable that the convergence result does not require any regularity of the exact solution and thus is complementary to error estimates for a sufficiently smooth exact solution.

We emphasize that the present analysis is not just a straightforward extension of the preceding analysis for the case that V_A is dense and continuously embedded in V_B (see [13]). Already for the unperturbed situation, the approach here is more intrigued and requires more involved techniques.

As further considerations would overburden the present work, we do not treat the case $1 < p < 2$. This extension is left to future work. Also examples and applications (e.g., the description of a vibrating membrane, see [13, Eq. (1)]) are left for a separate study.

Let $\{V_m\}_{m\in\mathbb{N}}$ be a Galerkin scheme for V (recall that V is the intersection of the separable spaces V_A and V_B). For given $m \in \mathbb{N}$ and a variable time grid

$$\mathbb{I}\colon\quad 0 = t_0 < t_1 < \cdots < t_N = T, \qquad \tau_n = t_n - t_{n-1} \quad (n = 1, 2, \ldots, N \in \mathbb{N}), \tag{1.2}$$

we look for a fully discrete approximation $\{u^n\}_{n=0}^N \subset V_m$ with $u^n \approx u(t_n)$ such that for all $\varphi \in V_m$,

$$\frac{2}{\tau_{n+1}+\tau_n}\left(\frac{u^{n+1}-u^n}{\tau_{n+1}} - \frac{u^n-u^{n-1}}{\tau_n}, \varphi\right) + \left\langle A(t_n)\frac{u^{n+1}-u^n}{\tau_{n+1}}, \varphi\right\rangle + \big\langle B(t_n)u^n, \varphi\big\rangle = \big\langle f^n, \varphi\big\rangle,$$
$$n = 1, 2, \ldots, N-1, \tag{1.3}$$

where $u^0 \approx u_0$, $v^0 = (u^1 - u^0)/\tau_1 \approx v_0$, $\{f^n\} \approx f$ are given approximations for the initial data and right-hand side. (By $\langle\cdot,\cdot\rangle$, we denote the duality pairing.)

If $A \equiv 0$, the above time discretization is known as the leap-frog scheme falling into the class of Newmark schemes that can be interpreted as a partitioned Runge–Kutta method (here as the Störmer–Verlet method).

Indeed, writing (1.1) as the first-order system

$$\begin{cases} u'(t) - v(t) = 0, \\ v'(t) + A(t)v(t) + B(t)u(t) = f(t), \end{cases} \tag{1.4}$$

and applying the explicit and implicit Euler scheme to the first and second equation, respectively, gives for all $\varphi \in V_m$,

$$\begin{cases} \dfrac{1}{\tau_{n+1}}\big(u^{n+1} - u^n, \varphi\big) - \big(v^n, \varphi\big) = 0, & n = 0, 1, \ldots, N-1, \\ \dfrac{2}{\tau_{n+1}+\tau_n}\big(v^n - v^{n-1}, \varphi\big) + \big\langle A(t_n)v^n, \varphi\big\rangle + \big\langle B(t_n)u^n, \varphi\big\rangle = \big\langle f^n, \varphi\big\rangle, & n = 1, 2, \ldots, N-1. \end{cases} \tag{1.5}$$

Inserting the first into the second equation leads to the scheme (1.3).

At this point it is worth to mention that our analysis essentially relies upon the reformulation of (1.1) as the integro-differential equation

$$v'(t) + A(t)v(t) + B(t)\big(u_0 + Kv(t)\big) = f(t) \quad \text{with } Kv(t) := \int_0^t v(s)\,ds, \tag{1.6}$$

which follows from integrating the first differential equation in (1.4) and inserting it into the second one. (With a slight abuse of notation, we only write $Kv(t)$ instead of $(Kv)(t)$ although K is a nonlocal operator.)

In a similar manner, we obtain, from summing up the first equation in (1.5) and inserting it into the second one, the discretized integro-differential equation

$$\frac{2}{\tau_{n+1}+\tau_n}\big(v^n - v^{n-1}, \varphi\big) + \big\langle A(t_n)v^n, \varphi\big\rangle + \big\langle B(t_n)\big(u^0 + K_{\mathbb{I}}v^n\big), \varphi\big\rangle = \big\langle f^n, \varphi\big\rangle$$

$$\text{with } K_{\mathbb{I}}v^n := \sum_{j=0}^{n-1} \tau_{j+1}v^j, \tag{1.7}$$

for $n = 1, 2, \dots, N-1$ and all $\varphi \in V_m$. This is, indeed, a reformulation of the method (1.3) under consideration.

Formally (for sufficiently smooth solutions), the time discretization scheme (1.5) is of first order; in the case $A(t) \equiv 0$ with constant step sizes it is of second order. In the linear case, error estimates for the above full discretization (combining the Newmark scheme with a finite element method) are provided by [21, Ch. 8]. Recently, Runge–Kutta time discretizations were studied in [28].

For a particular class of nonlinear problems of the type (1.1) and with more restrictive assumptions on the problem data, the convergence of the temporal semi-discretization by the above scheme but on equidistant time grids has been studied in [8]. More precisely, the convergence analysis in [8] applies to the (less general and indeed much less involved) case $V_A = V_B$ with A_0 being a time-independent maximal monotone operator, B_0 being time-independent, linear, bounded, symmetric, and (up to some additive shift) strongly positive, and $A_1 = B_1 = 0$.

In the present work, we prove weak convergence of a subsequence of piecewise constant or linear prolongations with respect to time of fully discrete solutions to (1.3) towards a weak solution to (1.1) whenever the maximum time steps of the underlying sequence of variable time grids (1.2) tend to zero and the spatial discretization parameter m goes to infinity (see Theorems 4 and 12 as well as Remark 1 below). Essential conditions on the admissibility of the time grids are that the quantities

$$\max_n\left(\frac{1}{\tau_n}\max\left(0, \frac{\tau_{n-1}}{\tau_n} - \frac{\tau_{n-2}}{\tau_{n-1}}\right)\right), \qquad \sum_n \frac{(\tau_n - \tau_{n-1})^2}{(\tau_n + \tau_{n-1})^3}$$

remain bounded, which signifies that the deviation of the time grids from an equidistant time grid cannot be too large. An example is given by $\tau_n = \tau_{n-1}(1 + c\tau_{n-1})$ for some $c > 0$. Similar restrictions are also known in the context of the convergence of time discretization methods for nonlinear parabolic problems (see [11,12]). Moreover, a suitable coupling of the maximum time step size and the spatial discretization parameter m is required.

Nevertheless, it is of importance for practical issues to substantiate the use of variable time stepping as this is the basis of any adaptive step size control.

Our convergence result not only justifies the numerical approximation of the problem under consideration, especially in the case where regularity of the exact solution and thus error estimates are not at hand, it also provides existence of a solution to the continuous problem, which is, to our best knowledge, new in this general framework.

The proof of convergence relies upon monotonicity and compactness arguments, employing also the stability of the time discretization in terms of a discrete analogue of the integration-by-parts formula. An essential auxiliary result is a certain integration-by-parts formula on the continuous level, which we prove with the help of the Steklov average. Moreover, in the case of perturbations of the principle parts, we derive uniform a priori estimates in abstract fractional Sobolev spaces, in order to apply a generalization of the Lions–Aubin lemma. Here, we need to impose that H is an intermediate space of class $\underline{\mathscr{K}}_\eta(V^*, V_A)$ for some $\eta \in (0,1)$ in the sense of Lions and Peetre and that the couple V, H possesses a certain approximation property.

The paper is organized as follows: In Section 2, we study (1.1) neglecting perturbations. We specify the assumptions on the principle parts as well as on the discretization, provide existence and a priori estimates for the fully discrete solution, and prove a convergence result from which existence of a weak solution to (1.1) follows. In Section 3, following the lines of the previous section, we then focus on additional nonlinear perturbations of the principle parts employing a priori estimates of the fully discrete solution in abstract Sobolev–Slobodetskii spaces.

2. Equations without perturbations

In this section, we consider (1.1) in the case that $A = A_0$ and $B = B_0$, i.e., $A_1 = B_1 = 0$, with an exponent $p \geqslant 2$ occurring in the coercivity of A_0. In order to avoid an additional compactness argument, we also assume that A_0 is monotone and coercive already without an additional additive shift. Such a shift will be handled later as a perturbation.

In the sequel, the space of Bochner integrable (for $r = \infty$ Bochner measurable and essentially bounded) abstract functions mapping $[0, T]$ into a (reflexive) Banach space X is denoted by $L^r(0, T; X)$ ($r \in [1, \infty]$) and equipped with the standard norm $\|\cdot\|_{L^r(0,T;X)}$. By u' and u'', we denote the first and second time derivative of the abstract function $u = u(t)$ in the distributional sense. Moreover, we denote by $\mathscr{C}^r([0, T]; X)$ ($r \in \mathbb{N}$, $\mathscr{C}^0 \equiv \mathscr{C}$) the space of uniformly continuous functions mapping $[0, T]$ into X with uniformly continuous time derivatives up to order r. By $\mathscr{C}_w([0, T]; X)$, we denote the space of abstract functions mapping $[0, T]$ into X that are continuous on $[0, T]$ with respect to the weak topology in X, i.e., demicontinuous functions. Finally, by c we denote a generic positive constant.

2.1. Assumptions on the continuous problem

Remember that $(V_C, \|\cdot\|_{V_C})$ ($C \in \{A, B\}$) denotes a real, reflexive, separable Banach space that is dense and continuously embedded in the Hilbert space $(H, (\cdot\,,\cdot), |\cdot|)$. Further, we have $V = V_A \cap V_B$ with norm $\|\cdot\| = \|\cdot\|_{V_A} + \|\cdot\|_{V_B}$. The space V is assumed to be dense in each of the spaces V_A and V_B. Obviously, V is also continuously embedded in each of the spaces V_A and V_B. The dual $V^* = V_A^* + V_B^*$ is equipped with the norm

$$\|f\|_* = \inf\bigl\{\max\bigl(\|f_A\|_{V_A^*}, \|f_B\|_{V_B^*}\bigr)\colon\ f = f_A + f_B \text{ with } f_A \in V_A^*,\ f_B \in V_B^*\bigr\}.$$

Observe that $V \subseteq H \subseteq V^*$ forms a Gelfand triple.

In what follows, we always assume $p \in [2, \infty)$ and set $p^* = p/(p-1)$. The duality pairing between $L^p(0, T; V) \ni v$ and $(L^p(0, T; V))^* = L^{p^*}(0, T; V^*) = L^{p^*}(0, T; V_A^*) + L^{p^*}(0, T; V_B^*) \ni f = f_A + f_B$ is given by

$$\langle f, v\rangle = \int_0^T \bigl\langle f(t), v(t)\bigr\rangle_{V^*\times V}\,dt = \int_0^T \bigl\langle f_A(t), v(t)\bigr\rangle_{V_A^*\times V_A}\,dt + \int_0^T \bigl\langle f_B(t), v(t)\bigr\rangle_{V_B^*\times V_B}\,dt;$$

it is independent of the particular decomposition. Moreover, we have $(L^1(0, T; H))^* = L^\infty(0, T; H)$ with the duality pairing

$$\langle f, v\rangle = \int_0^T \bigl(f(t), v(t)\bigr)\,dt.$$

The structural properties we assume for A_0 and B_0 read as follows:

Assumption (A_0). *$\{A_0(t)\}_{t\in[0,T]}$ is a family of monotone and hemicontinuous operators $A_0(t) : V_A \to V_A^*$ such that for all $v \in V_A$ the mapping $t \mapsto A_0(t)v : [0, T] \to V_A^*$ is continuous for almost all $t \in [0, T]$. For a suitable $p \in [2, \infty)$, there are constants $\mu_A, c > 0$, $\lambda \geqslant 0$ such that for all $t \in [0, T]$ and $v \in V_A$,*

$$\bigl\langle A_0(t)v, v\bigr\rangle \geqslant \mu_A\|v\|_{V_A}^p - \lambda, \qquad \bigl\|A_0(t)v\bigr\|_{V_A^*} \leqslant c\bigl(1 + \|v\|_{V_A}^{p-1}\bigr).$$

With $\{A_0(t)\}_{t\in[0,T]}$, we associate the Nemytskii operator A_0 that is defined by $(A_0v)(t) := A_0(t)v(t)$ ($t \in [0, T]$) for a function $v : [0, T] \to V_A$. Under Assumption (A_0), the Nemytskii operator A_0 maps $L^p(0, T; V_A)$ into its dual and is monotone, coercive, hemicontinuous and bounded.

Assumption ($\mathbf{B_0}$). *$B_0 : V_B \to V_B^*$ is a linear, bounded, symmetric, and strongly positive operator: There are constants $\mu_B, c_B > 0$ such that for all $v \in V_B$,*

$$\langle B_0 v, v\rangle \geqslant \mu_B \|v\|_{V_B}^2, \qquad \|B_0 v\|_{V_B^*} \leqslant c_B \|v\|_{V_B}.$$

Under Assumption (B_0), the operator $B_0 : V_B \to V_B^*$ extends to a linear, bounded, symmetric, and strongly positive operator mapping, e.g., $L^2(0,T;V_B)$ into its dual.

Remark 1. The above Assumptions (A_0) and (B_0) guarantee existence and uniqueness of a solution $u \in L^\infty(0,T;V_B)$ with $u' \in L^p(0,T;V_A) \cap L^\infty(0,T;H)$ and $u'' \in L^{p^*}(0,T;V_A^*) + L^\infty(0,T;V_B^*) \subseteq (L^p(0,T;V))^*$ to problem (1.1) for any $u_0 \in V_B$, $v_0 \in H$, and (at least) any $f \in (L^p(0,T;V_A))^*$, see [19, Thm. 2.1]. The differential equation is then fulfilled in the sense of equality in $(L^p(0,T;V))^*$. As one can also show that $u \in \mathscr{C}_w([0,T];V_B)$ and $u' \in \mathscr{C}_w([0,T];H)$, the initial conditions are satisfied in the sense that $u(t) \rightharpoonup u_0$ in V_B and $u'(t) \rightharpoonup v_0$ in H as $t \to 0$.

Since we prove a priori estimates for u' and its approximation in $L^\infty(0,T;H)$ it would also be possible to consider, as in [19], the somewhat more general case $f \in L^{p^*}(0,T;V_A^*) + L^1(0,T;H)$. However, in view of readability, we shall not consider this case.

2.2. Fully discrete problem and a priori estimates

Let $\{\varphi_i\}_{i\in\mathbb{N}}$ be a Galerkin basis of V. Then $\{V_m\}_{m\in\mathbb{N}}$ with $V_m := \operatorname{span}\{\varphi_1,\dots,\varphi_m\}$ forms a Galerkin scheme with the property of limited completeness, i.e.,

$$V = \operatorname{clos}_{\|\cdot\|} \bigcup_{m\in\mathbb{N}} V_m. \tag{2.1}$$

Since V is dense and continuously embedded in V_C ($C \in \{A,B\}$), $\{V_m\}_{m\in\mathbb{N}}$ is also a Galerkin scheme for V_C (with limited completeness w.r.t. $\|\cdot\|_{V_C}$).

With respect to the approximation of the function spaces, we make use of the following relation.

Relation ($\mathbf{V_B \leftarrow V_A}$). *For each $m \in \mathbb{N}$ there exists a positive constant $c_{V_B\leftarrow V_A}(m)$ such that*

$$\|v\|_{V_B} \leqslant c_{V_B\leftarrow V_A}(m)\|v\|_{V_A} \quad \text{for all } v \in V_m. \tag{2.2}$$

Because of the equivalence of all norms on a finite dimensional space, Relation $(V_B \leftarrow V_A)$ can always be established. Note that $c_{V_B\leftarrow V_A}(m)$ does not depend on m if $V_A \hookrightarrow V_B$. In general, (2.2) corresponds to an inverse inequality with $c_{V_B\leftarrow V_A}(m) \to \infty$ as $m \to \infty$ (see, e.g., [7, Sec. 17]). We remark that Relation $(V_B \leftarrow V_A)$ implies the corresponding (inverse) inequality for V and V_A,

$$\|v\| = \|v\|_{V_A} + \|v\|_{V_B} \leqslant \big(1 + c_{V_B\leftarrow V_A}(m)\big)\|v\|_{V_A} \quad \text{for all } v \in V_m. \tag{2.3}$$

Vice versa, the inverse inequality for V and V_A implies (for sufficiently large m) relation (2.2) whenever $V_A \not\subseteq V_B$, since then $c_{V\leftarrow V_A}(m) \to \infty$ as $m \to \infty$. If, however, $V_A \hookrightarrow V_B$, we do not need to employ any inverse inequality as was already shown in our previous work [13]. We also should remark that alternatively one may work with an analogous relation between the norms $\|\cdot\|_{V_B}$ and $\|\cdot\|_H$ on V_m, which, in certain cases, might yield weaker assumptions on the coupling of the maximum time step size and spatial discretization parameter.

We further consider an arbitrary time grid (1.2). We set $\tau_{n+1/2} := (\tau_n + \tau_{n+1})/2$, $t_{n+1/2} := t_n + \tau_{n+1}/2$, and denote by $r_{n+1} := \tau_{n+1}/\tau_n$ ($n = 1,2,\dots,N-1$) the ratio of adjacent step sizes. Moreover, we set

$$\tau_{\max} := \max_{n=1,2,\ldots,N} \tau_n, \qquad r_{\max} := \max\Bigl(1, \max_{n=2,3,\ldots,N} r_n\Bigr), \qquad r_{\min} := \min_{n=2,3,\ldots,N} r_n,$$

$$\gamma_n := \max\biggl(0, \frac{1}{r_n} - \frac{1}{r_{n-1}}\biggr) \quad (n = 3, \ldots, N), \qquad c_\gamma := \sum_{n=3}^{N} \gamma_n,$$

$$\theta := \sum_{n=2}^{N} \frac{1}{\tau_{n-1/2}} \biggl(\frac{r_n - 1}{r_n + 1}\biggr)^2 = 2 \sum_{n=2}^{N} \frac{(\tau_n - \tau_{n-1})^2}{(\tau_n + \tau_{n-1})^3}. \tag{2.4}$$

Representing u^n by $\{v^n\}$ by using the first equation in (1.5) gives

$$u^n = u^0 + \sum_{j=0}^{n-1} \bigl(u^{j+1} - u^j\bigr) = u^0 + \sum_{j=0}^{n-1} \tau_{j+1} v^j =: u^0 + K_{\mathbb{I}} v^n, \quad n = 0, 1, \ldots, N, \tag{2.5}$$

where $K_{\mathbb{I}}$ is a nonlocal operator acting on grid functions. We thus have that (1.3) is equivalent to (1.7) together with the first equation in (1.5). The relation (1.7) is the starting point for our analysis.

The solvability of the fully discrete problem will be based on the following auxiliary result.

Lemma 1. *Let $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ be continuous. If there is $R > 0$ such that $\Phi(\boldsymbol{v}) \cdot \boldsymbol{v} \geqslant 0$ for all $\boldsymbol{v} \in \mathbb{R}^m$ with $\|\boldsymbol{v}\|_{\mathbb{R}^m} = R$ then there exists $\bar{\boldsymbol{v}} \in \mathbb{R}^m$ with $\|\bar{\boldsymbol{v}}\|_{\mathbb{R}^m} \leqslant R$ and $\Phi(\bar{\boldsymbol{v}}) = 0$.*

Proof. The proof follows by contradiction from Brouwer's fixed point theorem (see, e.g., [16, Lemma 2.1 on p. 74]). □

Theorem 2 *(Existence and uniqueness of a discrete solution). Let Assumptions (A_0) and (B_0) be fulfilled and let $u^0, v^0 \in V_m$ and $\{f^n\}_{n=1}^{N-1} \subseteq V^*$ be given. Then there exists a unique solution $\{u^n\}_{n=1}^{N} \subseteq V_m$ to (1.3) with $\{v^n\}_{n=1}^{N-1} \subseteq V_m$ ($v^n = (u^{n+1} - u^n)/\tau_{n+1}$) being the solution to (1.7).*

Proof. There is a bijection between V_m and $\mathbb{R}^m$ given by the representation

$$v = \sum_{i=1}^{m} v_i \varphi_i \in V_m, \qquad \boldsymbol{v} = [v_i]_{i=1}^{m} \in \mathbb{R}^m.$$

Then $\|\boldsymbol{v}\|_{\mathbb{R}^m} := \|v\|$ defines a norm on $\mathbb{R}^m$.

The scheme (1.7) reduces, step by step, to the finite dimensional problem of determining $\boldsymbol{v}^n \in \mathbb{R}^m$ $(n = 1, 2, \ldots, N-1)$ such that

$$\Phi\bigl(\boldsymbol{v}^n\bigr) := \biggl[\frac{1}{\tau_{n+1/2}}\bigl(v^n - v^{n-1}, \varphi_i\bigr) + \bigl\langle A_0(t_n) v^n, \varphi_i\bigr\rangle + \bigl\langle B_0\bigl(u^0 + K_{\mathbb{I}} v^n\bigr), \varphi_i\bigr\rangle - \bigl\langle f^n, \varphi_i\bigr\rangle\biggr]_{i=1}^{m} = 0.$$

Remember here that $K_{\mathbb{I}} v^n$ only depends on $\{v^j\}_{j=0}^{n-1}$. Once $\{v^n\}_{n=0}^{N-1}$ is known, the solution $\{u^n\}_{n=1}^{N}$ can be calculated from (2.5). So it remains to prove the existence of a zero of Φ.

The function $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ is continuous, in particular, since $A_0(t_n) : V_A \to V_A^*$ is monotone and hemicontinuous and thus demicontinuous and since $B_0 : V_B \to V_B^*$ is linear and bounded.

For estimating

$$\Phi\bigl(\boldsymbol{v}^n\bigr) \cdot \boldsymbol{v}^n = \frac{1}{\tau_{n+1/2}}\bigl(v^n - v^{n-1}, v^n\bigr) + \bigl\langle A_0(t_n) v^n, v^n\bigr\rangle + \bigl\langle B_0\bigl(u^0 + K_{\mathbb{I}} v^n\bigr), v^n\bigr\rangle - \bigl\langle f^n, v^n\bigr\rangle,$$

we observe that

$$\left(v^n - v^{n-1}, v^n\right) \geqslant \left|v^n\right|^2 - \left\|v^{n-1}\right\|_* \left\|v^n\right\|.$$

Because of the coercivity condition on $A_0(t_n)$, we find

$$\left\langle A_0(t_n)v^n, v^n\right\rangle \geqslant \mu_A \left\|v^n\right\|_{V_A}^p - \lambda.$$

This, together with (2.3), yields

$$\left\langle A_0(t_n)v^n, v^n\right\rangle \geqslant \mu_A\left(1 + c_{V_B \leftarrow V_A}(m)\right)^{-p} \left\|v^n\right\|^p - \lambda.$$

Moreover, we have

$$\left\langle B_0\left(u^0 + K_{\mathbb{I}}v^n\right), v^n\right\rangle - \left\langle f^n, v^n\right\rangle \geqslant -\left(\left\|B_0\left(u^0 + K_{\mathbb{I}}v^n\right)\right\|_* + \left\|f^n\right\|_*\right)\left\|v^n\right\|.$$

Putting together the foregoing estimates shows that

$$\begin{aligned}\Phi\left(\boldsymbol{v}^n\right) \cdot \boldsymbol{v}^n \geqslant \Bigg(&\mu_A\left(1 + c_{V_B \leftarrow V_A}(m)\right)^{-p} \|v^n\|^{p-1} - \frac{1}{\tau_{n+1/2}}\left\|v^{n-1}\right\|_* \\ &- \left\|B_0\left(u^0 + K_{\mathbb{I}}v^n\right)\right\|_* - \left\|f^n\right\|_*\Bigg)\left\|v^n\right\| + \frac{1}{\tau_{n+1/2}}\left|v^n\right|^2 - \lambda. \qquad (2.6)\end{aligned}$$

Taking now $\|\boldsymbol{v}^n\|_{\mathbb{R}^m} = \|v^n\| = R$ for sufficiently large R implies $\Phi(\boldsymbol{v}^n) \cdot \boldsymbol{v}^n \geqslant 0$.

Lemma 1 now provides the existence of a zero of Φ.

With respect to the uniqueness, we only have to show that (1.7) possesses a unique solution. This is again done step-by-step. For n fixed, let $\{v^j\}_{j=0}^{n-1}$ and f^n be given. Assume that v_1^n and v_2^n are two different solutions to (1.7). We take the difference of the corresponding equations and test with $v_1^n - v_2^n$. Since $B_0(u^0 + K_{\mathbb{I}}v_1^n) - B_0(u^0 + K_{\mathbb{I}}v_2^n) = 0$, the monotonicity of $A_0(t_n)$ now provides $|v_1^n - v_2^n|^2 \leqslant 0$, which is in contradiction to our assumption. □

The following result provides uniform a priori estimates for the fully discrete solution.

Theorem 3 *(A priori estimates). In addition to the assumptions of Theorem* 2 *let* $\{f^n\}_{n=1}^{N-1} \subseteq V_A^*$ *and*

$$c_{V_B \leftarrow V_A}(m)^2 \tau_{\max} < \min\left(1, \frac{\mu_A}{c_B}\right). \qquad (2.7)$$

Then there holds for all $n = 1, 2, \ldots, N-1$,

$$\begin{aligned}&\left\|u^{n+1}\right\|_{V_B}^2 + \left|v^n\right|^2 + \sum_{j=1}^{n}\left|v^j - v^{j-1}\right|^2 + \sum_{j=1}^{n}\tau_{j+1/2}\left\|v^j\right\|_{V_A}^p \\ &\quad\leqslant c\left(\left\|u^0\right\|_{V_B}^2 + \left|v^0\right|^2 + \tau_1^2\left\|v^0\right\|_{V_B}^2 + \sum_{j=1}^{n}\tau_{j+1/2}\left\|f^j\right\|_{V_A^*}^{p^*} + T\right), \qquad (2.8)\end{aligned}$$

where $c > 0$ *is a function in* $1/r_{\min}$, c_γ, *and* $1/(\mu_A - c_B c_{V_B \leftarrow V_A}(m)^2 \tau_{\max})$ *that is bounded on bounded subsets.*

Proof. We test (1.7) with v^n. Since

$$(a-b)a = \frac{1}{2}\big(a^2 - b^2 + (a-b)^2\big), \quad a, b \in \mathbb{R}, \tag{2.9}$$

we have

$$\frac{1}{\tau_{n+1/2}}\big(v^n - v^{n-1}, v^n\big) = \frac{1}{2\tau_{n+1/2}}\big(\big|v^n\big|^2 - \big|v^{n-1}\big|^2 + \big|v^n - v^{n-1}\big|^2\big).$$

Because of the coercivity of $A_0(t_n)$, we find

$$\big\langle A_0(t_n)v^n, v^n\big\rangle \geqslant \mu_A \big\|v^n\big\|_{V_A}^p - \lambda.$$

With (2.5) and

$$(a-b)b = \frac{1}{2}\big(a^2 - b^2 - (a-b)^2\big), \quad a, b \in \mathbb{R}, \tag{2.10}$$

we obtain (with $\|\cdot\|_B := \langle B_0 \cdot, \cdot\rangle^{1/2}$ denoting the norm on V_B induced by B_0, which is equivalent to $\|\cdot\|_{V_B}$)

$$\begin{aligned}
\big\langle B_0\big(u^0 + K_{\mathbb{I}}v^n\big), v^n\big\rangle &= \frac{1}{\tau_{n+1}}\big\langle B_0\big(u^0 + K_{\mathbb{I}}v^n\big), \big(u^0 + K_{\mathbb{I}}v^{n+1}\big) - \big(u^0 + K_{\mathbb{I}}v^n\big)\big\rangle \\
&= \frac{1}{2\tau_{n+1}}\big(\big\|u^0 + K_{\mathbb{I}}v^{n+1}\big\|_B^2 - \big\|u^0 + K_{\mathbb{I}}v^n\big\|_B^2 \\
&\quad - \big\|\big(u^0 + K_{\mathbb{I}}v^{n+1}\big) - \big(u^0 + K_{\mathbb{I}}v^n\big)\big\|_B^2\big) \\
&= \frac{1}{2\tau_{n+1}}\big(\big\|u^{n+1}\big\|_B^2 - \big\|u^n\big\|_B^2 - \tau_{n+1}^2\big\|v^n\big\|_B^2\big).
\end{aligned}$$

Employing Young's inequality, we find

$$\big\langle f^n, v^n\big\rangle \leqslant \big\|f^n\big\|_{V_A^*}\big\|v^n\big\|_{V_A} \leqslant c\big\|f^n\big\|_{V_A^*}^{p^*} + \frac{\mu_A}{2}\big\|v^n\big\|_{V_A}^p.$$

Multiplying by $2\tau_{n+1/2}$, summing up, and taking into account (2.5) now gives

$$\begin{aligned}
&\big|v^n\big|^2 + \sum_{j=1}^n \big|v^j - v^{j-1}\big|^2 + \mu_A \sum_{j=1}^n \tau_{j+1/2}\big\|v^j\big\|_{V_A}^p \\
&\quad + \frac{1}{2}\left(1 + \frac{1}{r_{n+1}}\right)\big\|u^{n+1}\big\|_B^2 + \frac{1}{2}\sum_{j=2}^n \left(\frac{1}{r_j} - \frac{1}{r_{j+1}}\right)\big\|u^j\big\|_B^2 \\
&\leqslant \big|v^0\big|^2 + \frac{1}{2}\left(1 + \frac{1}{r_2}\right)\big\|u^1\big\|_B^2 + c\sum_{j=1}^n \tau_{j+1/2}\big\|f^j\big\|_{V_A^*}^{p^*} + \sum_{j=1}^n \tau_{j+1/2}\tau_{j+1}\big\|v^j\big\|_B^2 + c\lambda T.
\end{aligned} \tag{2.11}$$

With (2.2) there holds

$$\big\|v^j\big\|_B^2 \leqslant c_B\big\|v^j\big\|_{V_B}^2 \leqslant c_B c_{V_B \leftarrow V_A}(m)^2\big\|v^j\big\|_{V_A}^2 \leqslant c_B c_{V_B \leftarrow V_A}(m)^2\big(1 + \big\|v^j\big\|_{V_A}^p\big).$$

This, together with (2.7), a discrete Gronwall argument and the equivalence of $\|\cdot\|_B$ and $\|\cdot\|_{V_B}$, yields the estimate asserted. □

We remark that we are not able to derive a suitable estimate for $\{v^n\}_{n=0}^{N-1}$ in the V- or V_A-norm with constants that remain bounded for all $m \in \mathbb{N}$ under the more general assumption $\{f^n\}_{n=1}^{N-1} \subseteq V^*$. Therefore, we are not able to derive results on the solvability of the original problem for right-hand sides taking values in V^*. This is in accordance with the results in [19].

2.3. Convergence towards a weak solution

We often write $g(m, \mathbb{I})$ to emphasize the dependence of a quantity g on the finite dimensional space V_m and the time grid $\mathbb{I}$.

For the solution $\{u^n\}_{n=0}^{N} \subseteq V_m$, $\{v^n\}_{n=0}^{N-1} \subseteq V_m$ to (1.3) and (1.7) corresponding to a time grid $\mathbb{I}$, we define

$$u_{m,\mathbb{I}}(t) := \begin{cases} 0 & \text{for } t \in [0, t_{1/2}], \\ u^n & \text{for } t \in (t_{n-1/2}, t_{n+1/2}]\ (n = 1, 2, \ldots, N-1), \\ 0 & \text{for } t \in (t_{N-1/2}, t_N], \end{cases}$$

$$v_{m,\mathbb{I}}(t) := \begin{cases} 0 & \text{for } t \in [0, t_{1/2}], \\ v^n & \text{for } t \in (t_{n-1/2}, t_{n+1/2}]\ (n = 1, 2, \ldots, N-1), \\ 0 & \text{for } t \in (t_{N-1/2}, t_N], \end{cases}$$

$$\hat{v}_{m,\mathbb{I}}(t) := \begin{cases} v^0 & \text{for } t \in [0, t_{1/2}], \\ v^n + \frac{t - t_{n+1/2}}{\tau_{n+1/2}}(v^n - v^{n-1}) & \text{for } t \in (t_{n-1/2}, t_{n+1/2}]\ (n = 1, 2, \ldots, N-1), \\ v^{N-1} & \text{for } t \in (t_{N-1/2}, t_N]. \end{cases}$$

Note that $\hat{v}_{m,\mathbb{I}}$ is piecewise linear and continuous in time, and thus differentiable in the weak sense.

Without loss of generality, we assume $A_0(t)0 \equiv 0$ $(t \in [0, T])$ and thus $\lambda = 0$ in Assumption (A_0). This is allowed since otherwise we may replace $f(t)$ by $f(t) - A_0(t)0$ $(t \in [0, T])$. For the right-hand side, we restrict ourselves to the approximation

$$f^n := \frac{1}{\tau_{n+1/2}} \int_{t_{n-1/2}}^{t_{n+1/2}} f(t)\, dt, \quad n = 1, 2, \ldots, N-1, \tag{2.12}$$

which is well defined for $f \in L^{p^*}(0, T; V_A^*)$, and set

$$f_{\mathbb{I}}(t) := \begin{cases} 0 & \text{for } t \in [0, t_{1/2}], \\ f^n & \text{for } t \in (t_{n-1/2}, t_{n+1/2}]\ (n = 1, 2, \ldots, N-1), \\ 0 & \text{for } t \in (t_{N-1/2}, t_N], \end{cases}$$

$$A_{0,\mathbb{I}}(t) := \begin{cases} A_0(t_1) & \text{for } t \in [0, t_{1/2}], \\ A_0(t_n) & \text{for } t \in (t_{n-1/2}, t_{n+1/2}]\ (n = 1, 2, \ldots, N-1), \\ A_0(t_{N-1}) & \text{for } t \in (t_{N-1/2}, t_N]. \end{cases}$$

We now consider a sequence $\{(V_{m_\ell}, \mathbb{I}_\ell)\}_{\ell \in \mathbb{N}}$ consisting of finite dimensional spaces $V_{m_\ell} \in \{V_m\}_{m \in \mathbb{N}}$ and time grids $\mathbb{I}_\ell$ of type (1.2) fulfilling the following assumption (see also (2.4) for the notation and (2.7)):

Assumption ($\mathbf{V_m}, \mathbb{I}$). *The sequence* $\{(V_{m_\ell}, \mathbb{I}_\ell)\}_{\ell\in\mathbb{N}}$ *satisfies*

$$m_\ell \to \infty \quad \text{and} \quad \tau_{\max}(\mathbb{I}_\ell) \to 0 \quad \text{as } \ell \to \infty,$$
$$\sup_{\ell\in\mathbb{N}} c_{V_B\leftarrow V_A}(m_\ell)^2 \tau_{\max}(\mathbb{I}_\ell) < \min\left(1, \frac{\mu_A}{c_B}\right), \qquad c_{V_B\leftarrow V_A}(m_\ell)^2 \tau_{\max}(\mathbb{I}_\ell) \to 0 \quad \text{as } \ell \to \infty,$$
$$\sup_{\ell\in\mathbb{N}} r_{\max}(\mathbb{I}_\ell) < \infty, \qquad \inf_{\ell\in\mathbb{N}} r_{\min}(\mathbb{I}_\ell) > 0, \qquad \sup_{\ell\in\mathbb{N}} c_\gamma(\mathbb{I}_\ell) < \infty, \qquad \sup_{\ell\in\mathbb{N}} \theta(\mathbb{I}_\ell) < \infty.$$

With respect to the initial data, we require

Assumption (IC). *The initial values for* (1.3) *satisfy*

$$u^0(m_\ell, \mathbb{I}_\ell), v^0(m_\ell, \mathbb{I}_\ell) \in V_{m_\ell} \quad (\ell \in \mathbb{N}), \qquad \sup_{\ell\in\mathbb{N}} \tau_{\max}(\mathbb{I}_\ell) \left\| v^0(m_\ell, \mathbb{I}_\ell) \right\|_{V_A}^p < \infty,$$
$$u^0(m_\ell, \mathbb{I}_\ell) \to u_0 \quad \text{in } V_B \quad \text{and} \quad v^0(m_\ell, \mathbb{I}_\ell) \to v_0 \quad \text{in } H \text{ as } \ell \to \infty.$$

Remark 2. Assumption (IC) on the sequence $\{v^0(\mathbb{I}_\ell)\}_{\ell\in\mathbb{N}}$ can always be fulfilled for $v_0 \in H$ since V_A is dense in H. Assumption $(V_m, \mathbb{I})$ on θ and c_γ, i.e., on the ratios of adjacent step sizes, is obviously fulfilled for an equidistant partition but also for variable time grids that are a perturbation of an equidistant partition.

The main result in this section now reads as follows.

Theorem 4 *(Convergence towards the weak solution). Let Assumptions* (A_0), (B_0), $(V_m, \mathbb{I})$, *and* (IC) *be fulfilled, and let* $u_0 \in V_B$, $v_0 \in H$, *and* $f \in (L^p(0,T; V_A))^*$. *Then, as* $\ell \to \infty$, *the piecewise constant prolongations* $u_{m_\ell,\mathbb{I}_\ell}$ *of the fully discrete solutions to* (1.3) *converge weakly* in* $L^\infty(0,T; V_B)$ *towards the exact solution* $u \in \mathscr{C}_w([0,T]; V_B) \cap L^\infty(0,T; V_B)$ *to* (1.1) *with* $u' \in \mathscr{C}_w([0,T]; H) \cap L^\infty(0,T; H) \cap L^p(0,T; V_A)$ *and* $u'' \in (L^p(0,T; V))^*$. *Moreover, the piecewise constant prolongations* $v_{m_\ell,\mathbb{I}_\ell}$ *as well as the piecewise linear prolongations* $\hat{v}_{m_\ell,\mathbb{I}_\ell}$ *converge weakly in* $L^p(0,T; V_A)$ *and weakly* in* $L^\infty(0,T; H)$ *towards* u'.

We may also derive strong convergence results if V_A is compactly embedded in H. This is, indeed, necessary when dealing with perturbations of the monotone main part and shall, therefore, be dealt with in Section 3.

The proof of the above theorem relies upon the following auxiliary results:

Lemma 5. *Under the assumptions of Theorem* 4 *there is a subsequence, denoted by* ℓ', *and there are elements*

$$u \in L^\infty(0,T; V_B), \qquad v \in L^\infty(0,T; H) \cap L^p(0,T; V_A)$$

with

$$u - u_0 = Kv \in \mathscr{C}\big([0,T]; V_A\big) \cap L^\infty(0,T; V_B) \quad \textit{and} \quad u' = v \in L^p(0,T; V_A)$$

such that

$$u_{m_{\ell'},\mathbb{I}_{\ell'}} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T; V_B),$$
$$v_{m_{\ell'},\mathbb{I}_{\ell'}} \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0,T; H), \qquad v_{m_{\ell'},\mathbb{I}_{\ell'}} \rightharpoonup v \quad \text{in } L^p(0,T; V_A),$$
$$\hat{v}_{m_{\ell'},\mathbb{I}_{\ell'}} \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0,T; H), \qquad \hat{v}_{m_{\ell'},\mathbb{I}_{\ell'}} \rightharpoonup v \quad \text{in } L^p(0,T; V_A),$$

$$K v_{m_{\ell'},\mathbb{I}_{\ell'}} \overset{*}{\rightharpoonup} K v \quad \text{in } L^\infty(0,T;V_A),$$
$$u_{m_{\ell'},\mathbb{I}_{\ell'}} - u_0 - K v_{m_{\ell'},\mathbb{I}_{\ell'}} \to 0 \quad \text{in } L^2(0,T;V_B) \text{ as } \ell' \to \infty.$$

Proof. With respect to the right-hand side of (2.8), we first observe that c is bounded since, by Assumption $(V_m,\mathbb{I})$, the sequences $\{1/r_{\min}(\mathbb{I}_\ell)\}$, $\{c_\gamma(\mathbb{I}_\ell)\}$, and $\{1/(\mu_A - c_B c_{V_B\leftarrow V_A}(m_\ell)^2\tau_{\max}(\mathbb{I}_\ell))\}$ are bounded. Furthermore, by Assumption (IC), the sequence $\{u^0(m_\ell,\mathbb{I}_\ell)\}$ is bounded in V_B and $\{v^0(m_\ell,\mathbb{I}_\ell)\}$ is bounded in H. We also see, by Relation $(V_B \leftarrow V_A)$ and Assumptions $(V_m,\mathbb{I})$ and (IC), that $\{\tau_1(\mathbb{I}_\ell)v^0(m_\ell,\mathbb{I}_\ell)\}$ is bounded in V_B since

$$\begin{aligned}\tau_1(\mathbb{I}_\ell)\big\|v^0(m_\ell,\mathbb{I}_\ell)\big\|_{V_B} &\leqslant \tau_1(\mathbb{I}_\ell)c_{V_B\leftarrow V_A}(m_\ell)\big\|v^0(m_\ell,\mathbb{I}_\ell)\big\|_{V_A}\\ &= \tau_{\max}(\mathbb{I}_\ell)^{1/p^*}c_{V_B\leftarrow V_A}(m_\ell)\big(\tau_{\max}(\mathbb{I}_\ell)\big\|v^0(m_\ell,\mathbb{I}_\ell)\big\|_{V_A}^p\big)^{1/p}.\end{aligned} \tag{2.13}$$

Finally, it is easy to see with (2.12) that

$$\sum_{j=1}^{N(\mathbb{I}_\ell)-1} \tau_{j+1/2}(\mathbb{I}_\ell)\big\|f^j(\mathbb{I}_\ell)\big\|_{V_A^*}^{p^*} \leqslant \int_0^T \big\|f(t)\big\|_{V_A^*}^{p^*}\,dt. \tag{2.14}$$

Altogether, this shows the boundedness of the right-hand side of the a priori estimate (2.8), uniform with respect to the sequence $\{(V_{m_\ell},\mathbb{I}_\ell)\}$.

Recalling that

$$u^1(m_\ell,\mathbb{I}_\ell) = u^0(m_\ell,\mathbb{I}_\ell) + \tau_1(\mathbb{I}_\ell)v^0(m_\ell,\mathbb{I}_\ell),$$

Assumption (IC) together with (2.13) implies the boundedness of $\{u^1(m_\ell,\mathbb{I}_\ell)\}$ in V_B. Then, as a direct consequence of the a priori estimate (2.8), we observe the boundedness of $\{u_{m_\ell,\mathbb{I}_\ell}\}$ in $L^\infty(0,T;V_B)$. Moreover, the sequence $\{v_{m_\ell,\mathbb{I}_\ell}\}$ is bounded in $L^\infty(0,T;H)$ as well as in $L^p(0,T;V_A)$ as one can immediately infer from (2.8). Also the sequence $\{\hat v_{m_\ell,\mathbb{I}_\ell}\}$ is bounded in $L^\infty(0,T;H)$ as well as in $L^p(0,T;V_A)$. The first assertion is easily seen, whereas the second one is somewhat more involved. However, a straightforward calculation shows that

$$\begin{aligned}\|\hat v_{m_\ell,\mathbb{I}_\ell}\|_{L^p(0,T;V_A)}^p &\leqslant \frac{1}{2}\big(\tau_1(\mathbb{I}_\ell)+\tau_{3/2}(\mathbb{I}_\ell)\big)\big\|v^0(m_\ell,\mathbb{I}_\ell)\big\|_{V_A}^p\\ &\quad + \frac{1}{2}\sum_{j=1}^{N(\mathbb{I}_\ell)-2}\big(\tau_{j+1/2}(\mathbb{I}_\ell)+\tau_{j+3/2}(\mathbb{I}_\ell)\big)\big\|v^j(m_\ell,\mathbb{I}_\ell)\big\|_{V_A}^p\\ &\quad + \frac{1}{2}\big(\tau_{N(\mathbb{I}_\ell)-1/2}(\mathbb{I}_\ell)+\tau_{N(\mathbb{I}_\ell)}(\mathbb{I}_\ell)\big)\big\|v^{N(\mathbb{I}_\ell)-1}(m_\ell,\mathbb{I}_\ell)\big\|_{V_A}^p\\ &\leqslant \tau_{\max}(\mathbb{I}_\ell)\big\|v^0(m_\ell,\mathbb{I}_\ell)\big\|_{V_A}^p + \frac{r_{\max}(\mathbb{I}_\ell)+1}{2}\sum_{j=1}^{N(\mathbb{I}_\ell)-1}\tau_{j+1/2}(\mathbb{I}_\ell)\big\|v^j(m_\ell,\mathbb{I}_\ell)\big\|_{V_A}^p.\end{aligned}$$

This, together with Assumption (IC) and (2.8), shows the boundedness asserted.

Since $L^\infty(0,T;H)$ is the dual of the separable $L^1(0,T;H)$, $L^\infty(0,T;V_B)$ is the dual of the separable $L^1(0,T;V_B^*)$, and $L^p(0,T;V_A)$ is reflexive, by standard arguments (see, e.g., [6, Cor. III.26, Thm. III.27]), we thus have the existence of a subsequence, denoted by ℓ', and of elements $u \in L^\infty(0,T;V_B)$, $v \in L^\infty(0,T;H) \cap L^p(0,T;V_A)$, $\hat v \in L^\infty(0,T;H) \cap L^p(0,T;V_A)$ such that

$$u_{m_{\ell'},\mathbb{I}_{\ell'}} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;V_B),$$
$$v_{m_{\ell'},\mathbb{I}_{\ell'}} \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0,T;H), \qquad v_{m_{\ell'},\mathbb{I}_{\ell'}} \rightharpoonup v \quad \text{in } L^p(0,T;V_A),$$
$$\hat{v}_{m_{\ell'},\mathbb{I}_{\ell'}} \overset{*}{\rightharpoonup} \hat{v} \quad \text{in } L^\infty(0,T;H), \qquad \hat{v}_{m_{\ell'},\mathbb{I}_{\ell'}} \rightharpoonup \hat{v} \quad \text{in } L^p(0,T;V_A) \text{ as } \ell' \to \infty.$$

The definition of $v_{m_{\ell'},\mathbb{I}_{\ell'}}$ and $\hat{v}_{m_{\ell'},\mathbb{I}_{\ell'}}$ yields

$$\begin{aligned}
&\|\hat{v}_{m_{\ell'},\mathbb{I}_{\ell'}} - v_{m_{\ell'},\mathbb{I}_{\ell'}}\|^2_{L^2(0,T;H)} \\
&\leqslant \frac{\tau_1(\mathbb{I}_{\ell'})}{2}\big|v^0(m_{\ell'},\mathbb{I}_{\ell'})\big|^2 + \sum_{j=1}^{N(\mathbb{I}_{\ell'})-1} \frac{\tau_{j+1/2}(\mathbb{I}_{\ell'})}{3}\big|v^j(m_{\ell'},\mathbb{I}_{\ell'}) - v^{j-1}(m_{\ell'},\mathbb{I}_{\ell'})\big|^2 \\
&\quad + \frac{\tau_{N(\mathbb{I}_{\ell'})}(\mathbb{I}_{\ell'})}{2}\big|v^{N(\mathbb{I}_{\ell'})-1}(m_{\ell'},\mathbb{I}_{\ell'})\big|^2 \\
&\leqslant \tau_{\max}(\mathbb{I}_{\ell'})\Bigg(\big|v^0(m_{\ell'},\mathbb{I}_{\ell'})\big|^2 + \sum_{j=1}^{N(\mathbb{I}_{\ell'})-1} \big|v^j(m_{\ell'},\mathbb{I}_{\ell'}) - v^{j-1}(m_{\ell'},\mathbb{I}_{\ell'})\big|^2 \\
&\quad + \big|v^{N(\mathbb{I}_{\ell'})-1}(m_{\ell'},\mathbb{I}_{\ell'})\big|^2\Bigg).
\end{aligned} \tag{2.15}$$

The a priori estimate (2.8) shows that the right-hand side of the foregoing estimate converges towards zero as $\ell' \to \infty$. Hence, the weak limits v and $\hat{v}$ coincide.

We are now going to prove $Kv_{m_{\ell'},\mathbb{I}_{\ell'}} \overset{*}{\rightharpoonup} Kv$ in $L^\infty(0,T;V_A) = (L^1(0,T;V_A^*))^*$. For arbitrary $g \in L^1(0,T;V_A^*)$, we have (by a change of the integration variables)

$$\begin{aligned}
\langle Kv_{m_{\ell'},\mathbb{I}_{\ell'}} - Kv, g\rangle &= \int_0^T \left\langle g(t), \int_0^t \big(v_{m_{\ell'},\mathbb{I}_{\ell'}}(s) - v(s)\big)\,ds \right\rangle dt \\
&= \int_0^T \int_0^t \big\langle g(t), v_{m_{\ell'},\mathbb{I}_{\ell'}}(s) - v(s)\big\rangle\,ds\,dt \\
&= \int_0^T \int_s^T \big\langle g(t), v_{m_{\ell'},\mathbb{I}_{\ell'}}(s) - v(s)\big\rangle\,dt\,ds \\
&= \int_0^T \left\langle \int_s^T g(t)\,dt, v_{m_{\ell'},\mathbb{I}_{\ell'}}(s) - v(s) \right\rangle ds.
\end{aligned}$$

Since $s \mapsto \int_s^T g(t)\,dt \in L^\infty(0,T;V_A^*)$ and since $v_{m_{\ell'},\mathbb{I}_{\ell'}} \rightharpoonup v$ in $L^p(0,T;V_A)$ as $\ell' \to \infty$, the right-hand side of the foregoing identity converges towards zero.

Let us now show that $u_{m_\ell,\mathbb{I}_\ell} - u_0 - Kv_{m_\ell,\mathbb{I}_\ell}$ converges towards zero, strongly in $L^2(0,T;V_B)$. With Hölder's inequality, relation (2.5), and the Cauchy–Schwarz inequality, we find

$$
\begin{aligned}
&\|u_{m_\ell,\mathbb{I}_\ell} - u_0 - K v_{m_\ell,\mathbb{I}_\ell}\|^2_{L^2(0,T;V_B)} \\
&\quad = \int_0^{t_{1/2}(\mathbb{I}_\ell)} \|u_0\|^2_{V_B}\,dt \\
&\qquad + \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \int_{t_{n-1/2}(\mathbb{I}_\ell)}^{t_{n+1/2}(\mathbb{I}_\ell)} \Bigg\| u^n(m_\ell,\mathbb{I}_\ell) - u_0 - \sum_{j=1}^{n-1} \tau_{j+1/2}(\mathbb{I}_\ell) v^j(m_\ell,\mathbb{I}_\ell) \\
&\qquad - \big(t - t_{n-1/2}(\mathbb{I}_\ell)\big) v^n(m_\ell,\mathbb{I}_\ell) \Bigg\|^2_{V_B}\,dt \\
&\qquad + \int_{t_{N(\mathbb{I}_\ell)-1/2}(\mathbb{I}_\ell)}^{T} \Bigg\| u_0 + \sum_{j=1}^{N(\mathbb{I}_\ell)-1} \tau_{j+1/2}(\mathbb{I}_\ell) v^j(m_\ell,\mathbb{I}_\ell) \Bigg\|^2_{V_B}\,dt \\
&\quad \leqslant c\tau_{\max}(\mathbb{I}_\ell)\|u_0\|^2_{V_B} + c\big\|u^0(m_\ell,\mathbb{I}_\ell) - u_0\big\|^2_{V_B} + c\tau_{\max}(\mathbb{I}_\ell)^2\big\|v^0(m_\ell,\mathbb{I}_\ell)\big\|^2_{V_B} \\
&\qquad + c\Bigg(\sum_{j=1}^{N(\mathbb{I}_\ell)-2} \big|\tau_{j+1/2}(\mathbb{I}_\ell) - \tau_{j+1}(\mathbb{I}_\ell)\big|\big\|v^j(m_\ell,\mathbb{I}_\ell)\big\|_{V_B}\Bigg)^2 \\
&\qquad + c\tau_{\max}(\mathbb{I}_\ell)^2 \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \tau_{n+1/2}(\mathbb{I}_\ell)\big\|v^n(m_\ell,\mathbb{I}_\ell)\big\|^2_{V_B} \\
&\qquad + c\tau_{\max}(\mathbb{I}_\ell) \sum_{j=1}^{N(\mathbb{I}_\ell)-1} \tau_{j+1/2}(\mathbb{I}_\ell)\big\|v^j(m_\ell,\mathbb{I}_\ell)\big\|^2_{V_B}.
\end{aligned}
\tag{2.16}
$$

With Relation ($V_B \leftarrow V_A$) and Hölder's inequality, we further obtain

$$
\begin{aligned}
&\Bigg(\sum_{j=1}^{N(\mathbb{I}_\ell)-1} \big|\tau_{j+1/2}(\mathbb{I}_\ell) - \tau_{j+1}(\mathbb{I}_\ell)\big|\big\|v^j(m_\ell,\mathbb{I}_\ell)\big\|_{V_B}\Bigg)^2 \\
&\quad \leqslant \frac{1}{4} \sum_{j=1}^{N(\mathbb{I}_\ell)-1} \frac{(\tau_{j+1}(\mathbb{I}_\ell) - \tau_j(\mathbb{I}_\ell))^2}{\tau_{j+1/2}(\mathbb{I}_\ell)} \sum_{j=1}^{N(\mathbb{I}_\ell)-1} \tau_{j+1/2}(\mathbb{I}_\ell)\big\|v^j(m_\ell,\mathbb{I}_\ell)\big\|^2_{V_B} \\
&\quad \leqslant c\tau_{\max}(\mathbb{I}_\ell)^2 c_{V_B\leftarrow V_A}(m_\ell)^2 \theta(\mathbb{I}_\ell) \Bigg(\sum_{j=1}^{N(\mathbb{I}_\ell)-1} \tau_{j+1/2}(\mathbb{I}_\ell)\big\|v^j(m_\ell,\mathbb{I}_\ell)\big\|^p_{V_A}\Bigg)^{2/p}
\end{aligned}
\tag{2.17}
$$

and thus

$$
\begin{aligned}
&\|u_{m_\ell,\mathbb{I}_\ell} - u_0 - K v_{m_\ell,\mathbb{I}_\ell}\|^2_{L^2(0,T;V_B)} \\
&\quad \leqslant c\tau_{\max}(\mathbb{I}_\ell)\|u_0\|^2_{V_B} + c\big\|u^0(m_\ell,\mathbb{I}_\ell) - u_0\big\|^2_{V_B} \\
&\qquad + c\tau_{\max}(\mathbb{I}_\ell)^{2/p^*} c_{V_B\leftarrow V_A}(m_\ell)^2 \big(\tau_{\max}(\mathbb{I}_\ell)\big\|v^0(m_\ell,\mathbb{I}_\ell)\big\|^p_{V_A}\big)^{2/p}
\end{aligned}
$$

$$
\begin{aligned}
&+ c\tau_{\max}(\mathbb{I}_\ell) c_{V_B \leftarrow V_A}(m_\ell)^2 \big(\theta(\mathbb{I}_\ell)\tau_{\max}(\mathbb{I}_\ell) + \tau_{\max}(\mathbb{I}_\ell) + 1\big) \\
&\times \left(\sum_{j=1}^{N(\mathbb{I}_\ell)-1} \tau_{j+1/2}(\mathbb{I}_\ell) \big\| v^j(m_\ell, \mathbb{I}_\ell) \big\|_{V_A}^p \right)^{2/p}. \qquad (2.18)
\end{aligned}
$$

The assumptions together with the a priori estimate (2.8) now show the convergence asserted.

For proving $u - u_0 = Kv$, we conclude from what is shown before that

$$
\begin{aligned}
&u - u_0 - Kv = u_{m_{\ell'},\mathbb{I}_{\ell'}} - u_0 - Kv_{m_{\ell'},\mathbb{I}_{\ell'}} + u - u_{m_{\ell'},\mathbb{I}_{\ell'}} + Kv_{m_{\ell'},\mathbb{I}_{\ell'}} - Kv \rightharpoonup 0 \\
&\quad \text{in } L^2(0,T;V_A + V_B).
\end{aligned}
$$

Remember here that $V = V_A \cap V_B$ is dense in V_A and V_B and that V_A and V_B are dense and continuously embedded in H. We, therefore, obtain $u - u_0 = Kv \in \mathscr{C}([0,T];V_A)$ as well as $Kv = u - u_0 \in L^\infty(0,T;V_B)$.

Since $Kv = u - u_0$ is absolutely continuous as an abstract function with values in the reflexive space V_A and thus is differentiable almost everywhere (Theorem of Kōmura, see, e.g., [5, Cor. A.2]), we see that $u' = v \in L^p(0,T;V_A)$. □

For what follows, we need to introduce the Steklov average. Let $w \in L^p(0,T;X)$ ($p \in [1,\infty)$, X being a Banach space) be extended by zero outside $[0,T]$. Then we define for any (sufficiently small) $h > 0$,

$$
S_h^\pm w(t) := \pm \frac{1}{h} \int_t^{t\pm h} w(s)\,ds, \qquad S_h w(t) := \frac{1}{2}\big(S_h^+ w(t) + S_h^- w(t)\big) = \frac{1}{2h} \int_{t-h}^{t+h} w(s)\,ds.
$$

It is well known (see also [10, Thm. 9 on p. 49]) that

$$
\begin{aligned}
&S_h w \in L^p(0,T;X) \quad \text{with } \|S_h w\|_{L^p(0,T;X)} \leqslant \|w\|_{L^p(0,T;X)}, \\
&S_h w(t) \to w(t) \quad \text{in } X, \text{ a.e. in } (0,T) \ni t, \quad \text{and} \quad S_h w \to w \quad \text{in } L^p(0,T;X) \text{ as } h \to 0.
\end{aligned}
$$

Lemma 6. *Let $u_0 \in V_B$ and let $w \in L^p(0,T;V_A)$ with $Kw \in L^2(0,T;V_B)$ such that $w' + B_0(u_0 + Kw) \in L^{p^*}(0,T;V_A^*)$. Then for almost all $\alpha, \beta \in (0,T)$ with $\alpha < \beta$ there holds*

$$
\begin{aligned}
&\int_\alpha^\beta \big\langle \big(w' + B_0(u_0 + Kw)\big)(t), w(t) \big\rangle\,dt \\
&\quad = \frac{1}{2}\big|w(\beta)\big|^2 - \frac{1}{2}\big|w(\alpha)\big|^2 + \frac{1}{2}\big\|u_0 + Kw(\beta)\big\|_B^2 - \frac{1}{2}\big\|u_0 + Kw(\alpha)\big\|_B^2 \qquad (2.19)
\end{aligned}
$$

with $\|\cdot\|_B := \langle B_0 \cdot, \cdot \rangle^{1/2}$ denoting the norm on V_B induced by B_0. If in addition $w \in \mathscr{C}_w([0,T];H)$ with $Kw \in \mathscr{C}_w([0,T];V_B)$ then for almost all $\beta \in (0,T)$ there holds

$$
\begin{aligned}
&\int_0^\beta \big\langle \big(w' + B_0(u_0 + Kw)\big)(t), w(t) \big\rangle\,dt \\
&\quad \leqslant \frac{1}{2}\big|w(\beta)\big|^2 - \frac{1}{2}\big|w(0)\big|^2 + \frac{1}{2}\big\|u_0 + Kw(\beta)\big\|_B^2 - \frac{1}{2}\|u_0\|_B^2. \qquad (2.20)
\end{aligned}
$$

Remark 3. The main difficulty in proving Lemma 6 is that only the sum of w' and $B_0(u_0 + Kw)$ is in the dual of $L^p(0,T;V_A) \ni w$. Indeed, in the application later, we will only have $w' \in (L^p(0,T;V))^* \supset (L^p(0,T;V_A))^*$ and $B_0(u_0 + Kw) \in L^\infty(0,T;V_B^*)$. Therefore, it is not allowed to split the duality pairing and to perform an integration by parts separately.

Proof of Lemma 6. We commence with proving the assertion (2.19) for the Steklov average $S_h w$ instead of w. Let $h > 0$ be sufficiently small such that, in particular, $(\alpha,\beta) \subset (h, T-h)$. First, we recall that by construction $S_h w \in L^p(0,T;V_A)$. However, since

$$S_h w(t) = \frac{1}{2h}\big(Kw(t+h) - Kw(t-h)\big),$$

we also have, for fixed $h > 0$, that $S_h w \in L^2(0,T;V_B)$. We, therefore, can split the terms appearing and can carry out integration by parts (using $S_h w = (u_0 + KS_h w)'$) as follows:

$$\begin{aligned}
&\int_\alpha^\beta \big\langle \big((S_h w)' + B_0(u_0 + KS_h w)\big)(t), S_h w(t)\big\rangle\, dt \\
&= \int_\alpha^\beta \big\langle (S_h w)'(t), S_h w(t)\big\rangle\, dt + \int_\alpha^\beta \big\langle B_0(u_0 + KS_h w)(t), (u_0 + KS_h w)'(t)\big\rangle\, dt \\
&= \frac{1}{2}\big|S_h w(\beta)\big|^2 - \frac{1}{2}\big|S_h w(\alpha)\big|^2 + \frac{1}{2}\big\|u_0 + KS_h w(\beta)\big\|_B^2 - \frac{1}{2}\big\|u_0 + KS_h w(\alpha)\big\|_B^2. \qquad (2.21)
\end{aligned}$$

We now consider the difference between the formulas for w and $S_h w$. We have

$$\begin{aligned}
&\int_\alpha^\beta \big\langle \big(w' + B_0(u_0 + Kw)\big)(t), w(t)\big\rangle\, dt - \int_\alpha^\beta \big\langle \big((S_h w)' + B_0(u_0 + KS_h w)\big)(t), S_h w(t)\big\rangle\, dt \\
&= \int_\alpha^\beta \big\langle \big(w' + B_0(u_0 + Kw)\big)(t), (w - S_h w)(t)\big\rangle\, dt \\
&\quad + \int_\alpha^\beta \big\langle \big(w' + B_0(u_0 + Kw) - (S_h w)' - B_0(u_0 + KS_h w)\big)(t), S_h w(t)\big\rangle\, dt. \qquad (2.22)
\end{aligned}$$

The first term on the right-hand side converges towards zero as $h \to 0$ since $w' + B_0(u_0 + Kw) \in L^{p^*}(0,T;V_A^*)$ and since $S_h w$ converges towards w in $L^p(0,T;V_A)$ as h tends to zero.

The second term on the right-hand side in (2.22) is more involved. As is easily seen, S_h and differentiation commute,

$$S_h w'(t) = \frac{1}{2h}\big(w(t+h) - w(t-h)\big) = (S_h w)'(t), \quad t \in (\alpha,\beta).$$

It is allowed to split the term under consideration as follows (using $(S_h w)' = S_h w'$ and the linearity of B_0):

$$\int_\alpha^\beta \big\langle \big(w' + B_0(u_0 + Kw) - (S_h w)' - B_0(u_0 + KS_h w)\big)(t), S_h w(t)\big\rangle\, dt$$

$$= \int_\alpha^\beta \big\langle \big(w' + B_0(u_0 + Kw) - S_h\big(w' + B_0(u_0 + Kw)\big)\big)(t), S_h w(t)\big\rangle\, dt$$

$$+ \int_\alpha^\beta \big\langle \big(B_0(S_h K w - K S_h w)\big)(t), S_h w(t)\big\rangle\, dt.$$

Since $w' + B_0(u_0 + Kw) \in L^{p^*}(0, T; V_A^*)$, we have

$$S_h\big(w' + B_0(u_0 + Kw)\big) \to w' + B_0(u_0 + Kw) \quad \text{in } L^{p^*}\big(0, T; V_A^*\big) \text{ as } h \to 0.$$

This, together with $\|S_h w\|_{L^p(0,T;V_A)} \leqslant \|w\|_{L^p(0,T;V_A)}$, shows that the first term on the right-hand side of the foregoing identity vanishes as h tends to zero.

For the remaining term, we find (by changing the order of integration and using $w = (Kw)'$, integration by parts as well as $Kw(0) = 0$) the commutator relation

$$\begin{aligned}
(S_h K w - K S_h w)(t) &= \frac{1}{2h}\int_{t-h}^{t+h}\int_0^s w(r)\, dr\, ds - \frac{1}{2h}\int_0^t \int_{s-h}^{s+h} w(r)\, dr\, ds \\
&= \frac{1}{2h}\left(\int_0^{t-h}\int_{t-h}^{t+h} w(r)\, ds\, dr + \int_{t-h}^{t+h}\int_r^{t+h} w(r)\, ds\, dr\right) \\
&\quad - \frac{1}{2h}\left(\int_0^h\int_0^{r+h} w(r)\, ds\, dr + \int_h^{t-h}\int_{r-h}^{r+h} w(r)\, ds\, dr + \int_{t-h}^{t+h}\int_{r-h}^{t} w(r)\, ds\, dr\right) \\
&= \frac{1}{2h}\int_0^h (h - r) w(r)\, dr \\
&= \frac{1}{2h}\int_0^h (h - r)(Kw)'(r)\, dr \\
&= \frac{1}{2h}\int_0^h Kw(r)\, dr
\end{aligned}$$

which implies

$$(S_h K w - K S_h w)(t) \equiv S_h K w(0).$$

Note in particular that the commutator is independent of time.

Hence, we obtain (using again the above commutator relation as well as (2.10))

$$\int_\alpha^\beta \big\langle \big(B_0(S_hKw - KS_hw)\big)(t), S_hw(t)\big\rangle\, dt$$

$$= \left\langle B_0S_hKw(0), \int_\alpha^\beta S_hw(t)\right\rangle = \left\langle B_0S_hKw(0), \int_\alpha^\beta (KS_hw)'(t)\right\rangle$$

$$= \big\langle B_0S_hKw(0), KS_hw(\beta) - KS_hw(\alpha)\big\rangle$$

$$= \big\langle B_0S_hKw(0), u_0 + KS_hw(\beta)\big\rangle - \big\langle B_0S_hKw(0), u_0 + KS_hw(\alpha)\big\rangle$$

$$= \big\langle B_0\big(u_0 + S_hKw(\beta)\big) - B_0\big(u_0 + KS_hw(\beta)\big), u_0 + KS_hw(\beta)\big\rangle$$

$$\quad - \big\langle B_0\big(u_0 + S_hKw(\alpha)\big) - B_0\big(u_0 + KS_hw(\alpha)\big), u_0 + KS_hw(\alpha)\big\rangle$$

$$= \frac{1}{2}\big\|u_0 + S_hKw(\beta)\big\|_B^2 - \frac{1}{2}\big\|u_0 + KS_hw(\beta)\big\|_B^2 - \frac{1}{2}\big\|\big(u_0 + S_hKw(\beta)\big) - \big(u_0 + KS_hw(\beta)\big)\big\|_B^2$$

$$\quad - \frac{1}{2}\big\|u_0 + S_hKw(\alpha)\big\|_B^2 + \frac{1}{2}\big\|u_0 + KS_hw(\alpha)\big\|_B^2 + \frac{1}{2}\big\|\big(u_0 + S_hKw(\alpha)\big) - \big(u_0 + KS_hw(\alpha)\big)\big\|_B^2$$

$$\big(\text{with } \big(u_0 + S_hKw(\beta)\big) - \big(u_0 + KS_hw(\beta)\big) = S_hKw(0) = \big(u_0 + S_hKw(\alpha)\big) - \big(u_0 + KS_hw(\alpha)\big)\big)$$

$$= \frac{1}{2}\big\|u_0 + S_hKw(\beta)\big\|_B^2 - \frac{1}{2}\big\|u_0 + KS_hw(\beta)\big\|_B^2 - \frac{1}{2}\big\|u_0 + S_hKw(\alpha)\big\|_B^2 + \frac{1}{2}\big\|u_0 + KS_hw(\alpha)\big\|_B^2.$$

This, together with (2.21), proves the first assertion (2.19): Since $S_hw(t) \to w(t)$ in $V_A \hookrightarrow H$ and $S_hKw(t) \to Kw(t)$ in V_B for almost all $t \in [0, T]$ as $h \to 0$, the first and third term on the right-hand side of the foregoing relation converge towards the corresponding terms in (2.19), whereas the second and fourth term cancel in view of (2.21).

The second assertion follows by taking $\alpha \to 0$ in the first assertion and employing the weak lower semi-continuity of the norm. □

We shall remark that a result similar to (2.20) can also be found in [19, Lemma 2.1] (with a different proof).

Proof of Theorem 4. From the numerical scheme (1.5), we conclude

$$\begin{aligned}
&-\int_0^T \big(\hat{v}_{m_\ell,\mathbb{I}_\ell}(t), \varphi\big)\psi'(t)\, dt + \big(v^{N(\mathbb{I}_\ell)-1}(m_\ell, \mathbb{I}_\ell), \varphi\big)\psi(T) - \big(v^0(m_\ell, \mathbb{I}_\ell), \varphi\big)\psi(0) \\
&\quad + \int_0^T \big\langle (A_{0,\mathbb{I}_\ell} v_{m_\ell,\mathbb{I}_\ell})(t), \varphi\big\rangle\psi(t)\, dt + \int_0^T \big\langle B_0u_{m_\ell,\mathbb{I}_\ell}(t), \varphi\big\rangle\psi(t)\, dt \\
&= \int_0^T \big\langle f_{\mathbb{I}_\ell}(t), \varphi\big\rangle\psi(t)\, dt
\end{aligned} \tag{2.23}$$

for all $\varphi \in V_j$ with arbitrary $j \in \mathbb{N}$, all $\psi \in \mathscr{C}^1([0, T])$ and all $\ell \in \mathbb{N}$ with $m_\ell \geqslant j$ since with integration by parts

$$\int_0^T \big(\hat{v}'_{m_\ell,\mathbb{I}_\ell}(t),\varphi\big)\psi(t)\,dt = -\int_0^T \big(\hat{v}_{m_\ell,\mathbb{I}_\ell}(t),\varphi\big)\psi'(t)\,dt + \big(\hat{v}_{m_\ell,\mathbb{I}_\ell}(T),\varphi\big)\psi(T) - \big(\hat{v}_{m_\ell,\mathbb{I}_\ell}(0),\varphi\big)\psi(0)$$

and since, by definition, $\hat{v}_{m_\ell,\mathbb{I}_\ell}(0) = v^0(m_\ell,\mathbb{I}_\ell)$, $\hat{v}_{m_\ell,\mathbb{I}_\ell}(T) = v^{N(\mathbb{I}_\ell)-1}(m_\ell,\mathbb{I}_\ell)$.

We are going to employ the results of Lemma 5 and Assumption (IC). In addition, we observe the following.

The a priori estimate (2.8) shows that the sequence $\{v^{N(\mathbb{I}_\ell)-1}(m_\ell,\mathbb{I}_\ell)\}$ is bounded in H. By standard arguments, we can extract a subsequence (of the subsequence already given by Lemma 5, but still denoted by ℓ') such that

$$v^{N(\mathbb{I}_{\ell'})-1}(m_{\ell'},\mathbb{I}_{\ell'}) \rightharpoonup \xi \quad \text{in } H \text{ as } \ell' \to \infty \tag{2.24}$$

for some element $\xi \in H$.

The growth condition for A_0 shows that A_0 maps subsets bounded in $L^p(0,T;V_A)$ into subsets bounded in $(L^p(0,T;V_A))^*$. Therefore, $\{A_{0,\mathbb{I}_\ell} v_{m_\ell,\mathbb{I}_\ell}\}$ is bounded in $(L^p(0,T;V_A))^*$, and, by standard arguments, we have a subsequence (of the subsequence already chosen and still denoted by ℓ') and an element $a \in (L^p(0,T;V_A))^*$ such that

$$A_{0,\mathbb{I}_{\ell'}} v_{m_{\ell'},\mathbb{I}_{\ell'}} \rightharpoonup a \quad \text{in } \big(L^p(0,T;V_A)\big)^* \text{ as } \ell' \to \infty. \tag{2.25}$$

With respect to B_0, we see that B_0 is a linear and bounded mapping of $L^2(0,T;V_B)$ into $L^2(0,T;V_B^*)$ and thus is weakly–weakly continuous (see [6, Thm. III.9]). Since $u_{m_{\ell'},\mathbb{I}_{\ell'}}$ converges weakly* in $L^\infty(0,T;V_B)$ towards u as $\ell' \to \infty$, we also have $u_{m_{\ell'},\mathbb{I}_{\ell'}} \rightharpoonup u$ in $L^2(0,T;V_B)$ and, therefore,

$$B_0 u_{m_{\ell'},\mathbb{I}_{\ell'}} \rightharpoonup B_0 u \quad \text{in } L^2\big(0,T;V_B^*\big) \text{ as } \ell' \to \infty. \tag{2.26}$$

For the right-hand side in (2.23), a straightforward argument shows that

$$f_{\mathbb{I}_\ell} \to f \quad \text{in } \big(L^p(0,T;V_A)\big)^* \text{ as } \ell \to \infty. \tag{2.27}$$

Altogether, we thus obtain from (2.23) in the limit

$$\begin{aligned} &-\int_0^T \big(v(t),\varphi\big)\psi'(t)\,dt + (\xi,\varphi)\psi(T) - (v_0,\varphi)\psi(0) \\ &\quad + \int_0^T \big\langle a(t),\varphi\big\rangle\psi(t)\,dt + \int_0^T \big\langle B_0 u(t),\varphi\big\rangle\psi(t)\,dt \\ &= \int_0^T \big\langle f(t),\varphi\big\rangle\psi(t)\,dt \end{aligned} \tag{2.28}$$

for all $\varphi \in V_j$ with arbitrary $j \in \mathbb{N}$ and all $\psi \in \mathscr{C}^1([0,T])$.

Because of the limited completeness (2.1) of the Galerkin scheme, the foregoing relation (2.28) indeed holds for all $\varphi \in V$. Then it follows that $f - a - B_0 u \in L^{p^*}(0,T;V_A^*) + L^2(0,T;V_B^*) \subseteq L^1(0,T;V^*)$ is the weak derivative of $v \in L^p(0,T;V_A) \subseteq L^1(0,T;V^*)$ (see, e.g., [26, Lemma 1.1 on p. 250]). We, finally, obtain

$$v' + a + B_0 u = f \quad \text{in } \big(L^p(0,T;V)\big)^* \tag{2.29}$$

since $p \geqslant 2$ and since the set of functions $t \mapsto \varphi\psi(t)$ with $\varphi \in V$ and $\psi \in \mathscr{C}_c^1(0,T)$ is dense in $L^p(0,T;V)$ (remember also that $V^* = (V_A \cap V_B)^* = V_A^* + V_B^*$).

Note that $v \in L^p(0,T;V_A) \subseteq L^{p^*}(0,T;V^*)$ with $v' \in L^{p^*}(0,T;V^*)$ is absolutely continuous as an abstract function with values in V^* (see again, e.g., [26, Lemma 1.1 on p. 250]). Therefore, by taking $\psi(T) = 0$ and $\psi(0) = 0$, respectively, the relation (2.28) also shows

$$v(0) = v_0, \qquad v(T) = \xi. \tag{2.30}$$

Indeed, since $v \in L^\infty(0,T;H) \cap \mathscr{C}([0,T];V^*)$ it follows that $v \in \mathscr{C}_w([0,T];H)$ as H is dense and continuously embedded in V^* (see, e.g., [26, Lemma 1.4 on p. 263]). Similarly, since $u - u_0 = Kv \in L^\infty(0,T;V_B)$ but also in $\mathscr{C}([0,T];V_A) \hookrightarrow \mathscr{C}([0,T];H)$ it follows that $u - u_0 = Kv$ is, possibly after a change on a set of measure zero, continuous on $[0,T]$ with respect to the weak topology of V_B.

It remains to prove $a = A_0 v$ by employing the monotonicity of A_0 and the properties of B_0K. For arbitrary $w \in L^p(0,T;V_A)$, we obtain from testing the numerical scheme by $v_{m_\ell,\mathbb{I}_\ell}$ and because of the monotonicity of A_0 (thus of $A_{0,\mathbb{I}_\ell}$)

$$\begin{aligned}
0 &= \int_0^T \big\langle \hat{v}'_{m_\ell,\mathbb{I}_\ell}(t) + (A_{0,\mathbb{I}_\ell} v_{m_\ell,\mathbb{I}_\ell})(t) + B_0 u_{m_\ell,\mathbb{I}_\ell}(t) - f_{\mathbb{I}_\ell}(t), v_{m_\ell,\mathbb{I}_\ell}(t)\big\rangle\, dt \\
&\geqslant \int_0^T \big\langle \hat{v}'_{m_\ell,\mathbb{I}_\ell}(t) + (A_{0,\mathbb{I}_\ell} v_{m_\ell,\mathbb{I}_\ell})(t) + B_0 u_{m_\ell,\mathbb{I}_\ell}(t) - f_{\mathbb{I}_\ell}(t), v_{m_\ell,\mathbb{I}_\ell}(t)\big\rangle\, dt \\
&\quad - \int_0^T \big\langle (A_{0,\mathbb{I}_\ell} v_{m_\ell,\mathbb{I}_\ell})(t) - (A_{0,\mathbb{I}_\ell} w)(t), v_{m_\ell,\mathbb{I}_\ell}(t) - w(t)\big\rangle\, dt \\
&= \int_0^T \big\langle \hat{v}'_{m_\ell,\mathbb{I}_\ell}(t) + B_0 u_{m_\ell,\mathbb{I}_\ell}(t) - f_{\mathbb{I}_\ell}(t), v_{m_\ell,\mathbb{I}_\ell}(t)\big\rangle\, dt \\
&\quad + \int_0^T \big\langle (A_{0,\mathbb{I}_\ell} v_{m_\ell,\mathbb{I}_\ell})(t), w(t)\big\rangle\, dt + \int_0^T \big\langle (A_{0,\mathbb{I}_\ell} w)(t), v_{m_\ell,\mathbb{I}_\ell}(t) - w(t)\big\rangle\, dt.
\end{aligned} \tag{2.31}$$

For the term including the time derivative, we obtain with (2.9)

$$\begin{aligned}
\int_0^T \big\langle \hat{v}'_{m_\ell,\mathbb{I}_\ell}(t), v_{m_\ell,\mathbb{I}_\ell}(t)\big\rangle\, dt &= \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \int_{t_{n-1/2}(\mathbb{I}_\ell)}^{t_{n+1/2}(\mathbb{I}_\ell)} \left(\frac{v^n(m_\ell,\mathbb{I}_\ell) - v^{n-1}(m_\ell,\mathbb{I}_\ell)}{\tau_{n+1/2}(\mathbb{I}_\ell)}, v^n(m_\ell,\mathbb{I}_\ell)\right) dt \\
&= \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \big(v^n(m_\ell,\mathbb{I}_\ell) - v^{n-1}(m_\ell,\mathbb{I}_\ell), v^n(m_\ell,\mathbb{I}_\ell)\big) \\
&\geqslant \frac{1}{2}\big|v^{N(\mathbb{I}_\ell)-1}(m_\ell,\mathbb{I}_\ell)\big|^2 - \frac{1}{2}\big|v^0(m_\ell,\mathbb{I}_\ell)\big|^2.
\end{aligned} \tag{2.32}$$

For the term including B_0, we find with $v_{m_\ell,\mathbb{I}_\ell} = (u_0 + Kv_{m_\ell,\mathbb{I}_\ell})'$ and integration by parts

$$\begin{aligned}
\int_0^T \big\langle B_0 u_{m_\ell,\mathbb{I}_\ell}(t), v_{m_\ell,\mathbb{I}_\ell}(t)\big\rangle\,dt &= \int_0^T \big\langle B_0(u_0 + Kv_{m_\ell,\mathbb{I}_\ell})(t), (u_0 + Kv_{m_\ell,\mathbb{I}_\ell})'(t)\big\rangle\,dt \\
&\quad + \int_0^T \big\langle B_0(u_{m_\ell,\mathbb{I}_\ell} - u_0 - Kv_{m_\ell,\mathbb{I}_\ell})(t), v_{m_\ell,\mathbb{I}_\ell}(t)\big\rangle\,dt \\
&= \frac{1}{2}\big\|u_0 + Kv_{m_\ell,\mathbb{I}_\ell}(T)\big\|_B^2 - \frac{1}{2}\|u_0\|_B^2 \\
&\quad + \int_0^T \big\langle B_0(u_{m_\ell,\mathbb{I}_\ell} - u_0 - Kv_{m_\ell,\mathbb{I}_\ell})(t), v_{m_\ell,\mathbb{I}_\ell}(t)\big\rangle\,dt. \qquad (2.33)
\end{aligned}$$

This yields

$$\begin{aligned}
0 &\geqslant \frac{1}{2}\big|v^{N(\mathbb{I}_\ell)-1}(m_\ell, \mathbb{I}_\ell)\big|^2 - \frac{1}{2}\big|v^0(m_\ell, \mathbb{I}_\ell)\big|^2 + \frac{1}{2}\big\|u_0 + Kv_{m_\ell,\mathbb{I}_\ell}(T)\big\|_B^2 - \frac{1}{2}\|u_0\|_B^2 \\
&\quad + \int_0^T \big\langle B_0(u_{m_\ell,\mathbb{I}_\ell} - u_0 - Kv_{m_\ell,\mathbb{I}_\ell})(t), v_{m_\ell,\mathbb{I}_\ell}(t)\big\rangle\,dt - \int_0^T \big\langle f_{\mathbb{I}_\ell}(t), v_{m_\ell,\mathbb{I}_\ell}(t)\big\rangle\,dt \\
&\quad + \int_0^T \big\langle (A_{0,\mathbb{I}_\ell} v_{m_\ell,\mathbb{I}_\ell})(t), w(t)\big\rangle\,dt + \int_0^T \big\langle (A_{0,\mathbb{I}_\ell} w)(t), v_{m_\ell,\mathbb{I}_\ell}(t) - w(t)\big\rangle\,dt. \qquad (2.34)
\end{aligned}$$

We are now going to take the limit.

There holds (for a suitably chosen subsequence denoted by ℓ')

$$u_0 + Kv_{m_{\ell'},\mathbb{I}_{\ell'}}(T) \rightharpoonup u_0 + Kv(T) \quad \text{in } V_B \text{ as } \ell' \to \infty.$$

This might be shown by employing the weak–weak continuity of the trace operator $w \mapsto w(T)$, $W^{1,1}(0,T; V_A + V_B) \to V_A + V_B$ together with density arguments. However, we provide a simple direct proof.

A straightforward calculation shows that (1.5) implies

$$\begin{aligned}
&u_0 + Kv_{m_\ell,\mathbb{I}_\ell}(T) - u^{N(\mathbb{I}_\ell)}(m_\ell, \mathbb{I}_\ell) \\
&\quad = \sum_{n=1}^{N(\mathbb{I}_\ell)} \big(\tau_{n+1/2}(\mathbb{I}_\ell) - \tau_{n+1}(\mathbb{I}_\ell)\big) v^n(m_\ell, \mathbb{I}_\ell) + u_0 - u^0(m_\ell, \mathbb{I}_\ell) - \tau_1(\mathbb{I}_\ell) v^0(m_\ell, \mathbb{I}_\ell).
\end{aligned}$$

With Relation $(V_B \leftarrow V_A)$ and Hölder's inequality (recalling that $p \geqslant 2$ and thus $p^* = p/(p-1) \leqslant 2$ and recalling the definition of $\theta(\mathbb{I}_\ell)$ in (2.4)), we find (see also (2.17) and (2.13))

$$\begin{aligned}
&\big\|u_0 + Kv_{m_\ell,\mathbb{I}_\ell}(T) - u^{N(\mathbb{I}_\ell)}(m_\ell, \mathbb{I}_\ell)\big\|_{V_B} \\
&\quad \leqslant \frac{1}{2} c_{V_B \leftarrow V_A}(m_\ell) \sum_{n=1}^{N(\mathbb{I}_\ell)} \big|\tau_{n+1}(\mathbb{I}_\ell) - \tau_n(\mathbb{I}_\ell)\big| \big\|v^n(m_\ell, \mathbb{I}_\ell)\big\|_{V_A} + \big\|u_0 - u^0(m_\ell, \mathbb{I}_\ell)\big\|_{V_B}
\end{aligned}$$

$$
\begin{aligned}
&+ c_{V_B \leftarrow V_A}(m_\ell)\tau_{\max}(\mathbb{I}_\ell)\left\|v^0(m_\ell,\mathbb{I}_\ell)\right\|_{V_A} \\
&\leqslant cc_{V_B \leftarrow V_A}(m_\ell)\tau_{\max}(\mathbb{I}_\ell)\theta(\mathbb{I}_\ell)^{1/2}\left(\sum_{n=1}^{N(\mathbb{I}_\ell)}\tau_{n+1/2}(\mathbb{I}_\ell)\left\|v^n(m_\ell,\mathbb{I}_\ell)\right\|_{V_A}^p\right)^{1/p} + \left\|u_0 - u^0(m_\ell,\mathbb{I}_\ell)\right\|_{V_B} \\
&+ c_{V_B \leftarrow V_A}(m_\ell)\tau_{\max}(\mathbb{I}_\ell)^{1/p^*}\left(\tau_{\max}(\mathbb{I}_\ell)\left\|v^0(m_\ell,\mathbb{I}_\ell)\right\|_{V_A}^p\right)^{1/p}.
\end{aligned}
$$

This shows, because of the a priori estimate (2.8) and the assumptions, the strong convergence

$$
u_0 + Kv_{m_\ell,\mathbb{I}_\ell}(T) - u^{N(\mathbb{I}_\ell)}(m_\ell,\mathbb{I}_\ell) \to 0 \quad \text{in } V_B \text{ as } \ell \to \infty.
$$

The a priori estimate (2.8) now yields, together with the assumptions, the boundedness of the sequence $\{u^{N(\mathbb{I}_\ell)}(m_\ell,\mathbb{I}_\ell)\}$ in V_B. Therefore, also $\{u_0 + Kv_{m_\ell,\mathbb{I}_\ell}(T)\}$ is bounded in V_B, and we can extract the subsequence denoted by ℓ' in such a way that for some $\zeta \in V_B$,

$$
u_0 + Kv_{m_{\ell'},\mathbb{I}_{\ell'}}(T) \rightharpoonup \zeta \quad \text{in } V_B \text{ as } \ell' \to \infty. \tag{2.35}
$$

It remains to determine ζ.

Let $\varphi \in V_j \subseteq V \subseteq V_A^* \cap V_B^*$ with $j \in \mathbb{N}$ such that $j \leqslant m_\ell$. We then find with integration by parts, recalling that $(u_0 + Kv)' = v$, and inserting (2.23) with $\psi(t) = t^2/(2T)$,

$$
\begin{aligned}
\left(u_0 + Kv(T), \varphi\right) &= \int_0^T \left\langle \varphi, (u_0 + Kv)'(t)\right\rangle \frac{t}{T}\,dt + \int_0^T \left\langle \varphi, (u_0 + Kv)(t)\right\rangle \frac{1}{T}\,dt \\
&= \int_0^T \left\langle \varphi, v(t) - \hat{v}_{m_\ell,\mathbb{I}_\ell}(t)\right\rangle \frac{t}{T}\,dt + \int_0^T \left\langle \varphi, Kv(t) - Kv_{m_\ell,\mathbb{I}_\ell}(t)\right\rangle \frac{1}{T}\,dt \\
&\quad + \int_0^T \left\langle \varphi, \hat{v}_{m_\ell,\mathbb{I}_\ell}(t)\right\rangle \frac{t}{T}\,dt + \int_0^T \left\langle \varphi, u_0 + Kv_{m_\ell,\mathbb{I}_\ell}(t)\right\rangle \frac{1}{T}\,dt \\
&= \int_0^T \left\langle \varphi, v(t) - \hat{v}_{m_\ell,\mathbb{I}_\ell}(t)\right\rangle \frac{t}{T}\,dt + \int_0^T \left\langle \varphi, Kv(t) - Kv_{m_\ell,\mathbb{I}_\ell}(t)\right\rangle \frac{1}{T}\,dt \\
&\quad + \left(v^{N(\mathbb{I}_\ell)-1}(m_\ell,\mathbb{I}_\ell), \varphi\right)\frac{T}{2} + \int_0^T \left\langle (A_{0,\mathbb{I}_\ell} v_{m_\ell,\mathbb{I}_\ell})(t), \varphi\right\rangle \frac{t^2}{2T}\,dt \\
&\quad + \int_0^T \left\langle B_0 u_{m_\ell,\mathbb{I}_\ell}(t), \varphi\right\rangle \frac{t^2}{2T}\,dt - \int_0^T \left\langle f_{\mathbb{I}_\ell}(t), \varphi\right\rangle \frac{t^2}{2T}\,dt \\
&\quad + \left(u_0 + Kv_{m_\ell,\mathbb{I}_\ell}(T), \varphi\right) - \int_0^T \left\langle \varphi, (u_0 + Kv_{m_\ell,\mathbb{I}_\ell})'(t)\right\rangle \frac{t}{T}\,dt.
\end{aligned} \tag{2.36}
$$

Recalling $(u_0 + Kv_{m_\ell,\mathbb{I}_\ell})' = Kv_{m_\ell,\mathbb{I}_\ell}$ in the last term, taking now the limit employing the results of Lemma 5 and invoking (2.24) with (2.30), (2.25), (2.26), (2.27) with (2.29) as well as (2.35), we find (again with integration by parts)

$$\big(u_0 + Kv(T), \varphi\big) = \big(v(T), \varphi\big)\frac{T}{2} - \int_0^T \big\langle v'(t), \varphi\big\rangle \frac{t^2}{2T}\,dt + (\zeta, \varphi) - \int_0^T \big\langle \varphi, v(t)\big\rangle \frac{t}{T}\,dt = (\zeta, \varphi).$$

The limited completeness (2.1) of the Galerkin scheme and the density in the scale $V \subseteq V_B \subseteq H \subseteq V_B^*$ now shows that

$$\zeta = u_0 + Kv(T). \tag{2.37}$$

Unfortunately, we cannot take directly the limit in the term with B_0 in (2.34) since $B_0(u_{m_\ell,\mathbb{I}_\ell} - u_0 - Kv_{m_\ell,\mathbb{I}_\ell})$ converges strongly in $L^2(0,T;V_B^*)$ and $v_{m_{\ell'},\mathbb{I}_{\ell'}}$ converges weakly in $L^p(0,T;V_A)$ as $\ell' \to \infty$ but V_B^* and V_A do not match. With the definition of $u_{m_\ell,\mathbb{I}_\ell}$, $v_{m_\ell,\mathbb{I}_\ell}$ and with (1.5), we observe, however, that

$$\begin{aligned}
&\int_0^T \big\langle B_0(u_{m_\ell,\mathbb{I}_\ell} - u_0 - Kv_{m_\ell,\mathbb{I}_\ell})(t), v_{m_\ell,\mathbb{I}_\ell}(t)\big\rangle\,dt \\
&\quad= \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \int_{t_{n-1/2}(\mathbb{I}_\ell)}^{t_{n+1/2}(\mathbb{I}_\ell)} \Bigg\langle B_0\Bigg(u^n(m_\ell,\mathbb{I}_\ell) - u_0 - \sum_{j=1}^{n-1} \int_{t_{j-1/2}(\mathbb{I}_\ell)}^{t_{j+1/2}(\mathbb{I}_\ell)} v^j(m_\ell,\mathbb{I}_\ell)\,ds \\
&\qquad - \int_{t_{n-1/2}(\mathbb{I}_\ell)}^{t} v^n(m_\ell,\mathbb{I}_\ell)\,ds\Bigg), v^n(m_\ell,\mathbb{I}_\ell)\Bigg\rangle\,dt \\
&\quad= \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \sum_{j=1}^{n-1} \tau_{n+1/2}(\mathbb{I}_\ell)\big(\tau_{j+1}(\mathbb{I}_\ell) - \tau_{j+1/2}(\mathbb{I}_\ell)\big)\big\langle B_0 v^j(m_\ell,\mathbb{I}_\ell), v^n(m_\ell,\mathbb{I}_\ell)\big\rangle \\
&\qquad + \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \tau_{n+1/2}(\mathbb{I}_\ell)\big\langle B_0\big(\tau_1(\mathbb{I}_\ell)v^0(m_\ell,\mathbb{I}_\ell) + u^0(m_\ell,\mathbb{I}_\ell) - u_0\big), v^n(m_\ell,\mathbb{I}_\ell)\big\rangle \\
&\qquad - \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \frac{\tau_{n+1/2}(\mathbb{I}_\ell)^2}{2}\big\|v^n(m_\ell,\mathbb{I}_\ell)\big\|_B^2 \\
&\quad= \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \sum_{j=1}^{n-1} \tau_{n+1/2}(\mathbb{I}_\ell)\big(\tau_{j+1}(\mathbb{I}_\ell) - \tau_{j+1/2}(\mathbb{I}_\ell)\big)\big\langle B_0 v^j(m_\ell,\mathbb{I}_\ell), v^n(m_\ell,\mathbb{I}_\ell)\big\rangle \\
&\qquad - \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \big(\tau_{n+1}(\mathbb{I}_\ell) - \tau_{n+1/2}(\mathbb{I}_\ell)\big)\big\langle B_0\big(\tau_1(\mathbb{I}_\ell)v^0(m_\ell,\mathbb{I}_\ell) + u^0(m_\ell,\mathbb{I}_\ell) - u_0\big), v^n(m_\ell,\mathbb{I}_\ell)\big\rangle \\
&\qquad + \big\langle B_0\big(\tau_1(\mathbb{I}_\ell)v^0(m_\ell,\mathbb{I}_\ell) + u^0(m_\ell,\mathbb{I}_\ell) - u_0\big), u^{N(\mathbb{I}_\ell)-1}(m_\ell,\mathbb{I}_\ell) - \tau_1(\mathbb{I}_\ell)v^0(m_\ell,\mathbb{I}_\ell) - u^0(m_\ell,\mathbb{I}_\ell)\big\rangle \\
&\qquad - \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \frac{\tau_{n+1/2}(\mathbb{I}_\ell)^2}{2}\big\|v^n(m_\ell,\mathbb{I}_\ell)\big\|_B^2.
\end{aligned}$$

With Assumption (B_0), Relation $(V_B \leftarrow V_A)$, and Hölder's inequality (see also (2.17)), we find for the first term on the right-hand side of the foregoing identity

$$\begin{aligned}
&\Bigg| \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \sum_{j=1}^{n-1} \tau_{n+1/2}(\mathbb{I}_\ell)\big(\tau_{j+1}(\mathbb{I}_\ell) - \tau_{j+1/2}(\mathbb{I}_\ell)\big)\big\langle B_0 v^j(m_\ell, \mathbb{I}_\ell), v^n(m_\ell, \mathbb{I}_\ell)\big\rangle \Bigg| \\
&\quad \leqslant c \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \sum_{j=1}^{n-1} \tau_{n+1/2}(\mathbb{I}_\ell)\big|\tau_{j+1}(\mathbb{I}_\ell) - \tau_j(\mathbb{I}_\ell)\big| \big\| v^j(m_\ell, \mathbb{I}_\ell)\big\|_B \big\| v^n(m_\ell, \mathbb{I}_\ell)\big\|_B \\
&\quad \leqslant c c_B c_{V_B \leftarrow V_A}(m_\ell)^2 \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \tau_{n+1/2}(\mathbb{I}_\ell)\big\| v^n(m_\ell, \mathbb{I}_\ell)\big\|_{V_A} \sum_{j=1}^{N(\mathbb{I}_\ell)-1} \big|\tau_{j+1}(\mathbb{I}_\ell) - \tau_j(\mathbb{I}_\ell)\big| \big\| v^j(m_\ell, \mathbb{I}_\ell)\big\|_{V_A} \\
&\quad \leqslant c c_B c_{V_B \leftarrow V_A}(m_\ell)^2 \tau_{\max}(\mathbb{I}_\ell)\theta(\mathbb{I}_\ell)^{1/2} \Bigg(\sum_{n=1}^{N(\mathbb{I}_\ell)-1} \tau_{n+1/2}(\mathbb{I}_\ell)\big\| v^n(m_\ell, \mathbb{I}_\ell)\big\|_{V_A}^p \Bigg)^{2/p}.
\end{aligned}$$

The assumptions together with the a priori estimate (2.8) now show that this term vanishes as $\ell \to \infty$. For the second term, we similarly have

$$\begin{aligned}
&\Bigg| \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \big(\tau_{n+1}(\mathbb{I}_\ell) - \tau_{n+1/2}(\mathbb{I}_\ell)\big)\big\langle B_0\big(\tau_1(\mathbb{I}_\ell) v^0(m_\ell, \mathbb{I}_\ell) + u^0(m_\ell, \mathbb{I}_\ell) - u_0\big), v^n(m_\ell, \mathbb{I}_\ell)\big\rangle \Bigg| \\
&\quad \leqslant \big(\tau_1(\mathbb{I}_\ell)\| v^0(m_\ell, \mathbb{I}_\ell)\|_B + \big\| u^0(m_\ell, \mathbb{I}_\ell) - u_0\big\|_B\big) \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \big|\tau_{n+1}(\mathbb{I}_\ell) - \tau_n(\mathbb{I}_\ell)\big| \big\| v^n(m_\ell, \mathbb{I}_\ell)\big\|_B \\
&\quad \leqslant c c_B^{1/2} c_{V_B \leftarrow V_A}(m_\ell)\tau_{\max}(\mathbb{I}_\ell)\theta(\mathbb{I}_\ell)^{1/2} \\
&\qquad \times \big(c_B^{1/2} c_{V_B \leftarrow V_A}(m_\ell)\tau_1(\mathbb{I}_\ell)\big\| v^0(m_\ell, \mathbb{I}_\ell)\big\|_{V_A} + \big\| u^0(m_\ell, \mathbb{I}_\ell) - u_0\big\|_B\big) \\
&\qquad \times \Bigg(\sum_{n=1}^{N(\mathbb{I}_\ell)-1} \tau_{n+1/2}(\mathbb{I}_\ell)\big\| v^n(m_\ell, \mathbb{I}_\ell)\big\|_{V_A}^p \Bigg)^{1/p}.
\end{aligned}$$

Also this term converges towards zero as $\ell \to \infty$. For the next term, we observe, again with similar arguments as before, that

$$\begin{aligned}
&\big|\big\langle B_0\big(\tau_1(\mathbb{I}_\ell) v^0(m_\ell, \mathbb{I}_\ell) + u^0(m_\ell, \mathbb{I}_\ell) - u_0\big), u^{N(\mathbb{I}_\ell)-1}(m_\ell, \mathbb{I}_\ell) - \tau_1(\mathbb{I}_\ell) v^0(m_\ell, \mathbb{I}_\ell) - u^0(m_\ell, \mathbb{I}_\ell)\big\rangle\big| \\
&\quad \leqslant \big(c_B^{1/2} c_{V_B \leftarrow V_A}(m_\ell)\tau_1(\mathbb{I}_\ell)\big\| v^0(m_\ell, \mathbb{I}_\ell)\big\|_{V_A} + \big\| u^0(m_\ell, \mathbb{I}_\ell) - u_0\big\|_B\big) \\
&\qquad \times \big(\big\| u^{N(\mathbb{I}_\ell)-1}(m_\ell, \mathbb{I}_\ell)\big\|_B + c_B^{1/2} c_{V_B \leftarrow V_A}(m_\ell)\tau_1(\mathbb{I}_\ell)\big\| v^0(m_\ell, \mathbb{I}_\ell)\big\|_{V_A} + \big\| u^0(m_\ell, \mathbb{I}_\ell)\big\|_B\big),
\end{aligned}$$

and again this term vanishes as $\ell \to \infty$. Finally, we have that

$$\begin{aligned}
&\sum_{n=1}^{N(\mathbb{I}_\ell)-1} \frac{\tau_{n+1/2}(\mathbb{I}_\ell)^2}{2} \big\| v^n(m_\ell, \mathbb{I}_\ell)\big\|_B^2 \\
&\quad \leqslant c c_B c_{V_B \leftarrow V_A}(m_\ell)^2 \tau_{\max}(\mathbb{I}_\ell) \Bigg(\sum_{n=1}^{N(\mathbb{I}_\ell)-1} \tau_{n+1/2}(\mathbb{I}_\ell)\big\| v^n(m_\ell, \mathbb{I}_\ell)\big\|_{V_A}^p \Bigg)^{1/p}
\end{aligned}$$

vanishes as $\ell \to \infty$. Hence, we obtain

$$\int_0^T \bigl\langle B_0(u_{m_\ell,\mathbb{I}_\ell} - u_0 - Kv_{m_\ell,\mathbb{I}_\ell})(t), v_{m_\ell,\mathbb{I}_\ell}(t)\bigr\rangle\,dt \to 0 \quad \text{as } \ell \to \infty. \tag{2.38}$$

For the term including $A_{0,\mathbb{I}_\ell}$, we observe that by Assumption (A_0)

$$A_{0,\mathbb{I}_\ell}(t)w(t) \to A_0(t)w(t) \quad \text{in } V_A^*, \text{ a.e. in } (0,T) \ni t \text{ as } \ell \to \infty.$$

Moreover, from the growth condition, we find for almost all $t \in (0,T)$,

$$\bigl\| A_{0,\mathbb{I}_\ell}(t)w(t) - A_0(t)w(t)\bigr\|_{V_A^*}^{p^*} \leqslant c\bigl(1 + \|w(t)\|_{V_A}^p\bigr).$$

Lebesgue's theorem thus proves

$$A_{0,\mathbb{I}_\ell} w \to A_0 w \quad \text{in } L^{p^*}\bigl(0,T;V_A^*\bigr) \text{ as } \ell \to 0. \tag{2.39}$$

After all, inequality (2.34) now provides in the limit (replacing ℓ by ℓ' and taking $\ell' \to \infty$) because of Lemma 5, Assumption (IC), (2.24) with (2.30) and the weak lower semi-continuity of the norm $|\cdot|$, (2.35) with (2.37) and the weak lower semi-continuity of the norm $\|\cdot\|_B$, (2.38), (2.27), (2.25), (2.39), and (2.29) the inequality

$$\begin{aligned}
0 &\geqslant \frac{1}{2}\bigl|v(T)\bigr|^2 - \frac{1}{2}|v_0|^2 + \frac{1}{2}\bigl\|u_0 + Kv(T)\bigr\|_B^2 - \frac{1}{2}\|u_0\|_B^2 \\
&\quad - \int_0^T \bigl\langle f(t), v(t)\bigr\rangle\,dt + \int_0^T \bigl\langle a(t), w(t)\bigr\rangle\,dt + \int_0^T \bigl\langle (A_0 w)(t), v(t) - w(t)\bigr\rangle\,dt \\
&= \frac{1}{2}\bigl|v(T)\bigr|^2 - \frac{1}{2}|v_0|^2 + \frac{1}{2}\bigl\|u_0 + Kv(T)\bigr\|_B^2 - \frac{1}{2}\|u_0\|_B^2 \\
&\quad - \int_0^T \bigl\langle v'(t) + B_0 u(t), v(t)\bigr\rangle\,dt + \bigl\langle (A_0 w)(t) - a(t), v(t) - w(t)\bigr\rangle\,dt.
\end{aligned} \tag{2.40}$$

With $u = u_0 + Kv$ and applying Lemma 6 with $w = v$ (for the moment being, we suppose that the second assertion of Lemma 6 can be applied here with $\beta = T$) we thus find

$$0 \geqslant \int_0^T \bigl\langle (A_0 w)(t) - a(t), v(t) - w(t)\bigr\rangle\,dt.$$

With $w = v \pm sz$ ($z \in L^p(0,T;V_A)$) and $s \to 0+$, the hemicontinuity of A_0 immediately proves $a = A_0 v$ in $(L^p(0,T;V_A))^*$.

There remains, however, a problem with the application of Lemma 6 as the assertion only holds for almost all $\beta \in (0,T)$. Nevertheless, we can take a sequence $\{\beta_k\}_{k\in\mathbb{N}} \subset (0,T)$ such that the second assertion of Lemma 6 is fulfilled and such that $\beta_k \to T$ as $k \to \infty$. Without loss of generality, we can assume that $\beta_k > t_{1/2}(\mathbb{I}_\ell)$ for all k and all ℓ. There is then a number $N_k(\mathbb{I}_\ell) \in \{1, 2, \ldots, N(\mathbb{I}_\ell)\}$ such that $\beta_k \in (t_{N_k(\mathbb{I}_\ell)-3/2}, t_{N_k(\mathbb{I}_\ell)-1/2}]$ if $N_k(\mathbb{I}_\ell) \neq N(\mathbb{I}_\ell)$ or $\beta_k \in (t_{N(\mathbb{I}_\ell)-3/2}, T]$ if $N_k(\mathbb{I}_\ell) = N(\mathbb{I}_\ell)$.

It is easy to see that (2.31), (2.32), (2.33), (2.34), and (2.38) remain true if we replace T by $t_{N_k(\mathbb{I}_\ell)-1/2}$ and $N(\mathbb{I}_\ell)$ by $N_k(\mathbb{I}_\ell)$. The assumptions and (2.8) imply the boundedness of $\{v^{N_k(\mathbb{I}_\ell)-1}(m_\ell, \mathbb{I}_\ell)\}$ in H such that, for a suitably chosen subsequence denoted by ℓ', $v^{N_k(\mathbb{I}_{\ell'})-1}(m_\ell, \mathbb{I}_\ell)$ converges weakly in H towards an element ξ_k as $\ell' \to \infty$. Following the same arguments as above (replacing again T by $t_{N_k(\mathbb{I}_\ell)-1/2}$ and $N(\mathbb{I}_\ell)$ by $N_k(\mathbb{I}_\ell)$ in (2.23), taking the limit, which then yields (2.28) with β_k instead of T since all integrals over $(t_{N_k(\mathbb{I}_\ell)-1/2}, \beta_k)$ vanish), one may show that $\xi_k = v(\beta_k)$. Following the same lines of argumentation as above (taking, in particular, β_k instead of T in (2.36)), we may also show that, for a suitably chosen subsequence denoted by ℓ', $u_0 + Kv_{m_{\ell'},\mathbb{I}_{\ell'}}(t_{N_k(\mathbb{I}_{\ell'})-1/2})$ converges weakly in V_B towards $u_0 + Kv(\beta_k)$ as $\ell' \to \infty$. We then come up with (2.40) again but with β_k instead of T. This shows $a = A_0 v$ on $(0, \beta_k)$ and thus on $(0, T)$.

After all, we have that u and v with $u = u_0 + Kv$ fulfill the initial conditions $u(0) = u_0$ and $v(0) = v_0$ as well as the equation

$$v' + A_0 v + B_0 u = f \text{ in } \big(L^p(0, T; V)\big)^*,$$

which shows that u is a solution to the original problem (1.1).

By contradiction, we can show that not only a subsequence but the whole sequence converges towards u and v, respectively, since a solution to (1.1) is unique. □

Let us note that the initial conditions $u(0) = u_0 \in V_B$ and $v(0) = v_0 \in H$ make sense since

$$u = u_0 + Kv \in \mathscr{C}_w\big([0, T]; V_B\big) \text{ and } v \in \mathscr{C}_w\big([0, T]; H\big).$$

3. Equations including non-monotone perturbations

In this section, we consider again (1.1) in the case $p \geqslant 2$ but allow perturbations of the monotone main parts A_0 and B_0. Such perturbations arise from semi-linear terms in the underlying partial differential equation.

The analysis in the case of the appearance of perturbations of the monotone and coercive main parts A_0 and B_0 relies upon the characterization of compact subsets of Bochner–Lebesgue spaces. Instead of the classical Lions–Aubin theorem, compact subsets of Bochner–Lebesgue spaces are characterized as subsets of Sobolev–Slobodetskii spaces for abstract functions (see [1,23,24]). The use of fractional Sobolev spaces and thus showing the boundedness of fractional time derivatives (with respect to a stronger norm in space) instead of the classical Lions–Aubin theorem and showing the boundedness of first time derivatives (with respect to a weaker norm in space) seems to be very suited in the situation of a full discretization.

Nevertheless, at a certain point, we need to impose additional assumptions. In particular, we require that H is an intermediate space of class $\underline{\mathscr{K}}_\eta(V^*, V_A)$ for some $\eta \in (0, 1)$ in the sense of Lions and Peetre (see [18,20,25]).

We next give a definition of Sobolev–Slobodetskii spaces with Lebesgue exponent 2. For $\sigma \in (0, 1)$, let

$$H^\sigma(0, T; H) := \big\{w \in L^2(0, T; H)\colon |w|_{H^\sigma(0,T;H)} < \infty\big\},$$

$$\text{with } \|w\|_{H^\sigma(0,T;H)} := \big(\|w\|^2_{L^2(0,T;H)} + |w|^2_{H^\sigma(0,T;H)}\big)^{1/2},$$

$$|w|^2_{H^\sigma(0,T;H)} = \int_0^T \int_0^T \frac{|w(t) - w(s)|^2}{|t - s|^{1+2\sigma}}\, ds\, dt.$$

It is known (see [23, Cor. 2 on p. 82]) that there holds the compact embedding

$$L^p(0,T;V_A) \cap H^\sigma(0,T;H) \overset{c}{\hookrightarrow} L^r(0,T;H) \quad \text{for any } r \in \big[1, 2/(1-2\sigma)\big]$$
$$\text{if } V_A \overset{c}{\hookrightarrow} H,\ \sigma \in (0,1/2). \tag{3.1}$$

In what follows, we often require that the maximum time step size is sufficiently small. We shall not quantify this smallness, although it would easily be possible.

3.1. Assumptions on the continuous problem

In addition to the assumptions already settled in Section 2, we rely here upon the following structural assumptions.

Assumption (A₁). *$\{A_1(t)\}_{t\in[0,T]}$ is a family of operators $A_1(t): V_A \to V_A^*$ such that for all $v \in V_A$ the mapping $t \mapsto A_1(t)v : [0,T] \to V_A^*$ is continuous for almost all $t \in [0,T]$. There are constants $\varepsilon \in [0,1/4)$, $\varkappa \geqslant 0$, $\lambda_1 \geqslant 0$, $c > 0$ such that for all $t \in [0,T]$ and all $v \in V_A$,*

$$\langle A_1(t)v, v\rangle \geqslant -\varepsilon\mu_A\|v\|_{V_A}^p - \varkappa|v|^2 - \lambda_1, \qquad \big\|A_1(t)v\big\|_{V_A^*} \leqslant c\big(1 + \|v\|_{V_A}^{p-1}\big).$$

Moreover, there is a constant $\delta_A \in (0, p-1]$ such that for any $R > 0$ there is a constant $\alpha_A = \alpha_A(R) > 0$ and for all $t \in [0,T]$ and all $v, w \in V_A$ with $|v|, |w| \leqslant R$ there holds

$$\big\|A_1(t)v - A_1(t)w\big\|_{V_A^*} \leqslant \alpha_A(R)\big(1 + \|v\|_{V_A}^{p-1-\delta_A} + \|w\|_{V_A}^{p-1-\delta_A}\big)|v - w|^{\delta_A/p}.$$

With $\{A_1(t)\}_{t\in[0,T]}$, we associate the Nemytskii operator A_1 defined by $(A_1v)(t) := A_1(t)v(t)$ for a function $v : [0,T] \to V_A$. It is easy to show that, under the above assumption, A_1 maps $L^p(0,T;V_A)$ into $(L^p(0,T;V_A))^*$ and is bounded on bounded subsets. Since $V_A \hookrightarrow H$, $A_1(t): V_A \to V_A^*$ $(t \in [0,T])$ is continuous. If V_A is compactly embedded in H then $A_1(t): V_A \to V_A^*$ is strongly continuous, i.e., maps weakly convergent sequences into strongly convergent sequences (see [29, Def. 26.1 on p. 555]).

Sometimes, not $A_0(t): V_A \to V_A^*$ is monotone and coercive but only an additive shift $A_0(t) + \varkappa I : V_A \to V_A^*$. The above Assumption (A_1) allows to consider this case by taking $A_0(t) + \varkappa I$ instead of $A_0(t)$ and setting $A_1(t) = -\varkappa I$.

We shall remark that the assumption on the lower semi-boundedness of $A_1(t)$ $(t \in [0,T])$ follows, employing in particular Young's inequality, from the following (compared to Assumption (A_1)) more restrictive growth condition: There exist constants $\bar\delta_A \in (0, p-1]$, $c \geqslant 0$ such that for all $t \in [0,T]$ and all $v \in V_A$,

$$\big\|A_1(t)v\big\|_{V_A^*} \leqslant c\big(1 + \|v\|_{V_A}^{p-1-\bar\delta_A}|v|^{2\bar\delta_A/p}\big).$$

Assumption (B₁). *$\{B_1(t)\}_{t\in[0,T]}$ is a family of operators $B_1(t): V_B \to V_A^*$ such that for all $v \in V_B$ the mapping $t \mapsto B_1(t)v : [0,T] \to V_A^*$ is continuous for almost all $t \in [0,T]$. There is a constant $c > 0$ such that for all $t \in [0,T]$ and all $v \in V_B$,*

$$\big\|B_1(t)v\big\|_{V_A^*} \leqslant c\big(1 + \|v\|_{V_B}^{2(p-1)/p}\big).$$

Moreover, for any $R > 0$ there is a constant $\alpha_B = \alpha_B(R) > 0$ and for all $t \in [0,T]$ and all $v, w \in V_B$ with $\|v\|_{V_B}, \|w\|_{V_B} \leqslant R$ there holds

$$\big\|B_1(t)v - B_1(t)w\big\|_{V_A^*} \leqslant \alpha_B(R)|v - w|^{1-1/p}.$$

With $\{B_1(t)\}_{t\in[0,T]}$, we associate the Nemytskii operator B_1 defined by $(B_1 v)(t) := B_1(t)v(t)$ for a function $v : [0,T] \to V_B$. Under the above assumption, one may show that B_1 maps $L^2(0,T;V_B)$ into $(L^p(0,T;V_A))^*$ and is bounded on bounded subsets. Note that later we do not need the compact embedding of V_B into H and so we do not have that $B_1(t) : V_B \to V_A^*$ $(t \in [0,T])$ is strongly continuous. However, $B_1(t) : V_B \to V_A^*$ $(t \in [0,T])$ is continuous since $V_B \hookrightarrow H$.

We emphasize that the above Hölder-type conditions are needed only with arbitrarily small Hölder exponents and only on bounded subsets.

3.2. Fully discrete problem and a priori estimates

In the following, we state a result on the existence of a discrete solution and deduce an a priori estimate.

Theorem 7 *(Existence of a discrete solution). In addition to the assumptions of Theorem* 2 *let Assumptions* (A_1) *and* (B_1) *be fulfilled and let* $\tau_{\max}$ *be sufficiently small. Then there exists a solution* $\{u^n\}_{n=1}^N \subseteq V_m$ *to* (1.3) *with* $\{v^n\}_{n=1}^{N-1} \subseteq V_m$ $(v^n = (u^{n+1} - u^n)/\tau_{n+1})$ *being the solution to* (1.7).

Proof. The proof follows the same lines as that of Theorem 2. The mapping $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ remains continuous since $A_1(t) : V_A \to V_A^*$ and $B_1(t) : V_B \to V_A^*$ $(t \in [0,T])$ are continuous.

For the additional terms appearing, the assumptions yield

$$\big\langle A_1(t_n)v^n, v^n\big\rangle \geqslant -\varepsilon\mu_A \|v^n\|_{V_A}^p - \varkappa|v^n|^2 - \lambda_1 \tag{3.2}$$

as well as

$$\big\langle B_1(t_n)\big(u_0 + K_{\mathbb{I}}v^n\big), v^n\big\rangle \geqslant -\big\|B_1(t_n)\big(u_0 + K_{\mathbb{I}}v^n\big)\big\|_{V_A^*}\|v^n\|_{V_A} \geqslant -\big\|B_1(t_n)\big(u_0 + K_{\mathbb{I}}v^n\big)\big\|_{V_A^*}\|v^n\|.$$

Instead of (2.6), we therefore have

$$\begin{aligned}\Phi\big(\boldsymbol{v}^n\big)\cdot\boldsymbol{v}^n \geqslant \bigg(&(1-\varepsilon)\mu_A\big(1 + c_{V_B\leftarrow V_A}(m)\big)^{-p}\|v^n\|^{p-1} - \frac{1}{\tau_{n+1/2}}\|v^{n-1}\|_* - \big\|B_0\big(u^0 + K_{\mathbb{I}}v^n\big)\big\|_* \\ &- \big\|B_1(t_n)\big(u_0 + K_{\mathbb{I}}v^n\big)\big\|_{V_A^*} - \|f^n\|_*\bigg)\|v^n\| + \bigg(\frac{1}{\tau_{n+1/2}} - \varkappa\bigg)|v^n|^2 - \lambda - \lambda_1,\end{aligned}$$

and, by taking $\|v^n\| = R$ sufficiently large, Lemma 1 provides the existence of a zero of Φ and thus, step by step, of a solution to the discrete problem (1.7). □

In general, uniqueness cannot be expected and would require more restrictive assumptions on the perturbations (such that, e.g., strict monotonicity of the total operator is obtained).

Theorem 8 *(A priori estimates). In addition to the assumptions of Theorem* 3 *let Assumptions* (A_1) *and* (B_1) *be fulfilled and let* $\tau_{\max}$ *be sufficiently small. The assertion of Theorem* 3 *then remains true also for the perturbed problem.*

Proof. We follow the lines of the proof of Theorem 3. For the additional terms, we find (3.2) and, with Young's inequality,

$$\big\langle B_1(t_n)u^n, v^n\big\rangle \geqslant -\big\|B_1(t_n)u^n\big\|_{V_A^*}\|v^n\|_{V_A} \geqslant -c\big(1 + \|u^n\|_{V_B}^2\big) - \varepsilon\mu_A\|v^n\|_{V_A}^p.$$

After multiplication by $2\tau_{n+1/2}$ and summing up, the terms with $\|v^n\|_{V_A}^p$ can be absorbed within the left-hand side of (2.11). The terms with $|v^n|^2$ and $\|u^n\|_{V_B}^2$ require an application of a discrete Gronwall lemma. □

3.3. Convergence towards a weak solution

We next establish a convergence result analogous to Lemma 5.

Lemma 9. *In addition to the assumptions of Theorem 4 let Assumptions (A_1) and (B_1) be fulfilled and let $\tau_{\max}(\mathbb{I}_\ell)$ be sufficiently small. The assertion of Lemma 5 then remains true also for the perturbed problem.*

Proof. The assertion is an immediate consequence of the a priori estimate (Lemma 8) and follows the same lines as the proof of Lemma 5. □

The essential new ingredient in the perturbed situation we consider here is an a priori estimate in terms of a fractional Sobolev space implying then a result on the strong convergence. We first provide a result in the case that V_A is continuously and densly embedded in V_B.

Lemma 10. *In addition to the assumptions of Theorem 4 let the growth conditions in Assumptions (A_1) and (B_1) be fulfilled, let $\tau_{\max}(\mathbb{I}_\ell)$ be sufficiently small, assume that V_A is compactly embedded in H and that V_A is continuously and dense embedded in V_B. The sequence $\{v_{m_\ell,\mathbb{I}_\ell}\}$ is then bounded in $H^\sigma(0,T;H)$ for any $\sigma < 1/(2p)$. Moreover, there is a subsequence (of the subsequence of Lemma 9), denoted by ℓ', such that*

$$u_{m_{\ell'},\mathbb{I}_{\ell'}} \to u, \qquad v_{m_{\ell'},\mathbb{I}_{\ell'}} \to v, \qquad \hat{v}_{m_{\ell'},\mathbb{I}_{\ell'}} \to v \quad \text{in } L^r(0,T;H) \text{ for any } r \in [1,\infty) \text{ as } \ell' \to \infty.$$

Proof. We commence with the boundedness of $\{v_{m_\ell,\mathbb{I}_\ell}\}$ in $H^\sigma(0,T;H)$. We first recall that $\{v_{m_\ell,\mathbb{I}_\ell}\}$ is already bounded in $L^\infty(0,T;H)$ (see Lemma 9).

By definition, we have

$$\begin{aligned}
|v_{m_\ell,\mathbb{I}_\ell}|^2_{H^\sigma(0,T;H)} = {} & \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \int_0^{t_{1/2}(\mathbb{I}_\ell)} \int_{t_{n-1/2}(\mathbb{I}_\ell)}^{t_{n+1/2}(\mathbb{I}_\ell)} \frac{|v^n(m_\ell,\mathbb{I}_\ell)|^2}{|t-s|^{1+2\sigma}}\, ds\, dt \\
& + \sum_{j=1}^{N(\mathbb{I}_\ell)-1} \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \int_{t_{j-1/2}(\mathbb{I}_\ell)}^{t_{j+1/2}(\mathbb{I}_\ell)} \int_{t_{n-1/2}(\mathbb{I}_\ell)}^{t_{n+1/2}(\mathbb{I}_\ell)} \frac{|v^j(m_\ell,\mathbb{I}_\ell) - v^n(m_\ell,\mathbb{I}_\ell)|^2}{|t-s|^{1+2\sigma}}\, ds\, dt \\
& + \sum_{n=1}^{N(\mathbb{I}_\ell)-1} \int_{t_{N(\mathbb{I}_\ell)-1/2}}^{T} \int_{t_{n-1/2}(\mathbb{I}_\ell)}^{t_{n+1/2}(\mathbb{I}_\ell)} \frac{|v^n(m_\ell,\mathbb{I}_\ell)|^2}{|t-s|^{1+2\sigma}}\, ds\, dt \\
=: {} & S_1 + S_2 + S_3.
\end{aligned}$$

For the first term S_1, we immediately find (by calculating the appearing integrals exactly)

$$S_1 \leqslant \int_0^{t_{1/2}(\mathbb{I}_\ell)} \int_{t_{1/2}(\mathbb{I}_\ell)}^{t_{N(\mathbb{I}_\ell)-1/2}(\mathbb{I}_\ell)} \frac{1}{|t-s|^{1+2\sigma}}\, ds\, dt\, \|v_{m_\ell,\mathbb{I}_\ell}\|^2_{L^\infty(0,T;H)} \leqslant \frac{\tau_{\max}(\mathbb{I}_\ell)^{1-2\sigma}}{2\sigma(1-2\sigma)} \|v_{m_\ell,\mathbb{I}_\ell}\|^2_{L^\infty(0,T;H)}.$$

Analogously, we have

$$S_3 \leqslant \int_{t_{N(\mathbb{I}_\ell)-1/2}(\mathbb{I}_\ell)}^{T} \int_{t_{1/2}(\mathbb{I}_\ell)}^{t_{N(\mathbb{I}_\ell)-1/2}(\mathbb{I}_\ell)} \frac{1}{|t-s|^{1+2\sigma}}\, ds\, dt \, \|v_{m_\ell,\mathbb{I}_\ell}\|^2_{L^\infty(0,T;H)}$$

$$\leqslant \frac{\tau_{\max}(\mathbb{I}_\ell)^{1-2\sigma}}{2\sigma(1-2\sigma)} \|v_{m_\ell,\mathbb{I}_\ell}\|^2_{L^\infty(0,T;H)}.$$

For S_2, there holds

$$\begin{aligned} S_2 &= 2 \sum_{j=2}^{N(\mathbb{I}_\ell)-1} \int_{t_{j-1/2}(\mathbb{I}_\ell)}^{t_{j+1/2}(\mathbb{I}_\ell)} \int_{t_{j-3/2}(\mathbb{I}_\ell)}^{t_{j-1/2}(\mathbb{I}_\ell)} \frac{|v^j(m_\ell,\mathbb{I}_\ell) - v^{j-1}(m_\ell,\mathbb{I}_\ell)|^2}{|t-s|^{1+2\sigma}}\, ds\, dt \\ &\quad + 2 \sum_{j=3}^{N(\mathbb{I}_\ell)-1} \sum_{n=1}^{j-2} \int_{t_{j-1/2}(\mathbb{I}_\ell)}^{t_{j+1/2}(\mathbb{I}_\ell)} \int_{t_{n-1/2}(\mathbb{I}_\ell)}^{t_{n+1/2}(\mathbb{I}_\ell)} \frac{|v^j(m_\ell,\mathbb{I}_\ell) - v^{n}(m_\ell,\mathbb{I}_\ell)|^2}{|t-s|^{1+2\sigma}}\, ds\, dt \\ &=: S_{21} + S_{22}. \end{aligned} \tag{3.3}$$

Since

$$\begin{aligned} &\int_{t_{j-1/2}(\mathbb{I}_\ell)}^{t_{j+1/2}(\mathbb{I}_\ell)} \int_{t_{j-3/2}(\mathbb{I}_\ell)}^{t_{j-1/2}(\mathbb{I}_\ell)} \frac{1}{|t-s|^{1+2\sigma}}\, ds\, dt \\ &= \frac{1}{2\sigma(1-2\sigma)} \Big(\big(t_{j+1/2}(\mathbb{I}_\ell) - t_{j-1/2}(\mathbb{I}_\ell)\big)^{1-2\sigma} - \big(t_{j+1/2}(\mathbb{I}_\ell) - t_{j-3/2}(\mathbb{I}_\ell)\big)^{1-2\sigma} \\ &\quad + \big(t_{j-1/2}(\mathbb{I}_\ell) - t_{j-3/2}(\mathbb{I}_\ell)\big)^{1-2\sigma} \Big) \\ &\leqslant c\tau_{\max}(\mathbb{I}_\ell)^{1-2\sigma}, \end{aligned}$$

we find

$$S_{21} \leqslant c\tau_{\max}(\mathbb{I}_\ell)^{1-2\sigma} \sum_{j=2}^{N(\mathbb{I}_\ell)-1} \big| v^j(m_\ell,\mathbb{I}_\ell) - v^{j-1}(m_\ell,\mathbb{I}_\ell)\big|^2.$$

Because of the a priori estimate (2.8) (see also Lemma 9) the right-hand side is bounded (and indeed converges towards zero).

It remains to analyze S_{22}. We first observe that

$$\int_{t_{j-1/2}(\mathbb{I}_\ell)}^{t_{j+1/2}(\mathbb{I}_\ell)} \int_{t_{n-1/2}(\mathbb{I}_\ell)}^{t_{n+1/2}(\mathbb{I}_\ell)} \frac{1}{|t-s|^{1+2\sigma}}\, ds\, dt \leqslant \tau_{j+1/2}(\mathbb{I}_\ell)\tau_{n+1/2}(\mathbb{I}_\ell)\big(t_{j-1/2}(\mathbb{I}_\ell) - t_{n+1/2}(\mathbb{I}_\ell)\big)^{-1-2\sigma}. \tag{3.4}$$

With (1.5), we obtain for all $\varphi \in V_{m_\ell}$,

$$\big(v^j(m_\ell,\mathbb{I}_\ell) - v^n(m_\ell,\mathbb{I}_\ell),\varphi\big) = \sum_{k=n+1}^{j} \tau_{k+1/2}(\mathbb{I}_\ell)\big\langle g^k(m_\ell,\mathbb{I}_\ell),\varphi\big\rangle$$
$$\text{with } g^k(m_\ell,\mathbb{I}_\ell) := f^k(\mathbb{I}_\ell) - A\big(t_k(\mathbb{I}_\ell)\big)v^k(m_\ell,\mathbb{I}_\ell) - B\big(t_k(\mathbb{I}_\ell)\big)u^k(m_\ell,\mathbb{I}_\ell) \tag{3.5}$$

and thus with $\varphi = v^j(m_\ell,\mathbb{I}_\ell) - v^n(m_\ell,\mathbb{I}_\ell) \in V_{m_\ell}$,

$$\big|v^j(m_\ell,\mathbb{I}_\ell) - v^n(m_\ell,\mathbb{I}_\ell)\big|^2$$
$$\leqslant \sum_{k=n+1}^{j} \tau_{k+1/2}(\mathbb{I}_\ell)\big\|g^k(m_\ell,\mathbb{I}_\ell)\big\|_{V_A^*}\big(\big\|v^j(m_\ell,\mathbb{I}_\ell)\big\|_{V_A} + \big\|v^n(m_\ell,\mathbb{I}_\ell)\big\|_{V_A}\big). \tag{3.6}$$

Hölder's inequality now gives

$$\sum_{k=n+1}^{j} \tau_{k+1/2}(\mathbb{I}_\ell)\big\|g^k(m_\ell,\mathbb{I}_\ell)\big\|_{V_A^*}$$
$$\leqslant \big(t_{j+1/2}(\mathbb{I}_\ell) - t_{n+1/2}(\mathbb{I}_\ell)\big)^{1/p}\left(\sum_{k=n+1}^{j} \tau_{k+1/2}(\mathbb{I}_\ell)\big\|g^k(m_\ell,\mathbb{I}_\ell)\big\|_{V_A^*}^{p^*}\right)^{1/p^*}. \tag{3.7}$$

Recalling (2.14), the growth conditions for $A_0(t)$, $A_1(t)$, B_0, $B_1(t)$ ($t \in [0,T]$) (see Assumptions (A_0), (A_1), (B_0), (B_1)) and the boundedness of $\{v_{m_\ell,\mathbb{I}_\ell}\}$ in $L^p(0,T;V_A)$ and of $\{u_{m_\ell,\mathbb{I}_\ell}\}$ in $L^\infty(0,T;V_B)$ (see Lemma 9), we see, by using Minkowski's inequality and $V_B^* \hookrightarrow V_A^*$, that

$$\left(\sum_{k=n+1}^{j} \tau_{k+1/2}(\mathbb{I}_\ell)\big\|g^k(m_\ell,\mathbb{I}_\ell)\big\|_{V_A^*}^{p^*}\right)^{1/p^*}$$
$$\leqslant \left(\sum_{k=1}^{N(\mathbb{I}_\ell)-1} \tau_{k+1/2}(\mathbb{I}_\ell)\big\|g^k(m_\ell,\mathbb{I}_\ell)\big\|_{V_A^*}^{p^*}\right)^{1/p^*}$$
$$\leqslant \left(\sum_{k=1}^{N(\mathbb{I}_\ell)-1} \tau_{k+1/2}(\mathbb{I}_\ell)\big\|f^k(m_\ell,\mathbb{I}_\ell)\big\|_{V_A^*}^{p^*}\right)^{1/p^*} + \left(\sum_{k=1}^{N(\mathbb{I}_\ell)-1} \tau_{k+1/2}(\mathbb{I}_\ell)\big\|A\big(t_k(\mathbb{I}_\ell)\big)v^k(m_\ell,\mathbb{I}_\ell)\big\|_{V_A^*}^{p^*}\right)^{1/p^*}$$
$$+ \left(\sum_{k=1}^{N(\mathbb{I}_\ell)-1} \tau_{k+1/2}(\mathbb{I}_\ell)\big\|B_0u^k(m_\ell,\mathbb{I}_\ell)\big\|_{V_A^*}^{p^*}\right)^{1/p^*}$$
$$+ \left(\sum_{k=1}^{N(\mathbb{I}_\ell)-1} \tau_{k+1/2}(\mathbb{I}_\ell)\big\|B_1\big(t_k(\mathbb{I}_\ell)\big)u^k(m_\ell,\mathbb{I}_\ell)\big\|_{V_A^*}^{p^*}\right)^{1/p^*}$$
$$\leqslant \|f\|_{L^{p^*}(0,T;V_A^*)} + c\big(1 + \|v_{m_\ell,\mathbb{I}_\ell}\|_{L^p(0,T;V_A)}^{p-1}\big) + c\|u_{m_\ell,\mathbb{I}_\ell}\|_{L^{p^*}(0,T;V_B)}$$
$$+ c\big(1 + \|u_{m_\ell,\mathbb{I}_\ell}\|_{L^2(0,T;V_B)}^{2(p-1)/p}\big) \tag{3.8}$$

is bounded. The crucial point here is the boundedness of $B_0 : V_B \to V_B^* \hookrightarrow V_A^*$.

We, hence, deduce from (3.4), (3.6), (3.7), and (3.8) the estimate

$$\begin{aligned} S_{22} &\leqslant c \sum_{j=3}^{N(\mathbb{I}_\ell)-1} \sum_{n=1}^{j-2} a_{jn} \big(\big\| v^j(m_\ell, \mathbb{I}_\ell) \big\|_{V_A} + \big\| v^n(m_\ell, \mathbb{I}_\ell) \big\|_{V_A} \big) \\ &= c \sum_{j=3}^{N(\mathbb{I}_\ell)-1} \sum_{n=1}^{j-2} a_{jn} \big\| v^j(m_\ell, \mathbb{I}_\ell) \big\|_{V_A} + c \sum_{j=3}^{N(\mathbb{I}_\ell)-1} \sum_{n=1}^{j-2} a_{jn} \big\| v^n(m_\ell, \mathbb{I}_\ell) \big\|_{V_A} \\ &=: S_{221} + S_{222}, \end{aligned}$$

with

$$a_{jn} := \tau_{j+1/2}(\mathbb{I}_\ell) \tau_{n+1/2}(\mathbb{I}_\ell) \big(t_{j-1/2}(\mathbb{I}_\ell) - t_{n+1/2}(\mathbb{I}_\ell) \big)^{-1-2\sigma} \big(t_{j+1/2}(\mathbb{I}_\ell) - t_{n+1/2}(\mathbb{I}_\ell) \big)^{1/p}.$$

With Assumption $(V_m, \mathbb{I})$, we find

$$\tau_{n+1/2}(\mathbb{I}_\ell) = \frac{r_{n+1}(\mathbb{I}_\ell)^{-1} + 1}{1 + r_{n+2}(\mathbb{I}_\ell)} \tau_{n+3/2}(\mathbb{I}_\ell) \leqslant \big(1 + r_{\min}(\mathbb{I}_\ell)^{-1}\big) \tau_{n+3/2}(\mathbb{I}_\ell) \leqslant c \tau_{n+3/2}(\mathbb{I}_\ell) \tag{3.9}$$

as well as (recall here that $n = 1, \dots, j-2$)

$$\begin{aligned} t_{j+1/2}(\mathbb{I}_\ell) - t_{n+1/2}(\mathbb{I}_\ell) &= \left(1 + \frac{\tau_{j+1/2}(\mathbb{I}_\ell)}{t_{j-1/2}(\mathbb{I}_\ell) - t_{n+1/2}(\mathbb{I}_\ell)} \right) \big(t_{j-1/2}(\mathbb{I}_\ell) - t_{n+1/2}(\mathbb{I}_\ell) \big) \\ &\leqslant \left(1 + \frac{\tau_{j+1/2}(\mathbb{I}_\ell)}{\tau_{j-1/2}(\mathbb{I}_\ell)} \right) \big(t_{j-1/2}(\mathbb{I}_\ell) - t_{n+1/2}(\mathbb{I}_\ell) \big) \\ &\leqslant \big(2 + r_{\max}(\mathbb{I}_\ell) \big) \big(t_{j-1/2}(\mathbb{I}_\ell) - t_{n+1/2}(\mathbb{I}_\ell) \big) \\ &\leqslant c \big(t_{j-1/2}(\mathbb{I}_\ell) - t_{n+1/2}(\mathbb{I}_\ell) \big). \end{aligned} \tag{3.10}$$

Since $-1 - 2\sigma + 1/p < 0 < -2\sigma + 1/p$, we thus have

$$\begin{aligned} &\sum_{n=1}^{j-2} \tau_{n+1/2}(\mathbb{I}_\ell) \big(t_{j-1/2}(\mathbb{I}_\ell) - t_{n+1/2}(\mathbb{I}_\ell) \big)^{-1-2\sigma} \big(t_{j+1/2}(\mathbb{I}_\ell) - t_{n+1/2}(\mathbb{I}_\ell) \big)^{1/p} \\ &\quad \leqslant c \sum_{n=1}^{j-2} \tau_{n+3/2}(\mathbb{I}_\ell) \big(t_{j-1/2}(\mathbb{I}_\ell) - t_{n+1/2}(\mathbb{I}_\ell) \big)^{-1-2\sigma+1/p} \\ &\quad \leqslant c \sum_{n=1}^{j-2} \int_{t_{n+1/2}(\mathbb{I}_\ell)}^{t_{n+3/2}(\mathbb{I}_\ell)} \big(t_{j-1/2}(\mathbb{I}_\ell) - s \big)^{-1-2\sigma+1/p} \, ds \\ &\quad = c \int_{t_{3/2}(\mathbb{I}_\ell)}^{t_{j-1/2}(\mathbb{I}_\ell)} \big(t_{j-1/2}(\mathbb{I}_\ell) - s \big)^{-1-2\sigma+1/p} \, ds \\ &\quad = \frac{1}{-2\sigma + 1/p} \big(t_{j-1/2}(\mathbb{I}_\ell) - t_{3/2}(\mathbb{I}_\ell) \big)^{-2\sigma+1/p} \\ &\quad \leqslant c T^{-2\sigma+1/p}, \end{aligned} \tag{3.11}$$

and thus

$$S_{221} \leqslant c \sum_{j=3}^{N(\mathbb{I}_\ell)-1} \tau_{j+1/2}(\mathbb{I}_\ell) \left\| v^j(m_\ell, \mathbb{I}_\ell) \right\|_{V_A} \leqslant c \| v_{m_\ell,\mathbb{I}_\ell} \|_{L^1(0,T;V_A)}, \tag{3.12}$$

which shows that S_{221} is bounded. For S_{222}, we first change the order of summation and then argue analogously as before. This, finally, proves the a priori estimate asserted.

Since $\{v_{m_\ell,\mathbb{I}_\ell}\}$ is now bounded in $H^\sigma(0,T;H)$ for any $\sigma \in (0, 1/(2p))$ as well as in $L^p(0,T;V_A)$ (see Lemma 9) and since V_A is compactly embedded in H, we can extract a subsequence (of the subsequence already given by Lemma 9) such that $v_{m_{\ell'},\mathbb{I}_{\ell'}}$ converges strongly in $L^r(0,T;H)$ for any $r \in [1, 2/(1-1/p))$ as $\ell' \to \infty$ (see (3.1)), the limit can only be the weak-in-$L^p(0,T;V_A)$-limit v of Lemma 9. Since $\{v_{m_\ell,\mathbb{I}_\ell}\}$ is also bounded in $L^\infty(0,T;H)$, strong convergence follows for any $r \in [1,\infty)$.

Because of (2.15) showing that $\hat{v}_{m_\ell,\mathbb{I}_\ell} - v_{m_\ell,\mathbb{I}_\ell} \to 0$ in $L^2(0,T;H)$ as $\ell \to \infty$ and because of the boundedness of $\{\hat{v}_{m_\ell,\mathbb{I}_\ell}\}$ in $L^\infty(0,T;H)$, we also obtain the strong convergence of the piecewise linear prolongations.

Since $\{u_{m_\ell,\mathbb{I}_\ell}\}$ is bounded in $L^\infty(0,T;V_B) \hookrightarrow L^\infty(0,T;H)$, since $u_{m_\ell,\mathbb{I}_\ell} - u_0 - K v_{m_\ell,\mathbb{I}_\ell}$ converges strongly in $L^2(0,T;H)$ towards zero as $\ell \to \infty$ (see Lemma 9) and since

$$\| K v_{m_\ell,\mathbb{I}_\ell} - K v \|_{L^\infty(0,T;H)} \leqslant \| v_{m_\ell,\mathbb{I}_\ell} - v \|_{L^1(0,T;H)},$$

the strong convergence of $v_{m_{\ell'},\mathbb{I}_{\ell'}}$ towards v also implies the strong convergence of $u_{m_{\ell'},\mathbb{I}_{\ell'}}$ towards $u = u_0 + Kv$ in $L^r(0,T;H)$ for any $r \in [1,\infty)$ as $\ell' \to \infty$. $\square$

As one can infer from the proof above, the assumption $V_A \hookrightarrow V_B$ is only needed in order to estimate the term

$$\sum_{k=n+1}^{j} \tau_{k+1/2}(\mathbb{I}_\ell) \left\langle B_0 u^k(m_\ell, \mathbb{I}_\ell), v^j(m_\ell, \mathbb{I}_\ell) - v^n(m_\ell, \mathbb{I}_\ell) \right\rangle$$

in an appropriate way taking into account the boundedness of $\{v_{m_\ell,\mathbb{I}_\ell}\}$ in $L^p(0,T;V_A)$ and of $\{u_{m_\ell,\mathbb{I}_\ell}\}$ in $L^\infty(0,T;V_B)$.

The following result, however, allows to circumvent the assumption $V_A \hookrightarrow V_B$ if the Galerkin scheme satisfies an additional requirement. Lemma 11 below uses the concept of intermediate spaces of class $\underline{\mathscr{K}}_\eta$ in the sense of Lions and Peetre (following [18, pp. 27ff.] or, equivalently, of class J_η following [20, pp. 27ff.], see also [25, pp. 123ff.]). We recall that, by assumption, $V \subseteq V_A \subseteq H \subseteq V_A^* \subseteq V^*$ with dense and continuous embeddings.

Lemma 11. *In addition to the assumptions of Theorem* 4 *let Assumptions* (A_1) *and* (B_1) *be fulfilled, let* $\tau_{\max}(\mathbb{I}_\ell)$ *be sufficiently small, assume that* V_A *is compactly embedded in* H *and that* H *is an intermediate space of class* $\underline{\mathscr{K}}_\eta(V^*, V_A)$, *i.e., there is* $\eta \in (0,1)$ *and* $c > 0$ *such that for all* $v \in V_A$,

$$|v| \leqslant c \|v\|_{V_A}^{\eta} \|v\|_*^{1-\eta}. \tag{3.13}$$

Moreover, assume that the restriction on V *of the orthogonal projection* $P_{m_\ell} : H \to V_{m_\ell}$ *is bounded as an operator in* V *uniformly with respect to* m_ℓ. *The sequence* $\{v_{m_\ell,\mathbb{I}_\ell}\}$ *is then bounded in* $H^\sigma(0,T;H)$ *for any* $\sigma < (1-\eta)/p$. *Moreover, there is a subsequence (of the subsequence of Lemma* 9*), denoted by* ℓ', *such that*

$$u_{m_{\ell'},\mathbb{I}_{\ell'}} \to u, \qquad v_{m_{\ell'},\mathbb{I}_{\ell'}} \to v, \qquad \hat{v}_{m_{\ell'},\mathbb{I}_{\ell'}} \to v \quad \text{in } L^r(0,T;H) \text{ for any } r \in [1,\infty) \text{ as } \ell' \to \infty.$$

Proof. The first and last part of the proof follows exactly the same lines as that of Lemma 10. However, we estimate the term S_{22} (see (3.3)) in a different way.

Instead of (3.6), we find from (3.13)

$$\begin{aligned} &\left|v^j(m_\ell,\mathbb{I}_\ell)-v^n(m_\ell,\mathbb{I}_\ell)\right|^2 \\ &\quad\leqslant c\left\|v^j(m_\ell,\mathbb{I}_\ell)-v^n(m_\ell,\mathbb{I}_\ell)\right\|_{V_A}^{2\eta}\left\|v^j(m_\ell,\mathbb{I}_\ell)-v^n(m_\ell,\mathbb{I}_\ell)\right\|_*^{2(1-\eta)} \\ &\quad\leqslant c\left(\left\|v^j(m_\ell,\mathbb{I}_\ell)\right\|_{V_A}^{2\eta}+\left\|v^n(m_\ell,\mathbb{I}_\ell)\right\|_{V_A}^{2\eta}\right)\left\|v^j(m_\ell,\mathbb{I}_\ell)-v^n(m_\ell,\mathbb{I}_\ell)\right\|_*^{2(1-\eta)}. \end{aligned} \tag{3.14}$$

From the definition of P_{m_ℓ} and with (3.5), we obtain

$$\begin{aligned} \left\|v^j(m_\ell,\mathbb{I}_\ell)-v^n(m_\ell,\mathbb{I}_\ell)\right\|_* &= \sup_{w\in V,\|w\|=1}\left\langle v^j(m_\ell,\mathbb{I}_\ell)-v^n(m_\ell,\mathbb{I}_\ell),\, w\right\rangle \\ &= \sup_{w\in V,\|w\|=1}\left\langle v^j(m_\ell,\mathbb{I}_\ell)-v^n(m_\ell,\mathbb{I}_\ell),\, P_{m_\ell}w\right\rangle \\ &= \sup_{w\in V,\|w\|=1}\left\langle \sum_{k=n+1}^{j}\tau_{k+1/2}(\mathbb{I}_\ell)g^k(m_\ell,\mathbb{I}_\ell),\, P_{m_\ell}w\right\rangle \\ &\leqslant \sum_{k=n+1}^{j}\tau_{k+1/2}(\mathbb{I}_\ell)\left\|g^k(m_\ell,\mathbb{I}_\ell)\right\|_*\sup_{w\in V,\|w\|=1}\|P_{m_\ell}w\|. \end{aligned}$$

Since we assume that the operator norm of P_{m_ℓ} as an operator in V is bounded uniformly with respect to m_ℓ, this shows (possibly for sufficiently large m_ℓ), together with the definition of $\|\cdot\|_*$, with (3.5), and invoking (2.14) and the growth conditions for $A_0(t)$, $A_1(t)$, B_0, $B_1(t)$ ($t\in[0,T]$), that, instead of (3.7) and (3.8), there holds

$$\begin{aligned} &\left\|v^j(m_\ell,\mathbb{I}_\ell)-v^n(m_\ell,\mathbb{I}_\ell)\right\|_* \\ &\quad\leqslant c\sum_{k=n+1}^{j}\tau_{k+1/2}(\mathbb{I}_\ell)\left\|g^k(m_\ell,\mathbb{I}_\ell)\right\|_* \\ &\quad\leqslant c\sum_{k=n+1}^{j}\tau_{k+1/2}(\mathbb{I}_\ell)\Big(\left\|f^k(\mathbb{I}_\ell)\right\|_{V_A^*}+\left\|A\big(t_k(\mathbb{I}_\ell)\big)v^k(m_\ell,\mathbb{I}_\ell)\right\|_{V_A^*}+\left\|B_0u^k(m_\ell,\mathbb{I}_\ell)\right\|_{V_B^*} \\ &\qquad+\left\|B_1\big(t_k(\mathbb{I}_\ell)\big)u^k(m_\ell,\mathbb{I}_\ell)\right\|_{V_A^*}\Big) \\ &\quad\leqslant c\big(t_{j+1/2}(\mathbb{I}_\ell)-t_{n+1/2}(\mathbb{I}_\ell)\big)^{1/p}\big(\|f\|_{L^{p^*}(0,T;V_A^*)}+1+\|v_{m_\ell,\mathbb{I}_\ell}\|_{L^p(0,T;V_A)}^{p-1}+\|u_{m_\ell,\mathbb{I}_\ell}\|_{L^{p^*}(0,T;V_B)} \\ &\qquad+\|u_{m_\ell,\mathbb{I}_\ell}\|_{L^2(0,T;V_B)}^{2(p-1)/p}\big). \end{aligned}$$

The crucial difference between this estimate and (3.7), (3.8) is that we start with the norm in V^* and, therefore, can estimate the term with B_0 in the V_B-norm without using any embedding.

The foregoing estimate yields, because of the boundedness of $\{v_{m_\ell,\mathbb{I}_\ell}\}$ in $L^p(0,T;V_A)$ and of $\{u_{m_\ell,\mathbb{I}_\ell}\}$ in $L^\infty(0,T;V_B)$ and together with (3.3), (3.4), and (3.14) the estimate

$$S_{22} \leqslant c \sum_{j=3}^{N(\mathbb{I}_\ell)-1} \sum_{n=1}^{j-2} \tau_{j+1/2}(\mathbb{I}_\ell)\tau_{n+1/2}(\mathbb{I}_\ell)\big(t_{j-1/2}(\mathbb{I}_\ell) - t_{n+1/2}(\mathbb{I}_\ell)\big)^{-1-2\sigma}$$
$$\times \big(t_{j+1/2}(\mathbb{I}_\ell) - t_{n+1/2}(\mathbb{I}_\ell)\big)^{2(1-\eta)/p}\big(\big\|v^j(m_\ell,\mathbb{I}_\ell)\big\|_{V_A}^{2\eta} + \big\|v^n(m_\ell,\mathbb{I}_\ell)\big\|_{V_A}^{2\eta}\big).$$

We can now proceed as in the proof of Lemma 10: With (3.9), (3.10), and (3.11) (with $2(1-\eta)/p$ instead of $1/p$ and observing that, by assumption, $-1-2\sigma+2(1-\eta)/p < 0 < -2\sigma+2(1-\eta)/p$), we find

$$S_{221} \leqslant c \sum_{j=3}^{N(\mathbb{I}_\ell)-1} \tau_{j+1/2}(\mathbb{I}_\ell)\big\|v^j(m_\ell,\mathbb{I}_\ell)\big\|_{V_A}^{2\eta} \leqslant c\big(1 + \|v_{m_\ell,\mathbb{I}_\ell}\|_{L^2(0,T;V_A)}\big)$$

instead of (3.12). The rest of the proof follows the same steps as that of Lemma 10. □

Note that if $V_A \hookrightarrow V_B$ then $V = V_A$ as well as $V^* = V_A^*$ and H belongs to $\underline{\mathscr{K}}_{1/2}(V_A^*, V_A)$. The case $V_A \hookrightarrow V_B$ is thus a special case of the preceding lemma but goes along without any additional requirement on the Galerkin scheme.

We are now in the position to prove the main result for the perturbed problem, which is twofold and shows on the one hand the convergence towards a weak solution, on the other hand thus the existence of a weak solution. Uniqueness cannot be expected except the perturbations fulfill stronger continuity conditions.

Theorem 12 *(Existence of a weak solution. Convergence). Let, in addition to the assumptions of Theorem* 4*, Assumptions* (A_1) *and* (B_1) *be fulfilled, let* $\tau_{\max}(\mathbb{I}_\ell)$ *be sufficiently small, and assume that* V_A *is compactly embedded in* H*. Moreover, let* $V_A \hookrightarrow V_B$ *or, alternatively, let* $H \in \underline{\mathscr{K}}_\eta(V^*, V_A)$ *for some* $\eta \in (0,1)$ *and assume that the Galerkin scheme can be chosen in such a way that the operator norm in* V *of the corresponding orthogonal projection of* H *onto the finite dimensional subspaces is uniformly bounded.*

Then there exists an exact solution $u \in \mathscr{C}_w([0,T]; V_B) \cap L^\infty(0,T;V_B)$ *to* (1.1) *with* $u' \in \mathscr{C}_w([0,T]; H) \cap L^\infty(0,T;H) \cap L^p(0,T;V_A)$ *and* $u'' \in (L^p(0,T;V))^*$*, such that the differential equation in* (1.1) *is fulfilled in the sense of equality in* $(L^p(0,T;V))^*$ *and such that* $u(t) \rightharpoonup u_0$ *in* V_B *and* $u'(t) \rightharpoonup v_0$ *in* H *as* $t \to 0$*.*

As $\ell \to \infty$*, the piecewise constant prolongations* $u_{m_\ell,\mathbb{I}_\ell}$ *of the fully discrete solutions to* (1.3) *converge weakly* in* $L^\infty(0,T;V_B)$ *as well as strongly in* $L^r(0,T;H)$ *for any* $r \in [1,\infty)$ *towards* u*. Moreover, the piecewise constant prolongations* $v_{m_\ell,\mathbb{I}_\ell}$ *as well as the piecewise linear prolongations* $\hat{v}_{m_\ell,\mathbb{I}_\ell}$ *converge weakly in* $L^p(0,T;V_A)$*, weakly* in* $L^\infty(0,T;H)$ *as well as strongly in* $L^r(0,T;H)$ *for any* $r \in [1,\infty)$ *towards* u'*.*

Proof. Going through the proof of Theorem 4 with the obvious changes, we see that we only need to consider the additional terms arising from the perturbations.

Analogously to A_0, we introduce $A_{1,\mathbb{I}}$ and $B_{1,\mathbb{I}}$ as the piecewise-constant-in-time approximations of $\{A_1(t)\}_{t\in[0,T]}$ and $\{B_1(t)\}_{t\in[0,T]}$, respectively. On the left-hand side of (2.23), we then have to add

$$\int_0^T \big\langle (A_{1,\mathbb{I}_\ell} v_{m_\ell,\mathbb{I}_\ell})(t), \varphi\big\rangle \psi(t)\,dt + \int_0^T \big\langle (B_{1,\mathbb{I}_\ell} u_{m_\ell,\mathbb{I}_\ell})(t), \varphi\big\rangle \psi(t)\,dt.$$

Since the sequence $\{v_{m_\ell,\mathbb{I}_\ell}\}$ is bounded in $L^\infty(0,T;H) \cap L^p(0,T;V_A)$, Assumption (A_1) immediately shows with Hölder's inequality that

$$\|A_{1,\mathbb{I}_\ell} v_{m_\ell,\mathbb{I}_\ell} - A_{1,\mathbb{I}_\ell} v\|_{L^{p^*}(0,T;V_A^*)} \leqslant C\|v_{m_\ell,\mathbb{I}_\ell} - v\|_{L^1(0,T;H)}^{\delta_A/p},$$

where $C > 0$ depends on $\|v_{m_\ell,\mathbb{I}_\ell}\|_{L^\infty(0,T;H)}$, $\|v_{m_\ell,\mathbb{I}_\ell}\|_{L^p(0,T;V_A)}$, $\|v\|_{L^\infty(0,T;H)}$, and $\|v\|_{L^p(0,T;V_A)}$. The strong convergence result of Lemma 10 and 11, respectively, yields

$$A_{1,\mathbb{I}_\ell} v_{m_{\ell'},\mathbb{I}_{\ell'}} - A_{1,\mathbb{I}_{\ell'}} v \to 0 \quad \text{in } L^{p^*}\big(0,T;V_A^*\big) \text{ as } \ell' \to \infty.$$

Furthermore, Assumption (A_1) together with Lebesgue's theorem ensures that

$$A_{1,\mathbb{I}_{\ell'}} v \to Av \quad \text{in } L^{p^*}\big(0,T;V_A^*\big) \text{ as } \ell' \to \infty.$$

So we come up with

$$A_{1,\mathbb{I}_\ell} v_{m_{\ell'},\mathbb{I}_{\ell'}} \to A_1 v \quad \text{in } L^{p^*}\big(0,T;V_A^*\big) \text{ as } \ell' \to \infty. \tag{3.15}$$

Since $\{u_{m_\ell,\mathbb{I}_\ell}\}$ is bounded in $L^\infty(0,T;V_B)$, Assumption (B_1) immediately yields

$$\|B_{1,\mathbb{I}_\ell} u_{m_\ell,\mathbb{I}_\ell} - B_{1,\mathbb{I}_\ell} u\|_{L^{p^*}(0,T;V_A^*)} \leqslant C \|u_{m_\ell,\mathbb{I}_\ell} - u\|_{L^1(0,T;H)}^{1-1/p},$$

where $C > 0$ depends on $\|u_{m_\ell,\mathbb{I}_\ell}\|_{L^\infty(0,T;V_B)}$ and $\|u\|_{L^\infty(0,T;V_B)}$. Lemma 10 and 11, respectively, then implies

$$B_{1,\mathbb{I}_\ell} u_{m_{\ell'},\mathbb{I}_{\ell'}} - B_{1,\mathbb{I}_{\ell'}} u \to 0 \quad \text{in } L^{p^*}\big(0,T;V_A^*\big) \text{ as } \ell' \to \infty.$$

Furthermore, with Assumption (B_1) and Lebesgue's theorem, we have

$$B_{1,\mathbb{I}_{\ell'}} u \to Bu \quad \text{in } L^{p^*}\big(0,T;V_A^*\big) \text{ as } \ell' \to \infty$$

so that

$$B_{1,\mathbb{I}_\ell} u_{m_{\ell'},\mathbb{I}_{\ell'}} \to Bu \quad \text{in } L^{p^*}\big(0,T;V_A^*\big) \text{ as } \ell' \to \infty. \tag{3.16}$$

With (3.15) and (3.16), the rest of the proof is exactly the same as that of Theorem 4. We just have to add the corresponding additional terms such as $A_1 v + B_1 u$, e.g., on the left-hand side of (2.29). □

We shall remark that the assumption $H \in \underline{\mathcal{K}}_\eta(V^*, V_A)$ is not very restrictive and will be fulfilled in many applications. Also the additional assumption on the Galerkin scheme will be satisfied in many situations. Indeed, this assumption is a requirement on the couple V, H to possess a certain approximation property and has been studied in the context of the finite element method, e.g., for $V = W_0^{1,p}(\Omega)$ and $H = L^2(\Omega)$ in [9].

Moreover, our a priori estimates are in accordance with (but somewhat suboptimal with respect to the upper bound for σ compared to) results from interpolation theory as the following remark shows. With some modifications of the estimates above, we may enlarge the upper bound for σ in Lemma 10 and 11, which, however, would not change the main result.

Remark 4. Let H be in the class $\underline{\mathcal{K}}_\eta(V^*, V_A)$ for some $\eta \in (0,1)$. If $u' \in L^p(0,T;V_A)$ and $u'' \in L^{p^*}(0,T;V^*)$ then $u' \in H^\sigma(0,T;H)$ for any $\sigma < 1/2 + (1-2\eta)/p$. This follows from the result [1, Thm. 3.1, Cor. 4.3] on the interpolation of Besov spaces for vector-valued functions together with the characterization [20, Prop. 1.1.4 on p. 9, Prop. 1.3.2 on pp. 27ff.] of intermediate spaces of class $\underline{\mathcal{K}}_\eta$. Note that $0 < (1-\eta)/p < 1/2 + (1-2\eta)/p < 1$ if $\eta \in (0,1)$ and $p \geqslant 2$.

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