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Efficient time integration methods based on operator splitting and application to the Westervelt equation

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Efficient time integration methods based on operator splitting are introduced for the Westervelt equation, a nonlinear damped wave equation that arises in nonlinear acoustics as a mathematical model for the propagation of sound waves in high intensity ultrasound applications. A global error estimate is deduced for the first-order Lie–Trotter splitting method, confirming that the splitting method remains stable, and that the nonstiff convergence order is retained in situations where the problem data are sufficiently regular. Fundamental ingredients in the stability and error analysis are regularity results for the Westervelt equation and related linear evolution equations of hyperbolic and parabolic type. Numerical examples illustrate and complement the theoretical investigations.

Keywords: nonlinear evolution equations; Westervelt equation; regularity of solutions; time-splitting methods; stability; local error expansion; convergence.

1. Introduction

Scope of applications. High intensity ultrasound plays a crucial role in numerous practical settings ranging from medical treatment like lithotripsy or thermotherapy to industrial applications like ultrasound cleaning or welding and sonochemistry; as a small selection, we mention the contributions (Dreyer *et al.*, 2000; Kaltenbacher, 2007) and refer to the literature cited therein. The numerical simulation of high intensity ultrasound propagation is a valuable tool for the design and improvement of high intensity ultrasound devices, but poses major challenges due to the nonlinearity of the underlying partial differential equations. Time integration methods for the equations of nonlinear acoustics have been investigated in Dreyer *et al.* (2000) and Kaltenbacher *et al.* (2002); however, the use of transient simulations within the mathematical optimization of high intensity ultrasound devices still seems to be beyond the scope of these approaches. On the other hand, operator-splitting methods have proved to be efficient time integration methods in the context of other classes of nonlinear partial differential equations, see, for instance, Auzinger *et al.*, Descombes & Thalhammer (2012), Holden *et al.* (2010, 2013) and the references given therein. This motivates the investigation of time-splitting methods for the equations of nonlinear acoustics.

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Westervelt equation. In this work, we study time integration methods based on operator splitting for one of the classical models of nonlinear acoustics, the nondegenerate Westervelt equation

$$\partial_{tt}\psi(x,t) - b\Delta\partial_t\psi(x,t) - c^2\Delta\psi(x,t) = \frac{\beta_a}{c^2}\partial_t(\partial_t\psi(x,t))^2, \quad (x,t)\in\Omega\times(0,T]\subset\mathbb{R}^d\times\mathbb{R}_{>0}, \tag{1.1a}$$

describing the propagation of the acoustic velocity potential $\psi:\overline{\Omega}\times[0,T]\to\mathbb{R}$, where $c>0$ denotes the speed of sound, $b>0$ the diffusivity of sound and $\beta_a>1$ the parameter of nonlinearity; for details on the physical background as well as the derivation of the Westervelt equation, we refer to the original work Westervelt (1963), see also Kaltenbacher (2007). For our purposes, it is advantageous to employ a reformulation of (1.1a) as nonlinear evolutionary system

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = F(u(t)), \quad t\in(0,T], \tag{1.1b}$$

for the abstract function $u:[0,T]\to X = X_1\times X_2 : t\mapsto u(t) = \Psi(\cdot,t) = (\psi(\cdot,t),\partial_t\psi(\cdot,t))$ with values in the underlying Banach space $(X,\|\cdot\|_X)$.

Time-splitting methods. Our main concern is to introduce and analyse operator-splitting methods for the time integration of the nondegenerate Westervelt equation (1.1). Time-splitting methods for nonlinear evolution equations of the form (1.1b) utilize a natural decomposition of the defining operator

$$F = A + B, \tag{1.2a}$$

and rely on the presumption that efficient numerical solvers are available for the associated subproblems

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = A(v(t)), \quad \frac{\mathrm{d}}{\mathrm{d}t}w(t) = B(w(t)), \quad t\in(0,T]. \tag{1.2b}$$

We propose different decompositions for (1.1) which have in common that they require the numerical solution of a nonlinear diffusion equation in each time step.

Convergence analysis. For the decomposition with the best performance in numerical tests, we provide a detailed convergence analysis, adapting the approach exploited in Descombes & Thalhammer (2012) for the first-order Lie–Trotter splitting method applied to nonlinear Schrödinger equations, see also Auzinger *et al.* for an extension to the second-order Strang splitting method. A rigorous stability and error analysis of time-splitting methods for the Westervelt equation (1.1) is a complex task; on the one hand, two unbounded nonlinear operators $F = A + B$ are present, contrary to nonlinear Schrödinger equations comprising a linear differential operator and a nonlinear multiplication operator, and, on the other hand, the underlying Banach space $X = X_1\times X_2$ is composed of two different function spaces reflecting the regularity of the solution components $(\psi,\partial_t\psi)$. As this considerably simplifies the local error analysis, we focus on the first-order Lie–Trotter splitting method, given by

$$u_1 = \mathscr{S}_F(h,u_0) = \mathscr{E}_B(h,\mathscr{E}_A(h,u_0)) \approx u(h) = \mathscr{E}_F(h,u(0)),$$

and indicate the extension to higher-order splitting methods; here, we denote by $\mathscr{E}_F,\mathscr{E}_A,\mathscr{E}_B$ the evolution operators associated with the Westervelt equation (1.1) and the subproblems (1.2). In situations where the problem data are sufficiently regular, the obtained convergence result ensures that the Lie–Trotter splitting method remains stable and retains the nonstiff order of convergence. Fundamental ingredients in the stability and error analysis are regularity results for the Westervelt equation and related linear

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evolution equations of hyperbolic and parabolic type. Numerical examples illustrate and complement the theoretical investigations.

2. Westervelt equation

In this section, we introduce the initial-boundary value problem for the Westervelt equation and its abstract formulation as Cauchy problem. A regularity result, needed as a basic ingredient in the convergence analysis of time-splitting methods for the Westervelt equation, is stated in Section 7.

Westervelt equation. We study the following initial-boundary value problem for a function $\psi : \overline{\Omega} \times [0,T] \subset \mathbb{R}^d \times \mathbb{R}_{\geqslant 0} \to \mathbb{R} : (x,t) \mapsto \psi(x,t)$:

$$\begin{cases} \partial_{tt}\psi(x,t) - \alpha\Delta\partial_t\psi(x,t) - \beta\Delta\psi(x,t) \\ \quad = \gamma\,\partial_t(\partial_t\psi(x,t))^2 = \delta\,\partial_t\psi(x,t)\partial_{tt}\psi(x,t), \quad (x,t) \in \Omega \times (0,T], \\ \psi(x,t) \text{ given}, \quad (x,t) \in \partial\Omega \times [0,T], \\ \psi(x,0) \text{ given}, \quad \partial_t\psi(x,0) \text{ given}, \quad x \in \Omega, \end{cases} \tag{2.1a}$$

involving positive constants $\alpha, \beta, \gamma > 0$ and $\delta = 2\gamma > 0$. With regard to the time integration by first- and second-order splitting methods, we restrict ourselves to situations where a sufficiently regular solution to (2.1a) exists such that pointwise evaluations of the arising time and space derivatives of ψ are justified; in particular, we suppose the spatial domain and the prescribed initial data to be sufficiently regular.

Reformulation (Nondegenerate case). A regularity result for (2.1a), given in Section 7, ensures nondegeneracy of the Westervelt equation; that is, for any initial state of sufficiently small norm the relation $0 < 1 - \delta\partial_t\psi(x,t) < \infty$ holds for all $(x,t) \in \overline{\Omega} \times [0,T]$. As a consequence, the equivalent formulation

$$\begin{cases} \partial_{tt}\psi(x,t) = \alpha(1 - \delta\partial_t\psi(x,t))^{-1}\Delta\partial_t\psi(x,t) \\ \qquad\qquad + \beta(1 - \delta\partial_t\psi(x,t))^{-1}\Delta\psi(x,t), \quad (x,t) \in \Omega \times (0,T], \\ \psi(x,t) \text{ given}, \quad (x,t) \in \partial\Omega \times [0,T], \\ \psi(x,0) \text{ given}, \quad \partial_t\psi(x,0) \text{ given}, \quad x \in \Omega, \end{cases} \tag{2.1b}$$

is obtained.

Reformulation as first-order system. We next rewrite (2.1b) as a first-order system for the function $\Psi = (\Psi_1, \Psi_2) = (\psi, \partial_t\psi) : \overline{\Omega} \times [0,T] \to \mathbb{R}^2$

$$\begin{cases} \partial_t\Psi_1(x,t) = \Psi_2(x,t), \quad (x,t) \in \Omega \times (0,T], \\ \partial_t\Psi_2(x,t) = \alpha(1 - \delta\Psi_2(x,t))^{-1}\Delta\Psi_2(x,t) \\ \qquad\qquad + \beta(1 - \delta\Psi_2(x,t))^{-1}\Delta\Psi_1(x,t)), \quad (x,t) \in \Omega \times (0,T], \\ \Psi_1(x,t) \text{ given}, \quad (x,t) \in \partial\Omega \times [0,T], \\ \Psi_1(x,0) \text{ given}, \quad \Psi_2(x,0) \text{ given}, \quad x \in \Omega. \end{cases} \tag{2.1c}$$

Abstract Cauchy problem. In regard to the introduction and analysis of time-splitting methods for the Westervelt equation, it is convenient to formulate (2.1) as abstract Cauchy problem for the function

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$u : [0,T] \to X : t \mapsto u(t) = \Psi(\cdot, t)$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} u(t) = F(u(t)), & t \in (0,T], \\ u(0) \text{ given}, \end{cases} \tag{2.2a}$$

where $(X, \|\cdot\|_X)$ denotes the underlying Banach space and the nonlinear operator $F : D(F) \to X$ is given by

$$\begin{aligned} &F(v) = \begin{pmatrix} v_2 \\ \tilde{\alpha}(v_2)\,\Delta v_2 + \tilde{\beta}(v_2)\,\Delta v_1 \end{pmatrix}, \quad v = (v_1, v_2) \in D(F), \\ &\tilde{\alpha}(v_2) = \alpha(1 - \delta v_2)^{-1}, \quad \tilde{\beta}(v_2) = \beta(1 - \delta v_2)^{-1}. \end{aligned} \tag{2.2b}$$

According to the situation under consideration, the domain $D(F) \subset X$ is chosen such that it reflects the regularity requirements on the solution to the Westervelt equation as well as the imposed boundary conditions. It is notable that (2.2) can be cast into the form of a quasilinear problem, since

$$F(v) = \begin{pmatrix} 0 & I \\ \tilde{\beta}(v_2)\Delta & \tilde{\alpha}(v_2)\Delta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

where I denotes the identity operator.

3. Time-splitting methods

In this section, we introduce the general format of exponential operator-splitting methods for the time integration of nonlinear evolution equations, and specify the different decompositions considered for the Westervelt equation. Numerical examples comparing the accuracy and efficiency of the resulting time dicretizations are found in Section 6.

3.1 *Time-splitting methods for nonlinear evolution equations*

Nonlinear evolution equation. In accordance with the reformulation of the initial-boundary value problem for the Westervelt equation (2.1) as abstract Cauchy problem (2.2), we consider the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} u(t) = F(u(t)) = A(u(t)) + B(u(t)), & t \in (0,T], \\ u(0) \text{ given}, \end{cases} \tag{3.1a}$$

assuming that the nonlinear operators $A : D(A) \to X$ and $B : D(B) \to X$ are chosen such that the intersection $D(A) \cap D(B)$ coincides with the domain of the defining operator $F : D(F) \to X$. For the following investigations, it is useful to introduce the evolution operator associated with (3.1a)

$$u(t) = \mathscr{E}_F(t, u(0)), \quad t \in [0,T]. \tag{3.1b}$$

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Subproblems. Exponential operator-splitting methods for the time integration of the initial value problem (3.1a) rely on the numerical solution of the subproblems

$$\text{Subproblem A:} \quad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} v(t) = A(v(t)), \quad t \in (0,T], \\ v(0) \text{ given}, \end{cases}$$
$$\text{Subproblem B:} \quad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} w(t) = B(w(t)), \quad t \in (0,T], \\ w(0) \text{ given}. \end{cases} \tag{3.2a}$$

In accordance with (3.1b), the associated evolution operators are given by

$$v(t) = \mathscr{E}_A(t, v(0)), \quad w(t) = \mathscr{E}_B(t, w(0)), \quad t \in [0,T]. \tag{3.2b}$$

Time-discrete solution. For an initial approximation $u_0 \approx u(0)$ and a sequence of time grid points $0 = t_0 < t_1 < \cdots < t_N \leqslant T$ with corresponding time stepsizes $h_n = t_{n+1} - t_n$ for $0 \leqslant n \leqslant N-1$, the time-discrete solution to (3.1) is determined through a recurrence relation of the form

$$u_n = \mathscr{S}_F(h_{n-1}, u_{n-1}) \approx u(t_n) = \mathscr{E}_F(h_{n-1}, u(t_{n-1})), \quad 1 \leqslant n \leqslant N. \tag{3.3a}$$

The numerical evolution operator associated with a general splitting method of (nonstiff) order $p \geqslant 1$ can be cast into the format

$$\begin{aligned} u_n &= \mathscr{S}_F(h_{n-1}, u_{n-1}) = U_{n-1,s}, \\ U_{n-1,j} &= \mathscr{E}_{B_j}(h_{n-1}, \mathscr{E}_{A_j}(h_{n-1}, U_{n-1,j-1})), \quad 1 \leqslant j \leqslant s, \quad U_{n-1,0} = u_{n-1}, \end{aligned} \tag{3.3b}$$

involving certain coefficients $(a_j, b_j)_{j=1}^s$. Here, we employ the abbreviations

$$A_j = a_j A, \quad B_j = b_j B, \quad 1 \leqslant j \leqslant s; \tag{3.3c}$$

due to the fact that the considered evolution equation is autonomous, a scaling of the operators A, B corresponds to a scaling in time.

Lie–Trotter and Strang splitting methods. The Lie–Trotter splitting method of (nonstiff) order $p = 1$, given by

$$\mathscr{S}_F(t,v) = \mathscr{E}_B(t, \mathscr{E}_A(t,v)) \quad \text{or} \quad \mathscr{S}_F(t,v) = \mathscr{E}_A(t, \mathscr{E}_B(t,v)), \tag{3.4}$$

can be cast into the format (3.3) for the choices

$$\begin{aligned} &s = 1: \quad a_1 = 1, \ b_1 = 1 \quad \text{or} \\ &s = 2: \quad a_1 = 0, \ b_1 = 1, \ a_2 = 1, \ b_2 = 0, \end{aligned}$$

respectively. The widely used Strang splitting method of (nonstiff) order $p = 2$, given by

$$\begin{aligned} &\mathscr{S}_F(t,v) = \mathscr{E}_{A/2}(t, \mathscr{E}_B(t, \mathscr{E}_{A/2}(t,v))) \quad \text{or} \\ &\mathscr{S}_F(t,v) = \mathscr{E}_{B/2}(t, \mathscr{E}_A(t, \mathscr{E}_{B/2}(t,v))), \end{aligned} \tag{3.5}$$

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results for the choices

$$s=2: \quad a_1=\frac{1}{2}, \ b_1=1, \ a_2=\frac{1}{2}, \ b_2=0 \quad \text{or}$$
$$s=2: \quad a_1=0, \ b_1=\frac{1}{2}, \ a_2=1, \ b_2=\frac{1}{2},$$

respectively.

3.2 *Time-splitting methods for the Westervelt equation*

In the following, we propose different decompositions

$$F=A+B$$

of the nonlinear operator defining the Westervelt equation (2.1) and (2.2), and discuss the computational effort for the numerical solution of the associated subproblems (3.2).

3.2.1 *Decomposition* I

Decomposition. A first decomposition involves the nonlinear operators

$$A(v)=\begin{pmatrix} v_2 \\ \tilde{\alpha}(v_2)\Delta v_2 \end{pmatrix}, \quad B(v)=\begin{pmatrix} 0 \\ \tilde{\beta}(v_2)\Delta v_1 \end{pmatrix},$$
$$\tilde{\alpha}(v_2)=\alpha(1-\delta v_2)^{-1}, \quad \tilde{\beta}(v_2)=\beta(1-\delta v_2)^{-1}.$$

Subproblems. The resolution of the subproblem associated with A

$$\begin{cases} \partial_t\Psi_1(x,t)=\Psi_2(x,t), \\ \partial_t\Psi_2(x,t)=\alpha(1-\delta\Psi_2(x,t))^{-1}\Delta\Psi_2(x,t), \end{cases}$$

amounts to the numerical solution of a nonlinear diffusion equation for the second component $\Psi_2=\partial_t\psi$

$$\partial_t\Psi_2(x,t)=\alpha(1-\delta\Psi_2(x,t))^{-1}\Delta\Psi_2(x,t);$$

the first component $\Psi_1=\psi$ is then retained by integration

$$\Psi_1(x,t)=\Psi_1(x,0)+\int_0^t \Psi_2(x,\tau)\,d\tau.$$

For the subproblem associated with B

$$\begin{cases} \partial_t\Psi_1(x,t)=0, \\ \partial_t\Psi_2(x,t)=\beta(1-\delta\Psi_2(x,t))^{-1}\Delta\Psi_1(x,t), \end{cases}$$

the first component remains constant on the considered time interval

$$\Psi_1(x,t)=\Psi_1(x,0).$$

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Consequently, the second component is a solution to

$$\partial_t \Psi_2(x,t) = \beta(1-\delta\Psi_2(x,t))^{-1}\Delta\Psi_1(x,0),$$

with explicit representation

$$\Psi_2(x,t) = \frac{1}{\delta}(1-\sqrt{(1-\delta\Psi_2(x,0))^2 - 2\beta\delta t\Delta\Psi_1(x,0)}).$$

We note that in the nondegenerate case the relation $\delta\Psi_2(x,0) < 1$ holds; thus, the other admissable choice for a solution to the subproblem leads to a contradiction when evaluating at $t = 0$. Provided that the time increment $t > 0$ is chosen sufficiently small, it is ensured that $(1-\delta\Psi_2(x,0))^2 - 2\beta\delta t\Delta\Psi_1(x,0) > 0$, and hence $\Psi_2(x,t) \in \mathbb{R}$.

3.2.2 *Decomposition* II

Decomposition. A second decomposition involves the nonlinear operators

$$A(v) = \begin{pmatrix} \frac{1}{2}v_2 \\ \tilde{\alpha}(v_2)\Delta v_2 \end{pmatrix}, \quad B(v) = \begin{pmatrix} \frac{1}{2}v_2 \\ \tilde{\beta}(v_2)\Delta v_1 \end{pmatrix}.$$

Subproblems. The resolution of the subproblem associated with A

$$\begin{cases} \partial_t\Psi_1(x,t) = \frac{1}{2}\Psi_2(x,t), \\ \partial_t\Psi_2(x,t) = \alpha(1-\delta\Psi_2(x,t))^{-1}\Delta\Psi_2(x,t), \end{cases}$$

requires the numerical solution of a nonlinear diffusion equation for the second component

$$\partial_t\Psi_2(x,t) = \alpha(1-\delta\Psi_2(x,t))^{-1}\Delta\Psi_2(x,t);$$

the first component is then obtained by integration

$$\Psi_1(x,t) = \Psi_1(x,0) + \frac{1}{2}\int_0^t \Psi_2(x,\tau)\,\mathrm{d}\tau.$$

The resolution of the subproblem associated with B

$$\begin{cases} \partial_t\Psi_1(x,t) = \frac{1}{2}\Psi_2(x,t), \\ \partial_t\Psi_2(x,t) = \beta(1-\delta\Psi_2(x,t))^{-1}\Delta\Psi_1(x,t), \end{cases}$$

amounts to the numerical solution of a nonlinear wave equation for the first component

$$\partial_{tt}\Psi_1(t) = \frac{1}{2}\beta(1-2\delta\partial_t\Psi_1(x,t))^{-1}\Delta\Psi_1(x,t);$$

the second component is then given by

$$\Psi_2(x,t) = 2\partial_t\Psi_1(x,t).$$

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3.2.3 *Decomposition* III **Decomposition.** A third decomposition involves the nonlinear operators

$$A(v) = \begin{pmatrix} 0 \\ \tilde{\alpha}(v_2)\Delta v_2 \end{pmatrix}, \quad B(v) = \begin{pmatrix} v_2 \\ \tilde{\beta}(v_2)\Delta v_1 \end{pmatrix}.$$

Subproblems. The resolution of the subproblem associated with A

$$\begin{cases} \partial_t \Psi_1(x,t) = 0, \\ \partial_t \Psi_2(x,t) = \alpha(1 - \delta\Psi_2(x,t))^{-1}\Delta\Psi_2(x,t), \end{cases}$$

amounts to the numerical solution of a nonlinear diffusion equation for the second component

$$\partial_t \Psi_2(x,t) = \alpha(1 - \delta\Psi_2(x,t))^{-1}\Delta\Psi_2(x,t),$$

whereas the first component remains constant in time

$$\Psi_1(x,t) = \Psi_1(x,0).$$

The resolution of the subproblem associated with B

$$\begin{cases} \partial_t \Psi_1(x,t) = \Psi_2(x,t), \\ \partial_t \Psi_2(x,t) = \beta(1 - \delta\Psi_2(x,t))^{-1}\Delta\Psi_1(x,t), \end{cases}$$

requires the numerical solution of a nonlinear wave equation for the first component

$$\partial_{tt}\Psi_1(t) = \beta(1 - \delta\partial_t\Psi_1(x,t))^{-1}\Delta\Psi_1(x,t);$$

the second component is then given by

$$\Psi_2(x,t) = \partial_t\Psi_1(x,t).$$

3.2.4 *Decomposition* IV **Decomposition.** A fourth decomposition involves the nonlinear operators

$$A(v) = \begin{pmatrix} 0 \\ \tilde{\alpha}(v_2)\Delta v_2 + \delta v_2\tilde{\beta}(v_2)\Delta v_1 \end{pmatrix}, \quad B(v) = \begin{pmatrix} v_2 \\ \beta\Delta v_1 \end{pmatrix};$$

we note that the identity $\beta + \delta v_2\tilde{\beta}(v_2) = \tilde{\beta}(v_2)$ holds.

Subproblems. For the subproblem associated with A

$$\begin{cases} \partial_t \Psi_1(x,t) = 0, \\ \partial_t \Psi_2(x,t) = \alpha(1 - \delta\Psi_2(x,t))^{-1}\Delta\Psi_2(x,t) + \beta\delta\Psi_2(x,t)(1 - \delta\Psi_2(x,t))^{-1}\Delta\Psi_1(x,t), \end{cases}$$

the first component remains constant in time

$$\Psi_1(x,t) = \Psi_1(x,0),$$

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whereas the computation of the second component requires the numerical solution of a nonlinear reaction–diffusion equation

$$\partial_t \Psi_2(x,t) = \alpha(1-\delta\Psi_2(x,t))^{-1}\Delta\Psi_2(x,t) + \beta\delta\Psi_2(x,t)(1-\delta\Psi_2(x,t))^{-1}\Delta\Psi_1(x,0).$$

The resolution of the subproblem associated with B

$$\begin{cases} \partial_t \Psi_1(x,t) = \Psi_2(x,t), \\ \partial_t \Psi_2(x,t) = \beta\Delta\Psi_1(x,t), \end{cases}$$

amounts to the numerical solution of a linear wave equation for the first component

$$\partial_{tt}\Psi_1(x,t) = \beta\Delta\Psi_1(x,t);$$

the second component is then given by

$$\Psi_2(x,t) = \partial_t \Psi_1(x,t).$$

3.2.5 *Computational effort* **Subproblem A.** In all decompositions, the realization of the subproblem associated with the nonlinear operator A requires the numerical solution of a nonlinear diffusion equation for the second component $\Psi_2 = \partial_t \psi$, where in the case of Decomposition IV the nonlinear diffusion equation involves an additional zero-order term; the first component $\Psi_1 = \psi$ is obtained by integration or remains constant in time.

Subproblem B. In Decomposition I, the realization of the subproblem associated with the nonlinear operator B reduces to the application of the Laplace operator to the first component of the imposed initial state; the second component is then computed through an explicit representation. Contrary, in Decompositions II and III the realization of the subproblem amounts to the numerical solution of a nonlinear wave equation for the first component; the second component is then given by the time derivative $\partial_t \Psi_1$. We point out that the solvability of the arising nonlinear wave equations is an open question and that even local solvability in time cannot be guaranteed. As an alternative, we thus introduce Decomposition IV involving a linear wave equation.

4. Stability and error analysis

In this section, we describe the general approach for a convergence analysis of time-splitting methods, employing a framework of abstract Cauchy problems; details on the specialisation to the Westervelt equation are given in Section 5.

Uniform time grid. For notational simplicity, we henceforth restrict ourselves to constant time increments $h > 0$ with associated equidistant time grid points given by $t_n = nh$ for $0 \leqslant n \leqslant N$. However, it is straightforward to extend the arguments to a nonuniform time grid with stepsize ratios bounded from above and below; in the global error estimate (4.2e), the constant time increment $h > 0$ is then replaced with the maximum stepsize $h_{\max} = \max\{h_n : 0 \leqslant n \leqslant N-1\}$.

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Global and local error. In order to deduce a convergence result for time-splitting methods (3.3) applied to nonlinear evolution equations (3.1), we utilize the telescopic identity

$$u_N - u(t_N) = \mathscr{S}_F^N(h, u_0) - \mathscr{S}_F^N(h, u(0)) + \sum_{n=0}^{N-1} (\mathscr{S}_F^{N-n-1}(h, \mathscr{L}_F(h, u(t_n)) + \mathscr{E}_F(h, u(t_n))) - \mathscr{S}_F^{N-n-1}(h, \mathscr{E}_F(h, u(t_n)))), \tag{4.1a}$$

relating the global error to compositions of the splitting operator and local errors

$$\begin{aligned} &\mathscr{S}_F^{n+1}(h, v) = \mathscr{S}_F(h, \mathscr{S}_F^n(h, v)), \quad 0 \leqslant n \leqslant N-1, \quad \mathscr{S}_F^0(h, v) = v, \\ &\mathscr{L}_F(h, u(t_n)) = \mathscr{S}_F(h, u(t_n)) - \mathscr{E}_F(h, u(t_n)), \quad 0 \leqslant n \leqslant N-1. \end{aligned} \tag{4.1b}$$

Approach. Our aim is to deduce an estimate for the global error (4.1a) with respect to the norm of the underlying Banach space X or a suitable subspace $\tilde{X} \subset X$, respectively, such that the expected dependence on the time stepsize is retained under appropriate regularity requirements on the problem data. In the context of the Westervelt equation, we shall make use of the fact that the regularity result stated in Section 7 implies

$$u(0) \in D, \ \|u(0)\|_D \leqslant C_0 \implies u(t) \in D, \ \|u(t)\|_D \leqslant C, \quad t \in [0, T], \tag{4.2a}$$

for certain constants $C_0, C > 0$; the subspace $D \subset \tilde{X} \subset X$, chosen accordingly to the (nonstiff) order of the splitting method, captures additional regularity and compatibility requirements. Regularity results for the arising subproblems and associated variational equations shall ensure stability of the splitting procedure, that is, boundedness of compositions of the splitting operator

$$\|\mathscr{S}_F^n(h, v) - \mathscr{S}_F^n(h, \tilde{v})\|_{\tilde{X}} \leqslant e^{Ct_n} \|v - \tilde{v}\|_{\tilde{X}}, \quad 1 \leqslant n \leqslant N, \tag{4.2b}$$

provided that the arguments $v, \tilde{v} \in \tilde{D}$ remain bounded in a certain subspace $D \subset \tilde{D} \subset \tilde{X} \subset X$. Concerning a local error analysis, we restrict ourselves to the least technical case, the first-order Lie–Trotter splitting method given by

$$\mathscr{S}_F(h, v) = \mathscr{E}_B(h, \mathscr{E}_A(h, v)),$$

see also (3.4); here, a suitable expansion yields the representation

$$\mathscr{L}_F(h, v) = \int_0^h \int_0^{\tau_1} \partial_2 \mathscr{E}_F(h - \tau_1, \mathscr{S}_F(\tau_1, v)) \, \partial_2 \mathscr{E}_B(\tau_1, w) \, (\partial_2 \mathscr{E}_B(\tau_2, w))^{-1} \times [B, A](\mathscr{E}_B(\tau_2, w))|_{w = \mathscr{E}_A(\tau_1, v)} \, \mathrm{d}\tau_2 \, \mathrm{d}\tau_1, \tag{4.2c}$$

see Auzinger *et al.* and Descombes & Thalhammer (2012). Bounds for the evolution operators $\mathscr{E}_F, \mathscr{E}_A, \mathscr{E}_B$, their Fréchet derivatives with respect to the second argument $\partial_2 \mathscr{E}_F, \partial_2 \mathscr{E}_A, \partial_2 \mathscr{E}_B$, and the first Lie-commutator $[A, B]$ shall lead to the local error estimate

$$p = 1: \quad \|\mathscr{L}_F(h, v)\|_{\tilde{X}} \leqslant Ch^{p+1}. \tag{4.2d}$$

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Altogether, these ingredients allow to establish the global error estimate

$$\begin{aligned}\|u_N - u(t_N)\|_{\tilde{X}} &\leqslant \|\mathscr{S}_F^N(h,u_0) - \mathscr{S}_F^N(h,u(0))\|_{\tilde{X}} + \sum_{n=0}^{N-1} \|\mathscr{S}_F^{N-n-1}(h,\mathscr{L}_F(h,u(t_n)) + \mathscr{E}_F(h,u(t_n))) \\ &\quad - \mathscr{S}_F^{N-n-1}(h,\mathscr{E}_F(h,u(t_n)))\|_{\tilde{X}} \\ &\leqslant C\left(\|u_0 - u(0)\|_{\tilde{X}} + \sum_{n=1}^{N} \|\mathscr{L}_F(h,u(t_{n-1}))\|_{\tilde{X}}\right) \\ &\leqslant C(\|u_0 - u(0)\|_{\tilde{X}} + h^p), \end{aligned} \tag{4.2e}$$

with constant depending in particular on the bound for the exact solution values with respect to the norm in D, the bound for the initial approximation u_0 in $\tilde{D}$ and the final time; this shows that the nonstiff order of convergence is retained, provided that the prescribed initial state is sufficiently regular.

5. Global error estimate

In this section, we establish a global error estimate for the Lie–Trotter splitting method applied to the Westervelt equation. We include a detailed stability and error analysis for Decomposition I showing the best performance in numerical tests, see Section 3.2. In accordance with Section 7, we study the Westervelt equation in three space dimensions subject to homogeneous Dirichlet boundary conditions on a regular domain; we consider this to be the least technical case, as far as the boundary conditions are concerned, and the most relevant case, as far as the space dimension is concerned. Fundamental auxiliary results for the proof of Theorem 5.1 are deduced in Sections 5.1 and 5.2.

5.1 *Fréchet derivative and Lie-commutator*

In the following, we specify the Fréchet derivative of the operator defining (2.1) and (2.2), and determine the first Lie-commutator arising in the local error representation (4.2c).

Notation and auxiliary estimates. The product space $X = X_1 \times X_2$ of two Banach spaces is endowed with the norm $\|(x_1,x_2)\|_X = \|x_1\|_{X_1} + \|x_2\|_{X_2}$. We employ standard abbreviations for Sobolev spaces such as $(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}})$ and $H^k(\Omega) = W^{k,2}(\Omega)$ for $k \in \mathbb{N}_{\geqslant 0}$ and $p \in \mathbb{N}_{\geqslant 1}$, see, for instance, Adams & Fournier (2003); for notational simplicity, in the norms we do not indicate the dependence on the domain. For convenience, we recall Young's inequality

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}, \quad a,b \geqslant 0, \quad 1 < p,q < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{5.1a}$$

By means of Hölder's inequality

$$\|f_1 f_2\|_{L^1} \leqslant \|f_1\|_{L^p} \|f_2\|_{L^q}, \quad 1 \leqslant p,q \leqslant \infty, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{5.1b}$$

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and the continuous embeddings $H^2(\Omega) \hookrightarrow \mathscr{C}(\Omega)$ as well as $H^1(\Omega) \hookrightarrow L^4(\Omega), L^6(\Omega)$, valid for any bounded regular domain $\Omega \subset \mathbb{R}^3$, the relations

$$\begin{aligned}
\|f_1 f_2\|_{L^2} &\leqslant \|f_1\|_{L^\infty} \|f_2\|_{L^2} \leqslant C\|f_1\|_{H^2} \|f_2\|_{L^2}, \\
\|f_1 f_2\|_{L^2} &\leqslant \|f_1\|_{L^4} \|f_2\|_{L^4} \leqslant C\|f_1\|_{H^1} \|f_2\|_{H^1}, \\
\|f_1 f_2\|_{L^2} &\leqslant \|f_1\|_{L^3} \|f_2\|_{L^6} \leqslant C\|f_1\|_{H^1} \|f_2\|_{H^1}, \\
\|f_1 f_2 f_3\|_{L^2} &\leqslant \|f_1\|_{L^6} \|f_2\|_{L^6} \|f_3\|_{L^6} \leqslant C\|f_1\|_{H^1} \|f_2\|_{H^1} \|f_3\|_{H^1},
\end{aligned} \tag{5.1c}$$

follow. Moreover, we make use of the fact that the Sobolev space $H^{k+2}(\Omega)$ forms an algebra for arbitrary exponents $k \in \mathbb{N}_{\geqslant 0}$, that is, the relation

$$\|f_1 f_2\|_{H^{k+2}} \leqslant C\|f_1\|_{H^{k+2}} \|f_2\|_{H^{k+2}}, \quad k \in \mathbb{N}_{\geqslant 0}, \tag{5.1d}$$

holds. This in particular implies

$$\|\nabla f_1 \cdot \nabla f_2\|_{H^{k+2}} \leqslant C\|f_1\|_{H^{k+3}} \|f_2\|_{H^{k+3}}, \quad k \in \mathbb{N}_{\geqslant 0}. \tag{5.1e}$$

Defining operators. For the convenience of the reader, we recall the definitions of the nonlinear operators associated with Decomposition I

$$\begin{gathered}
F(v) = A(v) + B(v), \\
A(v) = \begin{pmatrix} v_2 \\ \tilde{\alpha}(v_2)\Delta v_2 \end{pmatrix}, \quad B(v) = \begin{pmatrix} 0 \\ \tilde{\beta}(v_2)\Delta v_1 \end{pmatrix}, \\
\tilde{\alpha}(v_2) = \alpha(1 - \delta v_2)^{-1}, \quad \tilde{\beta}(v_2) = \beta(1 - \delta v_2)^{-1}.
\end{gathered}$$

Under the nondegeneracy condition $0 < \underline{v} \leqslant 1 - \delta v_2(x) \leqslant \bar{v} < \infty$ for all $x \in \overline{\Omega}$, justified by Theorem 7.1 in Section 7, the auxiliary results in (5.1) imply

$$\|\tilde{\alpha}(v_2)\|_{H^{k+2}} + \|\tilde{\beta}(v_2)\|_{H^{k+2}} \leqslant C(\|v_2\|_{H^{k+2}}), \quad k \in \mathbb{N}_{\geqslant 0},$$

and further lead to the estimates

$$\begin{aligned}
\|\tilde{\alpha}(v_2)\Delta v_2\|_{L^2} &\leqslant C(\|v_2\|_{H^2}), \\
\|\tilde{\beta}(v_2)\Delta v_1\|_{L^2} &\leqslant C(\|v_1\|_{H^2}, \|v_2\|_{H^2}), \\
\|\tilde{\alpha}(v_2)\Delta v_2\|_{H^{k+2}} &\leqslant C(\|v_2\|_{H^{k+4}}), \quad k \in \mathbb{N}_{\geqslant 0}, \\
\|\tilde{\beta}(v_2)\Delta v_1\|_{H^{k+2}} &\leqslant C(\|v_1\|_{H^{k+4}}, \|v_2\|_{H^{k+2}}), \quad k \in \mathbb{N}_{\geqslant 0},
\end{aligned}$$

with constants $C(\cdot)$ depending on bounds for the arising norms of the solution components v_1, v_2.

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Fréchet derivatives. The Fréchet derivatives of the nonlinear operators associated with Decomposition I are given by

$$F'(v) = A'(v) + B'(v),$$
$$A'(v) = \begin{pmatrix} 0 & I \\ 0 & \tilde{\alpha}(v_2)\Delta + \tilde{\alpha}'(v_2)\Delta v_2 \end{pmatrix}, \quad B'(v) = \begin{pmatrix} 0 & 0 \\ \tilde{\beta}(v_2)\Delta & \tilde{\beta}'(v_2)\Delta v_1 \end{pmatrix},$$
$$\tilde{\alpha}'(v_2) = \alpha\delta(1-\delta v_2)^{-2} = \frac{\delta}{\alpha}(\tilde{\alpha}(v_2))^2, \quad \tilde{\beta}'(v_2) = \beta\delta(1-\delta v_2)^{-2} = \frac{\delta}{\beta}(\tilde{\beta}(v_2))^2;$$

more precisely, application to an element $w = (w_1, w_2)$ yields

$$A'(v)w = \begin{pmatrix} w_2 \\ \tilde{\alpha}(v_2)\Delta w_2 + \tilde{\alpha}'(v_2)\Delta v_2 w_2 \end{pmatrix},$$
$$B'(v)w = \begin{pmatrix} 0 \\ \tilde{\beta}(v_2)\Delta w_1 + \tilde{\beta}'(v_2)\Delta v_1 w_2 \end{pmatrix}.$$

Lie-commutator. In order to determine the first Lie-commutator

$$[A,B](v) = A'(v)B(v) - B'(v)A(v),$$

we employ the auxiliary relations

$$\nabla\tilde{\beta}(v_2) = \tilde{\beta}'(v_2)\nabla v_2,$$
$$\Delta\tilde{\beta}(v_2) = \tilde{\beta}''(v_2)\nabla v_2 \cdot \nabla v_2 + \tilde{\beta}'(v_2)\Delta v_2,$$
$$\Delta(\tilde{\beta}(v_2)\Delta v_1) = \Delta\tilde{\beta}(v_2)\Delta v_1 + 2\nabla\tilde{\beta}(v_2)\cdot\nabla\Delta v_1 + \tilde{\beta}(v_2)\Delta^2 v_1,$$
$$\tilde{\beta}''(v_2) = 2\beta\delta^2(1-\delta v_2)^{-3} = 2\left(\frac{\delta}{\beta}\right)^2(\tilde{\beta}(v_2))^3.$$

Noting that the identity $\tilde{\alpha}'(v_2)\tilde{\beta}(v_2) - \tilde{\alpha}(v_2)\tilde{\beta}'(v_2) = 0$ holds, a brief calculation yields

$$\begin{aligned}
[A,B](v) &= \begin{pmatrix} 0 & I \\ 0 & \tilde{\alpha}'(v_2)\Delta v_2 + \tilde{\alpha}(v_2)\Delta \end{pmatrix}\begin{pmatrix} 0 \\ \tilde{\beta}(v_2)\Delta v_1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \tilde{\beta}(v_2)\Delta & \tilde{\beta}'(v_2)\Delta v_1 \end{pmatrix}\begin{pmatrix} v_2 \\ \tilde{\alpha}(v_2)\Delta v_2 \end{pmatrix} \\
&= \begin{pmatrix} \tilde{\beta}(v_2)\Delta v_1 \\ (\tilde{\alpha}'(v_2)\tilde{\beta}(v_2) - \tilde{\alpha}(v_2)\tilde{\beta}'(v_2))\Delta v_1\Delta v_2 + \tilde{\alpha}(v_2)\Delta(\tilde{\beta}(v_2)\Delta v_1) - \tilde{\beta}(v_2)\Delta v_2 \end{pmatrix} \\
&= \begin{pmatrix} \tilde{\beta}(v_2)\Delta v_1 \\ \tilde{\alpha}(v_2)\Delta(\tilde{\beta}(v_2)\Delta v_1) - \tilde{\beta}(v_2)\Delta v_2 \end{pmatrix} \\
&= \begin{pmatrix} \tilde{\beta}(v_2)\Delta v_1 \\ \tilde{\alpha}(v_2)(\Delta\tilde{\beta}(v_2)\Delta v_1 + 2\nabla\tilde{\beta}(v_2)\cdot\nabla\Delta v_1 + \tilde{\beta}(v_2)\Delta^2 v_1) - \tilde{\beta}(v_2)\Delta v_2 \end{pmatrix},
\end{aligned}$$

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that is, we have

$$[A,B](v) = \begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}, \quad \zeta_1 = \tilde{\beta}(v_2)\Delta v_1,$$
$$\zeta_2 = \tilde{\alpha}(v_2)(\Delta\tilde{\beta}(v_2)\Delta v_1 + 2\nabla\tilde{\beta}(v_2)\cdot\nabla\Delta v_1 + \tilde{\beta}(v_2)\Delta^2 v_1) - \tilde{\beta}(v_2)\Delta v_2.$$

As a consequence, by means of the relations

$$\begin{aligned}
\|\zeta_1\|_{L^2} &\leqslant C(\|v_1\|_{H^2}, \|v_2\|_{H^2}),\\
\|\zeta_2\|_{L^2} &\leqslant C(\|v_1\|_{H^4}, \|v_2\|_{H^2}),\\
\|\zeta_1\|_{H^{k+2}} &\leqslant C(\|v_1\|_{H^{k+4}}, \|v_2\|_{H^{k+2}}), \quad k\in\mathbb{N}_{\geqslant 0},\\
\|\zeta_2\|_{H^{k+2}} &\leqslant C(\|v_1\|_{H^{k+6}}, \|v_2\|_{H^{k+4}}), \quad k\in\mathbb{N}_{\geqslant 0},
\end{aligned}$$

the following estimate is obtained:

$$\|[A,B](v)\|_{H^k\times H^k} + \|[A,B](v)\|_{H^{k+2}\times H^k} \leqslant C(\|v\|_{H^{k+4}\times H^{k+2}}), \quad k\in\mathbb{N}_{\geqslant 0},$$

see also (5.1); the special case $k=1$ is included by interpolation, see Adams & Fournier (2003, Theorem 7.23).

5.2 *Estimates for evolution operators*

In the following, we state estimates for the evolution operators associated with the Westervelt equation and the subproblems corresponding to Decomposition I as well as their derivatives, see (2.1), (2.2) and (3.2). For this purpose, we make use of the fact that the Fréchet derivative of the evolution operator $\mathscr{E}_G(\cdot, v)$ with respect to the initial state v, denoted by $\partial_2\mathscr{E}_G(\cdot, v)$, is a solution to the variational equation

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}\partial_2\mathscr{E}_G(t,v) = G'(\mathscr{E}_G(t,v))\partial_2\mathscr{E}_G(t,v), & t\in(0,T],\\ \partial_2\mathscr{E}_G(0,v) = I.\end{cases}$$

For the convenience of the reader, we collect the relations

$$\begin{aligned}
&F(v) = A(v) + B(v), \quad F'(v) = A'(v) + B'(v),\\
&A(v) = \begin{pmatrix} v_2\\ \tilde{\alpha}(v_2)\Delta v_2\end{pmatrix}, \quad A'(v) = \begin{pmatrix}0 & I\\ 0 & \tilde{\alpha}(v_2)\Delta + \tilde{\alpha}'(v_2)\Delta v_2\end{pmatrix},\\
&B(v) = \begin{pmatrix}0\\ \tilde{\beta}(v_2)\Delta v_1\end{pmatrix}, \quad B'(v) = \begin{pmatrix}0 & 0\\ \tilde{\beta}(v_2)\Delta & \tilde{\beta}'(v_2)\Delta v_1\end{pmatrix},\\
&\tilde{\alpha}(v_2) = \alpha(1-\delta v_2)^{-1}, \quad \tilde{\alpha}'(v_2) = \alpha\delta(1-\delta v_2)^{-2},\\
&\tilde{\beta}(v_2) = \beta(1-\delta v_2)^{-1}, \quad \tilde{\beta}'(v_2) = \beta\delta(1-\delta v_2)^{-2},
\end{aligned}$$

see also Section 5.1. We point out that suitable applications of the linear variation-of-constants formula and Gronwall's lemma imply that the constants arising in the estimates for the evolution operators are of the special form e^{Ct}, ensuring that repeated applications of the evolution operators remain bounded; this

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is crucial in view of the stability estimate (4.2b). For simplicity, we do not distinguish an abstract function $v = (v_1, v_2)$ and the associated function $v : \overline{\Omega} \times [0, T] \to \mathbb{R}^2 : (x, t) \mapsto v(x, t) = v(t)(x)$ in notation.

Estimates for evolution operators. Let $t \in [0, T]$.

(i) (a) In the situation of Theorem 7.1 in Section 7, the evolution operator $\mathscr{E}_F$ satisfies

$$\|\mathscr{E}_F(t, v)\|_{H^{k+6}\times H^{k+5}} \leqslant e^{Ct}\|v\|_{H^{k+6}\times H^{k+5}}, \quad k \in \mathbb{N}_{\geqslant 0}.$$

(b) For the associated derivative with respect to the initial state, the estimate

$$\|\partial_2\mathscr{E}_F(t, v)w\|_{H^{\ell+1}\times H^{\ell}} \leqslant e^{C(\|v\|_{H^4\times H^4})t}\|w\|_{H^{\ell+1}\times H^{\ell}}, \quad \ell = 0, 1, 2, 3,$$

is valid.

Proof. (a) The stated estimate follows from Theorem 7.1 and interpolation, see also Adams & Fournier (2003, Theorem 7.23).

(b) We denote by $v(t) = \mathscr{E}_F(t, v(0))$ the evolution operator associated with F and by $V(t) = \partial_2\mathscr{E}_F(t, v(0))$ the corresponding Fréchet derivative with respect to the initial state, which satisfies the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}V(t) = F'(v(t))V(t), & t \in (0, T], \\ V(0) = I; \end{cases}$$

due to linearity, the solution to the variational equation is given by a relation of the form

$$\partial_2\mathscr{E}_F(t, v(0))w = V(t)w = e^{D(t,0)}w, \quad t \in (0, T].$$

More precisely, using that

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \quad F' = \begin{pmatrix} 0 & I \\ F'_{21} & F'_{22} \end{pmatrix},$$
$$F'_{21}(v) = \tilde{\beta}(v_2)\Delta, \quad F'_{22}(v) = \tilde{\alpha}(v_2)\Delta + \tilde{\alpha}'(v_2)\Delta v_2 + \tilde{\beta}'(v_2)\Delta v_1,$$

we obtain the two decoupled systems

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}V_{11}(t) = V_{21}(t), & t \in (0, T], \\ \dfrac{\mathrm{d}}{\mathrm{d}t}V_{21}(t) = F'_{21}(v(t))V_{11}(t) + F'_{22}(v(t))V_{21}(t), & t \in (0, T], \\ V_{11}(0) = I, \quad V_{21}(0) = 0, \end{cases}$$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}V_{12}(t) = V_{22}(t), & t \in (0, T], \\ \dfrac{\mathrm{d}}{\mathrm{d}t}V_{22}(t) = F'_{21}(v(t))V_{12}(t) + F'_{22}(v(t))V_{22}(t), & t \in (0, T], \\ V_{12}(0) = 0, \quad V_{22}(0) = I, \end{cases}$$

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which correspond to the strongly damped linear wave equations

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}t^2} V_{11}(t) = F'_{21}(v(t))V_{11}(t) + F'_{22}(v(t))\frac{\mathrm{d}}{\mathrm{d}t}V_{11}(t), & t \in (0,T], \\ V_{11}(0) = I, \quad \frac{\mathrm{d}}{\mathrm{d}t}V_{11}(0) = V_{21}(0) = 0, \end{cases}$$

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}t^2} V_{12}(t) = F'_{21}(v(t))V_{12}(t) + F'_{22}(v(t))\frac{\mathrm{d}}{\mathrm{d}t}V_{12}(t), & t \in (0,T], \\ V_{12}(0) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}V_{12}(0) = V_{22}(0) = I. \end{cases}$$

Both equations can be cast into the form (7.4) with coefficient functions

$$a = \tilde{\alpha}(v_2), \quad b = \tilde{\alpha}'(v_2)\Delta v_2 + \tilde{\beta}'(v_2)\Delta v_1, \quad c = \tilde{\beta}(v_2), \quad f = 0.$$

We note that the assumptions of Proposition 7.2 are verified as follows: From Theorem 7.1(i) we have

$$0 < \underline{v} \leqslant 1 - \delta v_2(t) \leqslant \bar{v} < \infty, \quad t \in [0,T],$$

$$(v_1, v_2) \in \mathscr{C}([0,T], H^4(\Omega) \times H^4(\Omega)), \quad \partial_t v_2 \in \mathscr{C}([0,T], H^2(\Omega)),$$

$$\|(v_1(t), v_2(t), \partial_t v_2(t))\|_{H^4 \times H^4 \times H^2} \leqslant e^{Ct}\|(v_1(0), v_2(0))\|_{H^4 \times H^4}, \quad t \in [0,T],$$

where the first relation is understood pointwise for $x \in \overline{\Omega}$; these estimates can be made use of in

$$a(t) \geqslant \alpha \underline{v} > 0, \quad c(t) \geqslant \beta \underline{v} > 0, \quad t \in [0,T],$$

$$\|b\|_{L^2((0,T),L^3)} \leqslant \sqrt{T}\|b\|_{\mathscr{C}((0,T),H^1)} \leqslant \sqrt{T}C(\|(v_1,v_2)\|_{\mathscr{C}((0,T),H^3 \times H^3)}),$$

$$\|\partial_t c\|_{L^2((0,T),L^\infty)} \leqslant \sqrt{T}\|\partial_t c\|_{\mathscr{C}([0,T],L^\infty)} \leqslant \sqrt{T}C(\|\partial_t v_2\|_{\mathscr{C}([0,T],H^2)}).$$

Thus, Proposition 7.2 implies

$$\begin{aligned} \|V(t)w\|_{H^{\ell+1} \times H^\ell} &\leqslant e^{C(\|v\|_{\mathscr{C}([0,t],H^4 \times H^4)} + \|\partial_t v_2\|_{\mathscr{C}([0,t],H^2)})}\|w\|_{H^{\ell+1} \times H^\ell} \\ &\leqslant e^{C(\|v(0)\|_{H^4 \times H^4})t}\|w\|_{H^{\ell+1} \times H^\ell}, \quad \ell = 0,1,2,3, \end{aligned}$$

which proves the stated result. □

(ii) (a) Arguments based on the regularity result Evans (2010, Theorem 6, p. 386) for linear evolution equations of parabolic type imply

$$\|\mathscr{E}_A(t,v)\|_{H^{k+4} \times H^{k+2}} \leqslant e^{Ct}\|v\|_{H^{k+4} \times H^{k+2}}, \quad k \in \mathbb{N}_{\geqslant 0}.$$

(b) For any $k \in \mathbb{N}_{\geqslant 0}$ the estimate

$$\|\partial_2 \mathscr{E}_A(t,v)w\|_{H^k \times H^\ell} \leqslant \begin{cases} e^{C(\|v\|_{H^5 \times H^3})t}\|w\|_{H^k \times H^\ell}, & \ell = 0,1,2, \\ e^{C(\|v\|_{H^7 \times H^5})t}\|w\|_{H^k \times H^\ell}, & \ell \in \mathbb{N}_{\geqslant 3} \end{cases}$$

is valid.

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Proof. (a) For the following, we set $v(t) = \mathscr{E}_A(t, v(0))$. The second component of the solution, governed by the nonlinear diffusion equation

$$\partial_t v_2(t) = \alpha(1 - \delta v_2(t))^{-1} \Delta v_2(t)$$

inherites the regularity of the prescribed initial state; more precisely, an application of Proposition 7.3 with

$$a = \alpha(1 - \delta v_2)^{-1}, \quad b = 0, \quad f = 0,$$

see also Evans (2010, Theorem 6, p. 386), a fixed point argument, and Gronwall's lemma imply

$$\|v_2(t)\|_{H^{k+2}} \leqslant e^{Ct} \|v_2(0)\|_{H^{k+2}}, \quad k \in \mathbb{N}_{\geqslant 0}.$$

The regularity of the second component carries over to the first component

$$v_1(t) = v_1(0) + \int_0^t v_2(\tau)\, \mathrm{d}\tau,$$

and we thus obtain the estimate

$$\|v(t)\|_{H^{k+2} \times H^{k+2}} \leqslant e^{Ct} \|v(0)\|_{H^{k+2} \times H^{k+2}}, \quad k \in \mathbb{N}_{\geqslant 0}.$$

Furthermore, employing the identity

$$\partial_t \left(-\frac{1}{2\delta}(1 - \delta v_2(t))^2 \right) = (1 - \delta v_2(t)) \partial_t v_2(t) = \alpha \Delta v_2(t),$$

and performing integration

$$\begin{aligned} \frac{1}{2\delta}((1 - \delta v_2(0))^2 - (1 - \delta v_2(t))^2) &= \alpha \Delta \int_0^t v_2(\tau)\, \mathrm{d}\tau \\ &= \alpha \Delta (v_1(t) - v_1(0)), \end{aligned}$$

leads to the relation

$$\Delta v_1(t) = \Delta v_1(0) + \frac{1}{2\alpha\delta}((1 - \delta v_2(0))^2 - (1 - \delta v_2(t))^2).$$

Making use of the fact that $H^{k+2}(\Omega)$ forms an algebra for any $k \in \mathbb{N}_{\geqslant 0}$ leads to

$$\|v(t)\|_{H^{k+4} \times H^{k+2}} \leqslant e^{Ct} \|v(0)\|_{H^{k+4} \times H^{k+2}}, \quad k \in \mathbb{N}_{\geqslant 0},$$

and thus proves the statement.

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(b) Analogously to before, we denote $v(t) = \mathscr{E}_A(t, v(0))$ and consider the initial value problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} V(t) = A'(v(t))V(t), & t \in (0, T], \\ V(0) = I, \end{cases}$$

for $V(t) = \partial_2 \mathscr{E}_A(t, v(0))$. A reformulation in components

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \quad A' = \begin{pmatrix} 0 & I \\ 0 & A'_{22} \end{pmatrix}, \quad A'_{22}(u) = \tilde{\alpha}(v_2)\Delta + \tilde{\alpha}'(v_2)\Delta v_2,$$

leads to two decoupled systems involving a linear diffusion–reaction equation with time-dependent coefficients

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} V_{11}(t) = V_{21}(t), & t \in (0, T], \quad V_{11}(0) = I, \\ \frac{\mathrm{d}}{\mathrm{d}t} V_{21}(t) = A'_{22}(u(t))V_{21}(t), & t \in (0, T], \quad V_{21}(0) = 0, \end{cases}$$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} V_{12}(t) = V_{22}(t), & t \in (0, T], \quad V_{12}(0) = 0, \\ \frac{\mathrm{d}}{\mathrm{d}t} V_{22}(t) = A'_{22}(u(t))V_{22}(t), & t \in (0, T], \quad V_{22}(0) = I. \end{cases}$$

An application of Proposition 7.3 with

$$a = \tilde{\alpha}(v_2), \quad b = \tilde{\alpha}'(v_2)\Delta v_2, \quad f = 0,$$

leads to the following estimates:

$$\|V(t)w\|_{H^k \times H^\ell} \leqslant e^{C(\|v_2\|_{\mathscr{C}([0,t],H^3)})} \|w\|_{H^k \times H^\ell}, \quad k \in \mathbb{N}_{\geqslant 0}, \quad \ell = 0, 1, 2,$$
$$\|V(t)w\|_{H^k \times H^\ell} \leqslant e^{C(\|v_2\|_{\mathscr{C}([0,t],H^5)})} \|w\|_{H^k \times H^\ell}, \quad k \in \mathbb{N}_{\geqslant 0}, \quad \ell \in \mathbb{N}_{\geqslant 3},$$

which imply the stated result. □

(iii) (a) The evolution operator associated with B fulfils

$$\|\mathscr{E}_B(t, v)\|_{H^{k+2} \times H^k} \leqslant e^{Ct} \|v\|_{H^{k+2} \times H^k}, \quad k \in \mathbb{N}_{\geqslant 0}.$$

(b) The estimate

$$\|\partial_2 \mathscr{E}_B(t, v)w\|_{H^{k+2} \times H^k} \leqslant e^{C(\|v\|_{H^{k+4} \times H^{k+2}})t} \|w\|_{H^{k+2} \times H^k}, \quad k \in \mathbb{N}_{\geqslant 0},$$

is valid.

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Proof. (a) Let $v(t) = \mathscr{E}_B(t, v(0))$. The first component of the solution remains constant in time

$$v_1(t) = v_1(0);$$

using that the second component is given by the explicit representation

$$v_2(t) = \frac{1}{\delta}(1 - \sqrt{(1 - \delta v_2(0))^2 - 2\beta\delta t \Delta v_1(0)}),$$

the stated result follows.

(b) Differentiation of the explicit solution representation

$$v(t) = \begin{pmatrix} v_1(0) \\ \frac{1}{\delta}(1 - \sqrt{(1 - \delta v_2(0))^2 - 2\beta\delta t \Delta v_1(0)}) \end{pmatrix},$$

with respect to the initial state yields

$$\begin{aligned} V(t)w &= \begin{pmatrix} I & 0 \\ \dfrac{\beta t}{\sqrt{(1 - \delta v_2(0))^2 - 2\beta\delta t \Delta v_1(0)}}\Delta & \dfrac{1 - \delta v_2(0)}{\sqrt{(1 - \delta v_2(0))^2 - 2\beta\delta t \Delta v_1(0)}} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= \begin{pmatrix} w_1 \\ \dfrac{1}{\sqrt{(1 - \delta v_2(0))^2 - 2\beta\delta t \Delta v_1(0)}}(\beta t \Delta w_1 + (1 - \delta v_2(0))w_2) \end{pmatrix}, \end{aligned}$$

where $V(t) = \partial_2 \mathscr{E}_B(t, v(0))$. Suitable estimation proves the statement. □

5.3 *Main result*

In the following, we deduce a global error estimate for the Lie–Trotter splitting method applied to the Westervelt equation. Under the assumption that the prescribed initial state fulfils the regularity requirements

$$\begin{aligned} u(0) = (\psi(\cdot, 0), \partial_t \psi(\cdot, 0)) \in H^6(\Omega) \times H^5(\Omega), \\ \|u(0)\|_{H^6 \times H^5} = \|\psi(\cdot, 0)\|_{H^6} + \|\partial_t \psi(\cdot, 0)\|_{H^5} \leqslant C_0, \end{aligned} \tag{5.2}$$

with suitably chosen constant $C_0 > 0$ as well as certain compatibility conditions, Theorem 7.1 given in Section 7 guarantees that the solution to the Westervelt equation (2.1) and (2.2) satisfies

$$u(t) = (\psi(\cdot, t), \partial_t \psi(\cdot, t)) \in H^6(\Omega) \times H^5(\Omega), \quad t \in [0, T],$$

and remains bounded. As discussed in Section 4, the extension of the global error estimate to variable time stepsizes is straightforward.

THEOREM 5.1 (Lie–Trotter splitting method, Decomposition I) Assume that the initial state fulfils the regularity requirement (5.2) and that the initial approximation $u_0 \approx u(0)$ remains bounded in $H^5(\Omega) \times H^3(\Omega)$. Then the Lie–Trotter splitting method applied to the Westervelt equation (2.1) and (2.2) satisfies

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the global error estimate

$$\|u_N - u(t_N)\|_{H^3\times H^1} \leqslant C(\|u_0 - u(0)\|_{H^3\times H^1} + h), \quad 0 \leqslant t_N = Nh \leqslant T,$$

with constant depending on bounds for $\|u\|_{\mathscr{C}([0,t_N],\, H^6\times H^5)}$ as well as $\|u_0\|_{H^5\times H^3}$ and the final time t_N.

Proof. Let $t \in [0,T]$ and $0 \leqslant t_N = Nh \leqslant T$. In order to deduce the stated global error estimate for the Lie–Trotter splitting method based on Decomposition I, we follow the approach described in Section 4. For convenience, we collect the fundamental auxiliary results deduced in Sections 5.1 and 5.2, adapted to the present situation

$$\|\mathscr{E}_F(t,v)\|_{H^{k+6}\times H^{k+5}} \leqslant e^{Ct}\|v\|_{H^{k+6}\times H^{k+5}}, \quad k \in \mathbb{N}_{\geqslant 0}, \tag{5.3a}$$

$$\|\mathscr{E}_A(t,v)\|_{H^{k+4}\times H^{k+2}} \leqslant e^{Ct}\|v\|_{H^{k+4}\times H^{k+2}}, \quad k \in \mathbb{N}_{\geqslant 0}, \tag{5.3b}$$

$$\|\mathscr{E}_B(t,v)\|_{H^{k+2}\times H^{k}} \leqslant e^{Ct}\|v\|_{H^{k+2}\times H^{k}}, \quad k \in \mathbb{N}_{\geqslant 0}, \tag{5.3c}$$

$$\|\partial_2\mathscr{E}_F(t,v)w\|_{H^{\ell+1}\times H^{\ell}} \leqslant e^{C(\|v\|_{H^4\times H^4})t}\|w\|_{H^{\ell+1}\times H^{\ell}}, \quad \ell = 0,1,2,3, \tag{5.3d}$$

$$\|\partial_2\mathscr{E}_A(t,v)w\|_{H^{k+2}\times H^{k}} \leqslant \begin{cases} e^{C(\|v\|_{H^5\times H^3})t}\|w\|_{H^{k+2}\times H^{k}}, & k = 0,1,2, \\ e^{C(\|v\|_{H^7\times H^5})t}\|w\|_{H^{k+2}\times H^{k}}, & k \in \mathbb{N}_{\geqslant 3}, \end{cases} \tag{5.3e}$$

$$\|\partial_2\mathscr{E}_B(t,v)w\|_{H^{k+2}\times H^{k}} \leqslant e^{C(\|v\|_{H^{k+4}\times H^{k+2}})t}\|w\|_{H^{k+2}\times H^{k}}, \quad k \in \mathbb{N}_{\geqslant 0}, \tag{5.3f}$$

$$\|[A,B](v)\|_{H^{k+2}\times H^{k}} \leqslant C(\|v\|_{H^{k+4}\times H^{k+2}}), \quad k \in \mathbb{N}_{\geqslant 0}. \tag{5.3g}$$

(i) The basic regularity assumptions on the initial state and the initial numerical approximation

$$u(0) \in H^6(\Omega) \times H^5(\Omega), \quad u_0 \in H^5(\Omega) \times H^3(\Omega),$$

ensure that the exact and numerical solution values, given by

$$u(t) = \mathscr{E}_F(t, u(0)),$$
$$u_N = \mathscr{S}_F^N(h, u_0), \quad \mathscr{S}_F(h,v) = \mathscr{E}_B(h, \mathscr{E}_A(h,v)),$$

remain in the underlying product spaces; indeed, due to (5.3a) with $k = 0$ as well as (5.3b) and (5.3c) with $k = 1$ and $k = 3$, respectively, we obtain

$$\begin{aligned} \|\mathscr{E}_F(t,u(0))\|_{H^6\times H^5} &\leqslant e^{Ct}\|u(0)\|_{H^6\times H^5}, \\ \|\mathscr{S}_F^n(h,u_0)\|_{H^5\times H^3} &\leqslant e^{Ct_n}\|u_0\|_{H^5\times H^3}, \end{aligned} \tag{5.4}$$

for integer $1 \leqslant n \leqslant N$. We note that the constant arising in the estimation of the splitting operator

$$\begin{aligned} \|\mathscr{S}_F(h,v)\|_{H^5\times H^3} = \|\mathscr{E}_B(h,\mathscr{E}_A(h,v))\|_{H^5\times H^3} &\leqslant e^{Ch}\|\mathscr{E}_A(h,v)\|_{H^5\times H^3} \\ &\leqslant e^{Ch}\|v\|_{H^5\times H^3} \end{aligned}$$

is of the special form e^{Ch}, and thus repeated applications of the splitting method lead to the stated bound.

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(ii) In order to establish a stability estimate, see also (4.2b), we employ the expansion

$$\begin{aligned}\mathscr{E}_A(h,v) - \mathscr{E}_A(h,\tilde v) &= \mathscr{E}_A(h,\sigma_1 v + (1-\sigma_1)\tilde v)|_{\sigma_1=0}^1 \\ &= \int_0^1 \partial_2 \mathscr{E}_A(h,\sigma_1 v + (1-\sigma_1)\tilde v)\,\mathrm{d}\sigma_1 (v-\tilde v),\end{aligned}$$

and insert it into the analogous relation

$$\mathscr{E}_B(h,w) - \mathscr{E}_B(h,\tilde w) = \int_0^1 \partial_2 \mathscr{E}_B(h,\sigma_2 w + (1-\sigma_2)\tilde w)\,\mathrm{d}\sigma_2\,(w-\tilde w),$$
$$w = \mathscr{E}_A(h,v), \qquad \tilde w = \mathscr{E}_A(h,\tilde v);$$

this implies the identity

$$\begin{aligned}\mathscr{S}_F(h,v) - \mathscr{S}_F(h,\tilde v) &= \mathscr{E}_B(h,w) - \mathscr{E}_B(h,\tilde w) \\ &= \int_0^1 \partial_2 \mathscr{E}_B(h,\sigma_2 \mathscr{E}_A(t,v) + (1-\sigma_2)\mathscr{E}_A(t,\tilde v))\,\mathrm{d}\sigma_2 \\ &\quad\times \int_0^1 \partial_2 \mathscr{E}_A(h,\sigma_1 v + (1-\sigma_1)\tilde v)\,\mathrm{d}\sigma_1 (v-\tilde v).\end{aligned}$$

As a consequence, by (5.3f), a further application of (5.3b), and (5.3e) the estimates

$$\begin{aligned}k=0,1,2: \quad &\|\mathscr{S}_F(h,v) - \mathscr{S}_F(h,\tilde v)\|_{H^{k+2}\times H^k} \\ &\leqslant e^{C(\|v\|_{H^5\times H^3},\|v\|_{H^{k+4}\times H^{k+2}},\|\tilde v\|_{H^5\times H^3},\|\tilde v\|_{H^{k+4}\times H^{k+2}})h}\|v-\tilde v\|_{H^{k+2}\times H^k}, \\ k\geqslant 3: \quad &\|\mathscr{S}_F(h,v) - \mathscr{S}_F(h,\tilde v)\|_{H^{k+2}\times H^k} \\ &\leqslant e^{C(\|v\|_{H^7\times H^5},\|v\|_{H^{k+4}\times H^{k+2}},\|\tilde v\|_{H^7\times H^5},\|\tilde v\|_{H^{k+4}\times H^{k+2}})h}\|v-\tilde v\|_{H^{k+2}\times H^k},\end{aligned}$$

follow. This suggest the choice $k=1$ and implies the stability bound

$$\|\mathscr{S}_F^n(h,v) - \mathscr{S}_F^n(h,\tilde v)\|_{H^3\times H^1} \leqslant e^{C(\|v\|_{H^5\times H^3},\|\tilde v\|_{H^5\times H^3})t_n}\|v-\tilde v\|_{H^3\times H^1}.$$

(iii) Applying the above considerations to the global error representation (4.1a) yields

$$\begin{aligned}\|u_N - u(t_N)\|_{H^3\times H^1} &\leqslant \|\mathscr{S}_F^N(h,u_0) - \mathscr{S}_F^N(h,u(0))\|_{H^3\times H^1} \\ &\quad + \sum_{n=0}^{N-1} \|\mathscr{S}_F^{N-n-1}(h,\mathscr{L}_F(h,u(t_n)) + \mathscr{E}_F(h,u(t_n))) \\ &\quad - \mathscr{S}_F^{N-n-1}(h,\mathscr{E}_F(h,u(t_n)))\|_{H^3\times H^1} \\ &\leqslant C\left(\|u_0-u(0)\|_{H^3\times H^1} + \sum_{n=0}^{N-1}\|\mathscr{L}_F(h,u(t_n))\|_{H^3\times H^1}\right),\end{aligned}$$

with constant depending in particular on bounds for $\|u(0)\|_{H^5\times H^3}$ and $\|u_0\|_{H^5\times H^3}$.

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(iv) It remains to estimate the local error

$$\mathscr{L}_F(h,u(t_n)) = \int_0^h \int_0^{\tau_1} \partial_2\mathscr{E}_F(h-\tau_1,\mathscr{S}_F(\tau_1,u(t_n)))\,Z_1(\tau_1,\tau_2)\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1,$$
$$Z_1(\tau_1,\tau_2) = \partial_2\mathscr{E}_B(\tau_1,w)\,(\partial_2\mathscr{E}_B(\tau_2,w))^{-1}\,Z_2(\tau_1,\tau_2),$$
$$Z_2(\tau_1,\tau_2) = [B,A](\mathscr{E}_B(\tau_2,w)), \quad w = \mathscr{E}_A(\tau_1,u(t_n)),$$

see (4.2c). For this purpose, we first apply (5.3d) with $\ell = 2$

$$\begin{aligned}\|\mathscr{L}_F(h,u(t_n))\|_{H^3\times H^1} &\leqslant \|\mathscr{L}_F(h,u(t_n))\|_{H^3\times H^2}\\ &\leqslant C\int_0^h\int_0^{\tau_1}\|\partial_2\mathscr{E}_F(h-\tau_1,\mathscr{S}_F(\tau_1,u(t_n)))Z_1(\tau_1,\tau_2)\|_{H^3\times H^2}\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1\\ &\leqslant C\int_0^h\int_0^{\tau_1}\|Z_1(\tau_1,\tau_2)\|_{H^3\times H^2}\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1\\ &\leqslant C\int_0^h\int_0^{\tau_1}\|Z_1(\tau_1,\tau_2)\|_{H^4\times H^2}\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1,\end{aligned}$$

with constant depending in particular on a bound for

$$\|\mathscr{S}_F(h,u(t_n))\|_{H^4\times H^4} \leqslant \|\mathscr{S}_F(h,u(t_n))\|_{H^6\times H^4},$$

which by (5.3b) and (5.3c) reduces to a bound for

$$\|u(t_n)\|_{H^6\times H^4} \leqslant \|u(t_n)\|_{H^6\times H^5},$$

and further by (5.3a) to a bound for $\|u(0)\|_{H^6\times H^5}$; recall also the dependence on a bound for $\|u_0\|_{H^5\times H^3}$, cf. (5.4). Noting that the identity

$$(\partial_2\mathscr{E}_B(\tau_2,w))^{-1} = \partial_2\mathscr{E}_B(-\tau_2,\mathscr{E}_B(\tau_2,w))$$

holds, we next apply (5.3f) with $k = 2$ twice

$$\begin{aligned}\|\mathscr{L}_F(h,u(t_n))\|_{H^4\times H^2} &\leqslant C\int_0^h\int_0^{\tau_1}\|\partial_2\mathscr{E}_B(\tau_1,w)\partial_2\mathscr{E}_B(-\tau_2,\mathscr{E}_B(\tau_2,w))Z_2(\tau_1,\tau_2)\|_{H^4\times H^2}\\ &\leqslant C\int_0^h\int_0^{\tau_1}\|Z_2(\tau_1,\tau_2)\|_{H^4\times H^2},\end{aligned}$$

and finally (5.3g) with $k = 2$

$$\|\mathscr{L}_F(h,u(t_n))\|_{H^4\times H^2} \leqslant C\int_0^h\int_0^{\tau_1}\|[B,A](\mathscr{E}_B(\tau_2,w))\|_{H^4\times H^2} \leqslant Ch^2;$$

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the arising constant in addition depends on

$$\|w\|_{H^6\times H^4} \leqslant C\|u(t_n)\|_{H^6\times H^4} \leqslant C\|u(0)\|_{H^6\times H^5},$$
$$\|\mathscr{E}_B(\tau,w)\|_{H^6\times H^4} \leqslant C\|w\|_{H^6\times H^4} \leqslant C\|u(0)\|_{H^6\times H^5}, \quad \tau\in[0,h].$$

Altogether, this proves the stated global error estimate

$$\|u_N - u(t_N)\|_{H^3\times H^1} \leqslant C(\|u_0 - u(0)\|_{H^3\times H^1} + h),$$

with constant depending on bounds for $\|u(0)\|_{H^6\times H^5}$, $\|u_0\|_{H^5\times H^3}$ and the final time. We note that exchanging the roles of A and B, which corresponds to the application of the second scheme in (3.4), leads to the same estimate. □

6. Numerical examples

In this section, we include a numerical example comparing the accuracy of the Lie–Trotter and Strang splitting methods for the time integration of the Westervelt equation, based on the four decompositions that were introduced in Section 3.2; furthermore, we illustrate the numerical solution obtained for a problem with more realistic parameter values. As our focus is on the time integration and in order to facilitate the numerical computations, we restrict ourselves to the Westervelt equation in a single space dimension; in particular, the spatial grid width is chosen sufficiently fine such that the global error is dominated by the time discretization error. For the numerical solution of the subproblems, we apply explicit and implicit time integration methods of the same order as the underlying splitting method; in combination with the first-order Lie–Trotter splitting method we use the explicit and implicit Euler methods, and in combination with the second-order Strang splitting method we use a second-order explicit Runge–Kutta method and the Crank–Nicolson scheme. As expected, if an explicit solver is used for the numerical solution of the subproblems, sufficiently small time stepsizes are required to avoid instabilities; for a higher number of space grid points this unstable behaviour will change for the worse. We point out that for the chosen problem data a regular solution to the Westervelt equation exists such that no order reductions, and thus no loss of accuracy is encountered when applying the Lie–Trotter and Strang splitting methods. We report that the application of the linearly implicit and semi-implicit Euler methods leads to numerical results of essentially the same accuracy as the implicit Euler method, with less computational effort. For the space discretization of the first model problem we use the Finite Difference Method with equidistant grid points, which is simple to implement, such that it is rendered possible for the reader to reproduce the numerical results; the same qualitative behaviour is expected for a space discretization by the Finite Element Method, which will be the method of choice for practically relevant problems in two and three space dimensions.

Model problem. We consider the one-dimensional Westervelt equation

$$\partial_{tt}\psi(x,t) - \alpha\partial_{xxt}\psi(x,t) - \beta\partial_{xx}\psi(x,t) = \delta\partial_t\psi(x,t)\partial_{tt}\psi(x,t), \quad (x,t)\in(-a,a)\times(0,T], \tag{6.1a}$$

with parameter values

$$\alpha = 1, \quad \beta = 1, \quad \gamma = \frac{1}{2}, \quad \delta = 2\gamma = 1, \tag{6.1b}$$

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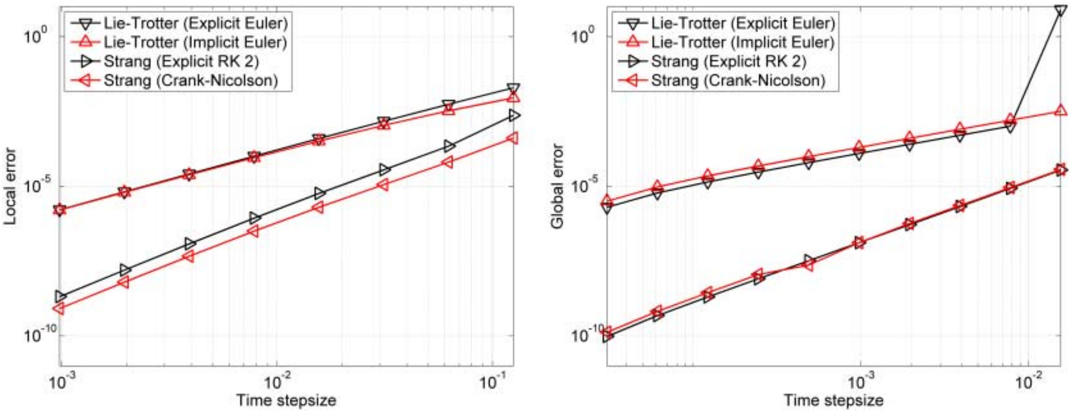

FIG. 1. Local (left) and global (right) errors for the Lie–Trotter and Strang splitting methods with respect to the $L^2 \times L^2$-norm obtained for Decomposition I. Comparison of different time integration methods for the numerical solution of the subproblems.

subject to homogeneous Dirichlet conditions

$$\psi(-a,t) = 0 = \psi(a,t), \quad t \in [0,T], \tag{6.1c}$$

and regular initial conditions

$$\psi(x,0) = e^{-x^2}, \quad \partial_t \psi(x,0) = -x\, e^{-x^2}, \quad x \in [-a,a]. \tag{6.1d}$$

For the numerical computations we set $a = 8$, $T = 1$, and choose $M = 100$ equidistant space grid points. In Figs 1 and 2 we display the local and global errors with respect to the $L^2 \times L^2$-norm, obtained for the Lie–Trotter and the Stang splitting methods based on Decompositions I–IV. Accordingly to Theorem 5.1, for Decomposition I we also include the errors with respect to a stronger norm, the $H^3 \times H^1$-norm, see Fig. 3. In all cases, the slopes of the lines reflect the convergence orders $p = 1$ for the Lie–Trotter splitting method and $p = 2$ for the Strang splitting method, respectively. From the numerical results, we conclude that the application of Decompositions II–IV does not improve the size of the errors; for this reason, we favour Decomposition I with the least computational effort.

Realistic parameter values. As a further illustration, we consider the one-dimensional Westervelt equation with more realistic parameter, geometry and excitation values

$$a = 15, \quad \alpha = 10^{-2}, \quad c = 10^3, \quad \beta = c^2 = 10^6, \quad \delta = 2 \times 10^{-4}, \tag{6.2}$$

$$\psi(x,0) = 0.5\, e^{-10(x-1)^2}, \quad x \in [-a,a], \tag{6.3}$$

$$\partial_t \psi(x,0) = c \partial_x \psi(x,0) = -10c(x-1)\, e^{-10(x-1)^2}, \quad x \in [-a,a], \tag{6.4}$$

in the MKS system of units. For the time discretization we apply the Lie–Trotter splitting method, based on Decomposition I and the explicit Euler method, and for the space discretization we use Fast Fourier techniques. In Fig. 4, we display the numerical result for the acoustic velocity potential ψ at time $t = 5 \times 10^{-3}$, obtained for $M = 6 \times 10^3$ space grid points and $N = 5 \times 10^4$ time steps. In order to reveal the effect of nonlinearity, we also display the profile of the solution to the linear wave equation $\partial_{tt}\psi - c^2 \Delta\, \psi = 0$; as typical in the nonlinear case, a slight one-sided steepening of the pulse over time while the wave travels to the left is observed, see also Kaltenbacher (2007, Section 5.6.4).

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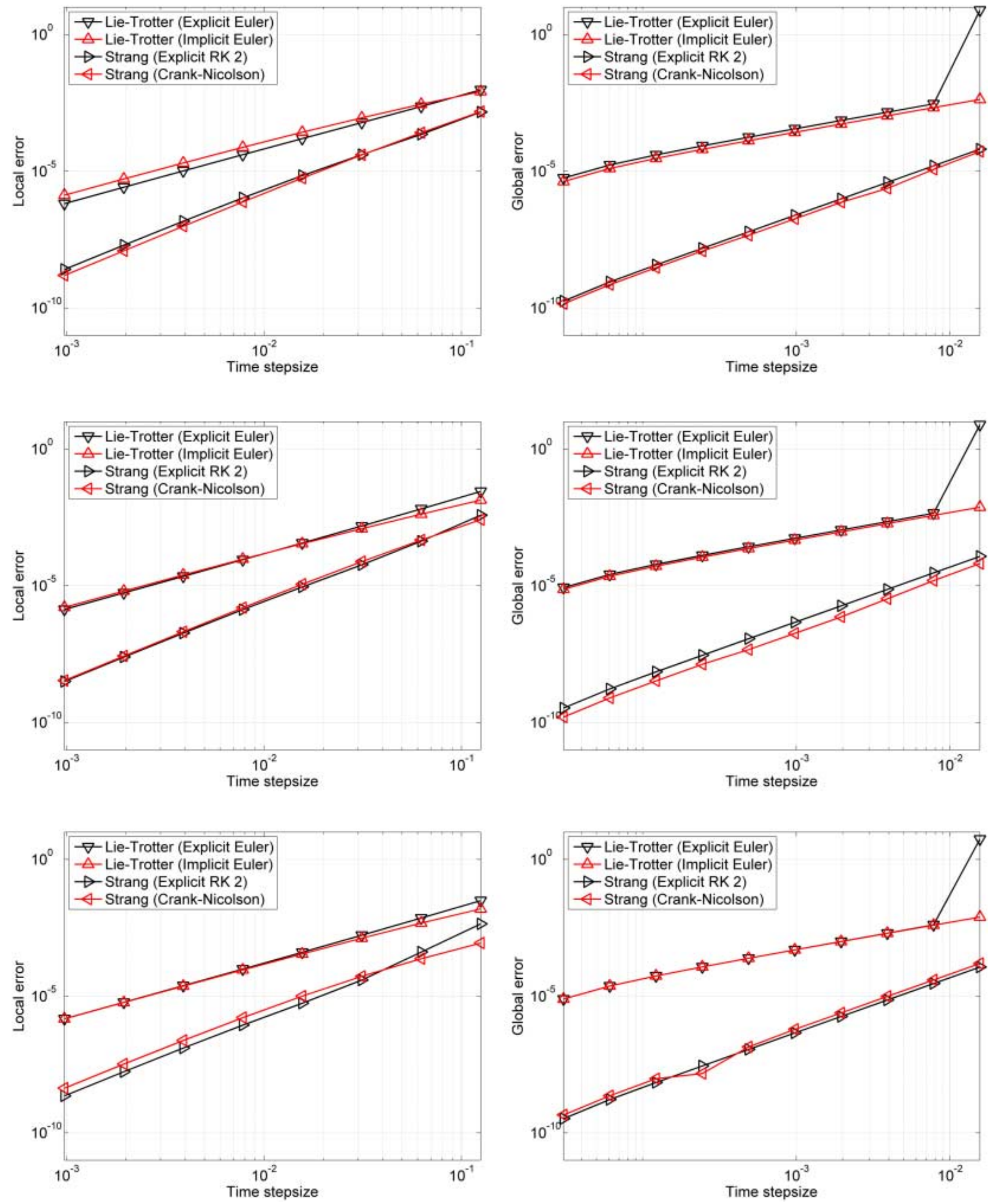

FIG. 2. Local (left) and global (right) errors for the Lie–Trotter and Strang splitting methods with respect to the $L^2 \times L^2$-norm obtained for Decomposition II (first row), Decomposition III (second row) and Decomposition IV (third row). Comparison of different time integration methods for the numerical solution of the subproblems.

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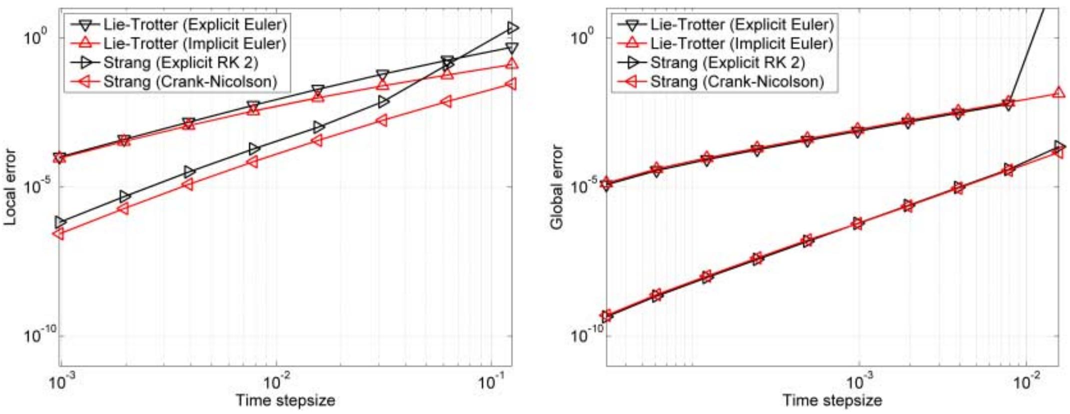

FIG. 3. Local (left) and global (right) errors for the Lie–Trotter and Strang splitting methods with respect to the $H^3 \times H^1$-norm obtained for Decomposition I. Comparison of different time integration methods for the numerical solution of the subproblems.

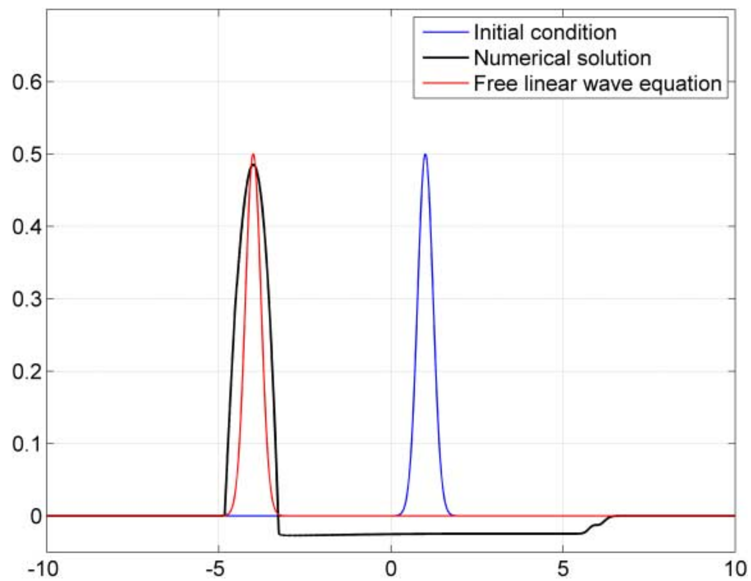

FIG. 4. Solution to the Westervelt equation with parameter values (6.2), computed by the Lie–Trotter splitting method based on Decomposition I. Comparison of the acoustic velocity potential with the solution to the linear wave equation at time 5×10^{-3}.

7. Regularity results

In this section, we state regularity results for the Westervelt equation and related evolution equations of hyperbolic and parabolic type. These results, ensuring that certain smoothness properties of the initial state are inherited by the solution to the original problem, the subproblems and the associated variational equations, respectively, are essential ingredients in our convergence analysis of operator-splitting methods for the time integration of the Westervelt equation. We study the Westervelt equation subject to homogeneous Dirichlet boundary conditions on a regular domain $\Omega \subset \mathbb{R}^3$. The results can be extended

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to other boundary conditions, in the context of the Westervelt equation, see, for instance, Clason *et al.* (2009); they literally carry over to the cases $\Omega \subset \mathbb{R}^d$ with $d = 1, 2$, where in a single space dimension the continuous embedding $H^1(\Omega) \hookrightarrow \mathscr{C}(\Omega)$ is utilized to deal with nonlinearities, replacing the continuous embedding $H^2(\Omega) \hookrightarrow \mathscr{C}(\Omega)$ for $\Omega \subset \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^3$, respectively. Our approach for the Westervelt equation utilizes a regularity result proved in Kaltenbacher & Lasiecka (2009) for a reformulation in terms of the acoustic pressure. The proofs of the stated regularity results for hyperbolic and parabolic equations are in the lines of Evans (2010, Theorem 6, p. 412) and Evans (2010, Theorem 6, p. 386), respectively. We note that estimates such as

$$\|(\psi, \partial_t\psi)\|_{\mathscr{C}([0,T],H^{k+5}\times H^{k+4})} \leqslant C, \quad k \in \mathbb{N}_{\geqslant 0},$$

are at first given for even exponents $k+5 = 2\ell$ with $\ell \in \mathbb{N}_{\geqslant 0}$; the corresponding assertions for noneven and, more generally, for noninteger exponents then follow from the exact interpolation theorem, which states that boundedness of $F: X \to Y$ and $F: \tilde{X} \to \tilde{Y}$ implies boundedness of $F: (X, \tilde{X})_{\vartheta,q} \to (Y, \tilde{Y})_{\vartheta,q}$ for $0 < \vartheta < 1$ and $1 < q < \infty$, see Adams & Fournier (2003, Theorem 7.23).

7.1 *Westervelt equation*

Initial-boundary value problem and reformulation. We consider (2.1a) under homogeneous Dirichlet boundary conditions on a regular domain

$$\begin{cases} (1-\delta\partial_t\psi(x,t))\partial_{tt}\psi(x,t) - \alpha\Delta\partial_t\psi(x,t) - \beta\Delta\psi(x,t) \\ \quad = 0, \quad (x,t) \in \Omega \times (0,T], \\ \psi(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \\ \psi(x,0) = \psi^0(x), \quad \partial_t\psi(x,0) = \psi^1(x), \quad x \in \Omega. \end{cases} \tag{7.1}$$

As a first step, we formulate the Westervelt equation in terms of the acoustic pressure, related to the acoustic velocity potential via

$$p(x,t) = \rho\partial_t\psi(x,t), \quad \partial_t\psi(x,t) = \frac{1}{\rho}p(x,t), \quad (x,t) \in \overline{\Omega} \times [0,T], \tag{7.2a}$$

for a positive constant $\rho > 0$. Differentiation of the Westervelt equation yields

$$-\delta(\partial_{tt}\psi(x,t))^2 + (1-\delta\partial_t\psi(x,t))\partial_{ttt}\psi(x,t) - \alpha\Delta\partial_{tt}\psi(x,t) - \beta\Delta\partial_t\psi(x,t) = 0;$$

substituting $\partial_t\psi = (1/\rho)p$ as well as $\partial_{tt}\psi = (1/\rho)\partial_t p$ and $\partial_{ttt}\psi = (1/\rho)\partial_{tt}p$ implies

$$-\frac{\delta}{\rho^2}(\partial_t p(x,t))^2 + \left(1 - \frac{\delta}{\rho}p(x,t)\right)\frac{1}{\rho}\partial_{tt}p(x,t) - \frac{\alpha}{\rho}\Delta\partial_t p(x,t) - \frac{\beta}{\rho}\Delta p(x,t) = 0.$$

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Altogether, we obtain the reformulation

$$\begin{cases} \left(1 - \dfrac{\delta}{\rho} p(x,t)\right) \partial_{tt} p(x,t) - \alpha \Delta \partial_t p(x,t) - \beta \Delta p(x,t) \\ \quad = \dfrac{\delta}{\rho} (\partial_t p(x,t))^2, \quad (x,t) \in \Omega \times (0,T], \\ p(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \\ p(x,0) = \rho \psi^1(x), \quad x \in \Omega, \\ \partial_t p(x,0) = \rho (1 - \delta \psi^1(x))^{-1} (\alpha \Delta \psi^1(x) + \beta \Delta \psi^0(x)), \quad x \in \Omega. \end{cases} \tag{7.2b}$$

Nondegeneracy and regularity result. Provided that the prescribed initial state satisfies the regularity assumptions

$$p(\cdot,0) \in H^2(\Omega), \quad \partial_t p(\cdot,0) \in H^1(\Omega), \quad \partial_{tt} p(\cdot,0) \in L^2(\Omega), \tag{7.3a}$$

with sufficiently small norms, the regularity result Kaltenbacher & Lasiecka (2009, Theorem 1.1) ensures nondegeneracy as well as local existence and uniqueness of a solution to a weak form of (7.2b), that is, there exist constants $0 < \underline{\nu} < \bar{\nu} < \infty$ such that the relation

$$\underline{\nu} \leqslant 1 - \frac{\delta}{\rho} p(x,t) \leqslant \bar{\nu}, \quad (x,t) \in \overline{\Omega} \times [0,T], \tag{7.3b}$$

holds and furthermore

$$p \in \mathscr{C}([0,T], H^2(\Omega)) \cap \mathscr{C}^1([0,T], H^1(\Omega)) \cap \mathscr{C}^2([0,T], L^2(\Omega)) \cap H^2((0,T), H^1(\Omega)). \tag{7.3c}$$

Regularity result. The above statement permits to deduce the following regularity result for the Westervelt equation. We note that the proof of Theorem 7.1 via Kaltenbacher & Lasiecka (2009, Theorems 1.2 and 1.3) ensures well-posedness, globally in time, that is, there exist positive constants $\rho > 0$ and $M > 0$ such that the energy norm remains bounded

$$E_{\psi,1}(t) = \frac{1}{2} (\|\partial_{ttt} \psi(\cdot,t)\|_{L^2}^2 + \|\nabla \partial_{tt} \psi(\cdot,t)\|_{L^2}^2 + \|\Delta \partial_t \psi(\cdot,t)\|_{L^2}^2) \leqslant M, \quad t \geqslant 0,$$

provided that $E_{\psi,1}(0) \leqslant \rho$; moreover, due to the strong damping present in the equation, the energy decays exponentially. However, for our purposes it suffices to employ local well-posedness and regularity.

THEOREM 7.1 Let $T > 0$ and $M > 0$ be arbitrary.

(i) There exists a constant $\rho > 0$ such that if the initial data fulfil the regularity and compatibility assumptions

$$\psi^0, \psi^1 \in H_0^1(\Omega) \cap H^4(\Omega), \quad \|(\psi^0, \psi^1)\|_{H^4 \times H^4} = \|\psi^0\|_{H^4} + \|\psi^1\|_{H^4} \leqslant \rho,$$
$$\partial_{tt} \psi(\cdot,0) = (1 - \delta \psi^1)^{-1} (\alpha \Delta \psi^1 + \beta \Delta \psi^0) \in H_0^1(\Omega) \cap H^2(\Omega),$$

then the solution to the Westervelt equation (7.1) exists and satisfies the nondegeneracy condition

$$\underline{\nu} \leqslant 1 - \delta \partial_t \psi(x,t) \leqslant \bar{\nu}, \quad (x,t) \in \overline{\Omega} \times [0,T],$$

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with some constants $0 < \underline{\nu} < \bar{\nu} < \infty$; moreover, the relation

$$\psi \in \mathscr{C}^1([0,T], H^4(\Omega)) \cap \mathscr{C}^2([0,T], H^2(\Omega)) \cap \mathscr{C}^3([0,T], L^2(\Omega)) \cap H^3((0,T), H^1(\Omega)),$$

and the bound

$$\|(\psi(\cdot,t), \partial_t\psi(\cdot,t), \partial_{tt}\psi(\cdot,t))\|_{H^4\times H^4\times H^2} \leqslant e^{Ct}\,\|(\psi^0,\psi^1)\|_{H^4\times H^4}, \quad t\in[0,T],$$

are valid.

(ii) For every $m \in \mathbb{N}_{\geqslant 2}$ there exists a constant $\rho > 0$ such that if the initial data fulfil the regularity and compatibility assumptions

$$(\psi^0,\psi^1) \in H^{2(m+1)}(\Omega) \times H^{2m+1}(\Omega), \quad \|(\psi^0,\psi^1)\|_{H^{2(m+1)}\times H^{2m+1}} \leqslant \rho,$$
$$\partial_{tt}\Delta^k\psi(\cdot,0) = (1-\delta\psi^1)^{-1}(\alpha\Delta^{k+1}\psi^1 + \beta\Delta^{k+1}\psi^0) + f_k(\cdot,0) \in H_0^1(\Omega),$$

for any integer $1 \leqslant k \leqslant m$, where

$$f_k = \Delta^k((1-\delta\partial_t\psi)^{-1}\Delta(\alpha\partial_t\psi + \beta\psi)) - (1-\delta\partial_t\psi)^{-1}\Delta^{k+1}(\alpha\partial_t\psi + \beta\psi),$$

then the solution to (7.1) satisfies the relation

$$(\psi, \partial_t\psi) \in \mathscr{C}([0,T], H^{2(m+1)}(\Omega) \times H^{2m+1}(\Omega)),$$

and the bound

$$\|(\psi(\cdot,t), \partial_t\psi(\cdot,t))\|_{H^{2(m+1)}\times H^{2m+1}} \leqslant e^{Ct}\|(\psi^0,\psi^1)\|_{H^{2(m+1)}\times H^{2m+1}}.$$

7.2 *Linear evolution equations of hyperbolic type*

Regularity result. In the following, we state a regularity result for a linear wave equation subject to homogeneous Dirichlet boundary conditions on a regular domain

$$\begin{cases} \partial_{tt}\psi(x,t) = a(x,t)\Delta\partial_t\psi(x,t) + b(x,t)\partial_t\psi(x,t) \\ \qquad\qquad +c(x,t)\Delta\psi(x,t) + f(x,t), \quad (x,t)\in\Omega\times(0,T], \\ \psi(x,t) = 0, \quad (x,t)\in\partial\Omega\times[0,T], \\ \psi(x,0) = \psi^0(x), \quad \partial_t\psi(x,0) = \psi^1(x), \quad x\in\Omega. \end{cases} \tag{7.4}$$

We note that in the present context the coefficient b may have arbitrary sign, and thus the term $b\,\partial_t\psi$ cannot be regarded as a damping term.

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PROPOSITION 7.2 Assume that the coefficient functions satisfy the positivity and regularity conditions

$$a(x,t), c(x,t) \geqslant \nu > 0, \quad (x,t) \in \Omega \times [0,T],$$
$$b \in L^2((0,T), L^3(\Omega)), \quad c \in W^{1,1}((0,T), L^\infty(\Omega)), \quad f \in L^2((0,T), L^2(\Omega)),$$

with $\|b\|_{L^2((0,T),L^3)}$ and $\|\partial_t c\|_{L^2((0,T),L^\infty)}$ sufficiently small.

(i) Then the solution to (7.4) satisfies the relation

$$(\psi, \partial_t \psi) \in \mathscr{C}([0,T], H^2(\Omega) \times H^1(\Omega))$$

and the bound

$$\|(\psi, \partial_t \psi)\|_{\mathscr{C}([0,T],H^2 \times H^1)} \leqslant C(\|(\psi^0, \psi^1)\|_{H^2 \times H^1} + \|f\|_{L^2((0,T),L^2)}),$$

with constant depending on the respective norms of b and c, but not on a.

(ii) If in addition the relations

$$a \in L^2((0,T), W^{1,\infty}(\Omega) \cap W^{2,3}(\Omega)), \quad b \in L^2((0,T), H^2(\Omega)),$$
$$c \in L^2((0,T), H^2(\Omega)) \cap H^1((0,T), L^\infty(\Omega)),$$
$$f \in \mathscr{C}([0,T], H_0^1(\Omega)) \cap L^2((0,T), H^2(\Omega)), \quad \psi^1 \in H_0^1(\Omega),$$
$$\partial_{tt}\psi(\cdot,0) = a(\cdot,0)\Delta\psi^1 + b(\cdot,0)\psi^1 + c(\cdot,0)\Delta\psi^0 + f(\cdot,0) \in H_0^1(\Omega),$$

are satisfied and the norms

$$\|a\|_{L^2((0,T),W^{1,\infty})}, \quad \|b\|_{L^2((0,T),H^1)}, \quad \|c\|_{L^2((0,T),H^2)},$$

are sufficiently small, then the solution to (7.4) satisfies the relation

$$(\psi, \partial_t \psi) \in \mathscr{C}([0,T], H^4(\Omega) \times H^3(\Omega)),$$

and the bound

$$\|(\psi, \partial_t \psi)\|_{\mathscr{C}([0,T],H^4 \times H^3)} \leqslant C(\|(\psi^0, \psi^1)\|_{H^4 \times H^3} + \|f\|_{L^2((0,T),H^2)}),$$

with constant depending on the respective norms of a, b, c. If additionally

$$b \in L^2((0,T), W^{1,\infty}(\Omega) \cap W^{2,3}(\Omega)),$$

then smallness of $\|b\|_{L^2((0,T),H^1)}$ is not required.

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7.3 *Linear evolution equations of parabolic type*

Regularity result. In the following, we state a regularity result for a linear reaction–diffusion equation subject to homogeneous Dirichlet boundary conditions on a regular domain

$$\begin{cases} \partial_t \psi(x,t) = a(x,t)\Delta\psi(x,t) + b(x,t)\psi(x,t) + f(x,t), & (x,t) \in \Omega \times (0,T], \\ \psi(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T], \\ \psi(x,0) = \psi^0(x), & x \in \Omega. \end{cases} \tag{7.5}$$

PROPOSITION 7.3 Assume that the coefficient functions satisfy the positivity and regularity conditions

$$0 < \nu \leqslant a(x,t) \leqslant \mu < \infty, \quad (x,t) \in \Omega \times [0,T],$$
$$a \in \mathscr{C}([0,T], W^{1,3}(\Omega)), \quad b \in L^2((0,T), H^1(\Omega)), \quad f \in L^2((0,T), H^1(\Omega)),$$

and that the norms of a and b are sufficiently small.

(i) Then the solution to (7.5) satisfies the relation

$$\psi \in \mathscr{C}([0,T], H^2(\Omega))$$

and the bound

$$\|\psi\|_{\mathscr{C}([0,T],H^2)} + \|\partial_t\psi\|_{L^2((0,T),H^1)} \leqslant C(\|\psi^0\|_{H^2} + \|f\|_{L^2((0,T),H^1)}),$$

with constant depending on the respective norms of a and b.

(ii) If in addition the relations

$$a \in L^2((0,T), H^3(\Omega)), \quad b \in L^2((0,T), H^3(\Omega)),$$
$$f \in \mathscr{C}([0,T], H_0^1(\Omega)) \cap L^2((0,T), H^3(\Omega)),$$
$$\partial_t\psi(\cdot,0) = a(\cdot,0)\Delta\psi^0 + b(\cdot,0)\psi^0 + f^0(\cdot,0) \in H_0^1(\Omega),$$

hold and the norms

$$\|a\|_{L^2((0,T),W^{1,\infty})}, \quad \|a\|_{L^2((0,T),W^{2,3})}, \quad \|b\|_{L^2((0,T),H^2)}$$

are sufficiently small, the solution to (7.5) satisfies the relation

$$\psi \in \mathscr{C}([0,T], H^4(\Omega))$$

and the bound

$$\|\psi\|_{\mathscr{C}([0,T],H^4)} + \|\partial_t\psi\|_{L^2((0,T),H^3)} \leqslant C(\|\psi^0\|_{H^4} + \|f\|_{L^2((0,T),H^3)}),$$

with constant depending on the respective norms of a and b.

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(ii) If for some $m \in \mathbb{N}_{\geqslant 2}$ the regularity and compatibility conditions

$$a \in L^2((0,T), H^5(\Omega)),$$
$$f^{m-1} \in \mathscr{C}([0,T], H_0^1(\Omega)) \cap L^2((0,T), H^{2m+1}(\Omega)),$$
$$\Delta^j \psi(\cdot,0) \in H_0^1(\Omega),$$
$$\partial_t \Delta^j \psi(\cdot,0) = a(\cdot,0)\Delta^{j+1}\psi^0 + b^j(\cdot,0)\Delta^j\psi^0 + f^j(\cdot,0) \in H_0^1(\Omega),$$

are fulfilled for any integer $1 \leqslant j \leqslant m$, where $b^j = b^{j-1} + \Delta a$ and

$$f^j = \Delta f^{j-1} + \Delta b^{j-1}\Delta^{j-1}\psi + 2\nabla b^{j-1}\nabla\Delta^{j-1}\psi + 2\nabla a \nabla \Delta^j \psi,$$

for $1 \leqslant j \leqslant m-1$, and if the norm of a is sufficiently small, then the solution to (7.5) satisfies the relation

$$\psi \in \mathscr{C}([0,T], H^{2(m+1)}(\Omega))$$

and the bound

$$\|\psi\|_{\mathscr{C}([0,T],H^{2(m+1)})} + \|\partial_t\psi\|_{L^2((0,T),H^{2m+1})} \leqslant C(\|\psi^0\|_{H^{2(m+1)}} + \|f\|_{L^2((0,T),H^{2m+1})}),$$

with constant depending on the respective norms of a and b.

8. Conclusions

In this work, we have introduced and investigated operator-splitting methods for the efficient time integration of the Westervelt equation modelling the propagation of high intensity ultrasound in nonlinear acoustics. We have provided numerical comparisons for the first-order Lie–Trotter and second-order Strang splitting methods based on four decompositions, using explicit and implicit solvers for the numerical solution of the subproblems. The numerical examples confirm that time-splitting methods remain stable and retain their nonstiff orders of convergence for sufficiently regular problem data. For the Lie–Trotter splitting method based on the computationally most favourable decomposition, we have carried out a rigorous stability and error analysis.

Future work shall be concerned with an extension of the error analysis to the second-order Strang splitting method, justifying the use of an adaptive time stepsize control combining the first-order Lie–Trotter and second-order Strang splitting methods. Also, it remains to analyse the effect of additional errors caused by the numerical solution of the subproblems. Furthermore, it is of interest to study absorbing boundary conditions used for tackling unbounded domains or the excitation by Neumann boundary conditions, and more advanced models of nonlinear acoustics such as Kuznetsov's equation.

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