

HIGHER-ORDER EXPONENTIAL INTEGRATORS FOR QUASI-LINEAR PARABOLIC PROBLEMS. PART I: STABILITY*

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Abstract. Explicit exponential integrators based on general linear methods are studied for the time discretization of quasi-linear parabolic initial-boundary value problems. Compared to other exponential integrators encountering rather severe order reductions, in general, the considered class of exponential general linear methods provides the possibility to construct schemes that retain higher-order accuracy in time when applied to quasi-linear parabolic problems. Employing an abstract framework, the considered problems take the form of initial value problems on Banach spaces: $u'(t) = Q(u(t))u(t)$, $t \in (0, T)$, $u(0)$ given. A fundamental requirement for the stability and error analysis is that the domains of the defining sectorial operators $Q(v) : D = D(Q(v)) \rightarrow X$ are independent of $v \in V \subset X$. The scope of applications in particular includes quasi-linear parabolic evolution equations subject to Dirichlet boundary conditions. The work is divided into two parts. In Part I, stability bounds in the norms of certain intermediate spaces between the domain D and the underlying Banach space X are deduced. In view of practical applications, the stability estimates are stated for variable time stepsizes, under mild restrictions on the ratios of subsequent stepsizes. The stability results provide a basic ingredient for the convergence analysis given in Part II.

Key words. quasi-linear parabolic problems, exponential integrators, general linear methods, variable stepsizes, stability, convergence

AMS subject classifications. 35K55, 35K90, 65M12

DOI. 10.1137/140961845

1. Introduction. Our present work on efficient time integration methods for quasi-linear parabolic problems comprises two parts. In this first part, we introduce the considered class of variable stepsize explicit exponential general linear methods and study their stability behavior. A second part shall be concerned with the convergence analysis of these exponential integrators.

Scope of applications. Quasi-linear parabolic initial-boundary value problems typically arise in the modeling of minimal surfaces and mean curvature flow, in the study of fluids in porous media and sharp fronts in polymers, and for the description of thin fluid films and diffusion processes with state-dependent diffusivity; see, for instance, [3, 4, 12, 14, 15].

Analytical framework. Employing the abstract framework of sectorial operators and analytic semigroups (see [2, 19, 22, 25]), we may cast a quasi-linear parabolic

*Received by the editors March 21, 2014; accepted for publication (in revised form) December 8, 2014; published electronically March 3, 2015. The research of the authors was supported by the Spanish government under project DGCYT MTM2010-18246-C03-01 and the FWF Austrian Science Fund under project P21620-N13.

<http://www.siam.org/journals/sinum/53-2/96184.html>

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initial-boundary value problem into the form of an initial value problem on a Banach space $(X, \|\cdot\|_X)$,

$$(1.1) \quad \begin{cases} u'(t) = Q(u(t)) u(t), & t \in (0, T), \\ u(0) \text{ given.} \end{cases}$$

Here, our basic requirement is that the sectorial operators $Q(v) : D(Q(v)) \rightarrow X$ are defined on a domain $D = D(Q(v)) \subset X$ that is independent of $v \in V$, where $V \subseteq X_\gamma$ denotes an open subset of some intermediate space $D \subset X_\gamma \subseteq X$; see [2]. The scope of applications in particular includes quasi-linear parabolic evolution equations subject to Dirichlet boundary conditions.

Exponential integrators. During the last few years, there has been a lot of research interest in time integration methods that are based on the evaluation of exponentials, so-called exponential integrators. As a small selection, we mention the recent contributions [6, 8, 9, 10, 11, 13, 16, 17, 18, 20, 21, 23, 24, 26] and refer to the references given therein. The stiffness of systems occurring in the spatial discretization of partial differential equations in general requires the application of implicit time integration methods such as implicit Runge–Kutta or linear multistep methods. Theoretical investigations and numerical comparisons confirm, however, that in situations where the arising matrix-exponentials can be realized efficiently, explicit exponential integrators may provide a favorable alternative to standard implicit methods.

Exponential general linear methods. Our main concern is to introduce and analyze explicit exponential integrators based on general linear methods [7, 24] for quasi-linear parabolic problems. The considered class of time integration methods combines the benefits of exponential Runge–Kutta and exponential Adams–Bashforth methods and allows the construction of explicit higher-order schemes that possess favorable stability properties for stiff differential equations. In the context of semilinear parabolic problems, the convergence analysis given in [24] shows that higher-order explicit general linear methods retain their nonstiff orders of convergence, provided that certain regularity and compatibility requirements are satisfied. This gives the motivation to suitably adapt explicit exponential general linear methods to quasi-linear parabolic problems and to analyze their stability and convergence behavior for the more involved quasi-linear case.

Magnus-type integrators. In our former work [18], a detailed stability and convergence analysis is given for an explicit exponential integrator based on the midpoint rule, which is related to a Magnus integrator for nonautonomous linear problems [17, 20]. In particular, second-order convergence is proved for quasi-linear parabolic problems satisfying certain regularity and compatibility requirements. However, as shown in [26], a severe order reduction has to be expected when a Magnus-type or commutator-free method of order $p \geq 3$ is applied to a problem of parabolic type.

Alternative approach. The losses of Magnus-type integrators and related methods justify the consideration of a different approach that relies on suitable linearizations of quasi-linear parabolic equations and the application of explicit exponential general linear methods, which proved to be favorable for semilinear parabolic problems [24].

Let us illustrate our approach for the time discretization of quasi-linear problems (1.1) by means of the explicit exponential Euler method. In view of practical applications, our investigations include the case of variable time stepsizes. In order to construct an approximation $u_{n+1} \approx u(t_{n+1})$ to the exact solution value at time

$t_{n+1} = t_n + h_n$ with $h_n > 0$, assuming that the proceeding approximation $u_n \approx u(t_n)$ is available, we rewrite (1.1) as

$$(1.2a) \quad u'(t) = Q(u(t)) u(t) = Q_n u(t) + \Theta_n(u(t)),$$

where the nonlinear operator $\Theta_n : D \rightarrow X$ is defined through

$$(1.2b) \quad \Theta_n(v) = (Q(v) - Q_n) v.$$

The application of the explicit exponential Euler method yields

$$u_{n+1} = e^{h_n Q_n} u_n + h_n \varphi_1(h_n Q_n) \Theta_n(u_n), \quad \varphi_1(h_n Q_n) = \int_0^{h_n} e^{\xi Q_n} d\xi.$$

Explicit exponential general linear methods including examples of two-step schemes are constructed in [24]. We note that in the present part the stability behavior of exponential general linear methods is analyzed. Thus, a suitable choice for the operator Q_n is the following:

$$(1.2c) \quad Q_n = Q(u_n);$$

alternatively, the operator $Q(u_n)$ can be replaced by any operator satisfying Hypotheses 2.1 and 2.3.

Quasi-linear versus semilinear parabolic problems. Within the analytical framework of sectorial operators and analytic semigroups, a fundamental tool for the derivation of existence results for nonlinear parabolic problems and the analysis of time discretization methods is Banach’s fixed point theorem; see [2, 19, 22, 25]. In the context of a quasi-linear parabolic problem (1.1), a fixed-point iteration based on the relation

$$u'_{\text{new}}(t) = Q(u_{\text{old}}(t)) u_{\text{new}}(t)$$

is employed (see [2]); as a preliminary step, it is thus essential to study the properties of the evolution operator associated with a nonautonomous linear problem

$$(1.3) \quad u'_{\text{new}}(t) = A(t) u_{\text{new}}(t).$$

In contrast, for a semilinear parabolic evolution equation of the form

$$w'(t) = A w(t) + f(w(t)),$$

the application of a fixed-point iteration permits us to reduce the problem to the study of the linear variation-of-constants formula for an inhomogeneous linear evolution equation,

$$(1.4) \quad w'_{\text{new}}(t) = A w_{\text{new}}(t) + f(w_{\text{old}}(t)) = A w_{\text{new}}(t) + b(t);$$

see [19, 25]. We note that the evolution operator to (1.4) is given by the analytic semigroup $(e^{tA})_{t \geq 0}$, whereas the construction of the evolution operator associated with (1.3) and in particular the justification that the evolution operator is defined on the whole underlying Banach space is an involved task; see [2, 22]. In view of the employed linearization and the resulting reformulation (1.2), we further point out that the operator $(Q(v) - Q_n) : D \rightarrow X$ is in general *not* well-defined on an intermediate space and thus (1.2) *cannot* be considered as a semilinear parabolic problem.

Outline. In this paper, we deduce stability bounds for variable stepsize exponential general linear methods based on the linearization (1.2) of a quasi-linear parabolic problem. The stability results are obtained under very weak restrictions on the ratios of subsequent time stepsizes. The paper has the following structure. In sections 2 and 3, we introduce the analytical framework and state the general scheme of explicit exponential general linear methods for quasi-linear parabolic problems. Section 4 is devoted to the derivation of our main result, Theorem 4.1. Fundamental stability bounds for nonautonomous linear parabolic problems are given in section 5. Concluding remarks are finally given in section 6.

Notation. In the following, we employ standard abbreviations for Lebesgue and Sobolev spaces. The operator norm associated with a linear operator between normed spaces $F : (W_1, \|\cdot\|_{W_1}) \rightarrow (W_2, \|\cdot\|_{W_2})$ is denoted by $\|\cdot\|_{W_2 \leftarrow W_1}$. In order to simplify the notation, we do not distinguish the arising constants. In particular, the quantities $C, K, L, M > 0$ may have different values at different occurrences.

2. Analytical framework. In section 2.1, we introduce the fundamental hypotheses on the initial value problem (1.1). In section 2.2, we illustrate the analytical framework by quasi-linear parabolic initial-boundary value problems subject to Dirichlet boundary conditions; in addition, we exemplify the notion of intermediate spaces within a Hilbert space setting. Detailed information on the theory of analytic semigroups and quasi-linear evolution equations is found in [2, 19, 22, 25], and for an overview on interpolation theory we refer to [5, 27].

2.1. Quasi-linear parabolic problems. Throughout, we consider a complex Banach space $(X, \|\cdot\|_X)$ and a dense, continuously embedded subspace $(D, \|\cdot\|_D)$. For exponents $\mu \in [0, 1]$ we denote by $(X_\mu, \|\cdot\|_{X_\mu})$ interpolation spaces between D and X such that

$$\|x\|_{X_\mu} \leq K \|x\|_X^{1-\mu} \|x\|_D^\mu$$

holds with a constant $K > 0$ for $x \in D$. The closed ball in X_μ with radius $\varrho > 0$ and center $v^* \in X_\mu$ is denoted by $B_\mu(v^*, \varrho) = \{v \in X_\mu : \|v - v^*\|_{X_\mu} \leq \varrho\} \subset X_\mu$. In particular, we set

$$X_0 = X, \quad X_1 = D.$$

Suitable choices for interpolation spaces are real interpolation spaces or intermediate Calderón spaces, respectively.

We suppose that the right-hand side of the differential equation in (1.1) is defined by a family of operators $(Q(v))_{v \in V}$, where $V \subseteq X_\gamma$ is an open subset of an interpolation space X_γ with exponent $\gamma \in [0, 1)$. Our fundamental hypotheses on Q are as follows.

Hypothesis 2.1. Let $v \in V$.

- (i) The closed linear operator $Q(v) : X_1 \rightarrow X_0$ is sectorial, uniformly for $v \in V$, that is, there exist constants $a \in \mathbb{R}$, $\phi \in (0, \frac{\pi}{2})$, and $M > 0$ such that for every element $v \in V$ and for any complex number $\lambda \in \mathbb{C}$ in the complement of the sector $S_\phi(a) = \{a\} \cup \{z \in \mathbb{C} : |\arg(a - z)| \leq \phi\}$ the resolvent estimate

$$\left\| (\lambda I - Q(v))^{-1} \right\|_{X_0 \leftarrow X_0} \leq \frac{M}{|\lambda - a|}$$

is satisfied.

- (ii) The graph norm of $Q(v)$ and the norm in X_1 are equivalent, that is, the relation

$$K^{-1}\|x\|_{X_1} \leq \|x\|_{X_0} + \|Q(v)x\|_{X_0} \leq K\|x\|_{X_1}$$

holds with a constant $K > 0$ for all elements $x \in X_1$.

- (iii) For some exponent $\vartheta \in [0, 1)$ the interpolation space $X_{1+\vartheta}$ between X_1 and the domain of $(Q(v))^2$ does not depend on $v \in V$. Moreover, the mapping $Q : V \rightarrow L(X_{1+\vartheta}, X_\vartheta)$ is Lipschitz-continuous, that is, the estimate

$$\|Q(v) - Q(w)\|_{X_\vartheta \leftarrow X_{1+\vartheta}} \leq L\|v - w\|_{X_\gamma}$$

is valid with a constant $L > 0$ for all elements $v, w \in V$.

Remark 2.2.

- (i) Under the above assumptions with $\vartheta = 0$, it is proved in [2] that the quasi-linear differential equation (1.1) defines a semiflow in $X_\beta \cap V$ for any $\beta \in (\gamma, 1]$.
- (ii) Henceforth, we tacitly suppose Hypothesis 2.1 to be valid with $\vartheta \leq \gamma$. This additional requirement is often fulfilled in applications.

Whenever the considered explicit exponential general linear methods are based on a linearization involving the first derivative of Q , we further impose the following regularity requirement with exponent $\vartheta \in [0, 1)$ chosen according to Hypothesis 2.1.

Hypothesis 2.3. The map Q belongs to $\mathcal{C}^1(V, L(X_{1+\vartheta}, X_\vartheta))$ and its derivative $Q' : V \rightarrow L(X_{1+\vartheta}, X_\vartheta)$ is Lipschitz-continuous, that is, the bound

$$\|Q'(v) - Q'(w)\|_{X_\vartheta \leftarrow X_{1+\vartheta}} \leq L\|v - w\|_{X_\gamma}$$

is valid with a constant $L > 0$ for all elements $v, w \in V$.

Basic estimates. Let $v \in V$. Evidently, for any linear operator $F : X_0 \rightarrow X_1$ Hypothesis 2.1 implies the bounds

$$\begin{aligned} \|Q(v)F\|_{X_0 \leftarrow X_0} &\leq K\|F\|_{X_1 \leftarrow X_0}, \\ \|F\|_{X_1 \leftarrow X_0} &\leq K\left(1 + \|Q(v)F\|_{X_0 \leftarrow X_0}\right). \end{aligned}$$

Moreover, the relation

$$(2.1) \quad \left\|t^{\nu-\mu}(\lambda I - tQ(v))^{-1}\right\|_{X_\nu \leftarrow X_\mu} \leq \frac{M}{|\lambda - at|}$$

holds for positive times $t > 0$, complex numbers $\lambda \in \mathbb{C} \setminus S_\phi(a)$, and exponents $\mu, \nu \in [0, 1]$ such that $\mu \leq \nu$.

It is well known [19, 22, 25] that the sectorial operator $Q(v) : X_1 \rightarrow X_0$ generates an analytic semigroup $(e^{tQ(v)})_{t \in [0, \infty)}$ on X_0 and that the evolution operator can be represented by means of Cauchy's integral formula

$$e^{tQ(v)} = \begin{cases} I, & t = 0, \\ \frac{1}{2\pi i} \int_\Gamma e^\lambda (\lambda I - tQ(v))^{-1} d\lambda, & t > 0, \end{cases}$$

where Γ denotes a path that surrounds the spectrum of $tQ(v)$. Furthermore, the estimate

$$(2.2) \quad \left\|t^{\nu-\mu} e^{tQ(v)}\right\|_{X_\nu \leftarrow X_\mu} + \left\|t^{\nu-\mu} (e^{tQ(v)} - I)\right\|_{X_\nu \leftarrow X_\mu} \leq K$$

is valid for $t \in [0, T]$ and $\mu, \nu \in [0, 1]$ such that $\mu \leq \nu$. In view of section 3, we define a family of complex functions $(\varphi_j)_{j \in \mathbb{N}}$ by

$$(2.3) \quad \varphi_j : \mathbb{C} \longrightarrow \mathbb{C} : z \longmapsto \varphi_j(z) = \begin{cases} e^z, & j = 0, \\ \int_0^1 e^{(1-\tau)z} \frac{\tau^{j-1}}{(j-1)!} d\tau, & j \geq 1. \end{cases}$$

The above relation (2.2) for the analytic semigroup implies the bound

$$\|t^{\nu-\mu} \varphi_j(tQ(v))\|_{X_\nu \leftarrow X_\mu} \leq K$$

for $t \in [0, T]$, $\mu, \nu \in [0, 1]$ such that $\mu \leq \nu$, and $j \in \mathbb{N}$.

2.2. Illustration. The quasi-linear parabolic initial-boundary value problem that is given below illustrates the analytical framework of subsection 2.1. We recall that applications of practical relevance were mentioned in the introduction. Numerical experiments for this kind of quasi-linear parabolic initial-boundary value problem shall be provided in Part II.

Example 2.4. Let $d, k, m, D \in \mathbb{N}$ be positive integers such that $k \leq 2m - 1$, reflecting the spatial dimension, the highest degrees of partial derivatives, and the number of partial differential equations. Let $\Omega \subset \mathbb{R}^d$ denote an open and bounded domain of class \mathcal{C}^{2m} , which implies that the boundary $\partial\Omega$ is a \mathcal{C}^{2m} surface in the embedding space. Furthermore, let Y be an open subset of $\mathbb{K}^{D(k)}$, where

$$D(k) = D \sum_{|\eta| \leq k} 1$$

and $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, respectively. For a multi-index $\eta = (\eta_1, \dots, \eta_d) \in \mathbb{N}^d$ of non-negative integers and elements $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we set $|\eta| = \eta_1 + \dots + \eta_d$ and $x^\eta = x_1^{\eta_1} \dots x_d^{\eta_d}$ as well as $\partial_x^\eta = \partial_{x_1}^{\eta_1} \dots \partial_{x_d}^{\eta_d}$.

We consider the following initial-boundary value problem for a real-valued function $U : \overline{\Omega} \times [0, T] \rightarrow \mathbb{R}$:

$$(2.4a) \quad \begin{cases} \partial_t U(x, t) = \mathcal{Q}(x, U(x, t)) U(x, t), & (x, t) \in \Omega \times (0, T), \\ \mathcal{B}U(x, t) = 0, & (x, t) \in \partial\Omega \times [0, T], \\ U(x, 0) = U_0(x), & x \in \overline{\Omega}. \end{cases}$$

For sufficiently often differentiable functions $v, w : \overline{\Omega} \rightarrow \mathbb{R}$ the differential operator \mathcal{Q} is defined by

$$(2.4b) \quad \mathcal{Q}(x, v(x)) w(x) = \sum_{|\eta| \leq 2m} \varrho_\eta(x, v(x), \partial_x v(x), \dots, \partial_x^k v(x)) \partial_x^\eta w(x).$$

We suppose the coefficient functions $\varrho_\eta : \overline{\Omega} \times Y \rightarrow \mathbb{R} : (x, y) \mapsto \varrho_\eta(x, y)$ to be continuous in the first variable $x \in \overline{\Omega}$ and Lipschitz-continuous with respect to the remaining variables $y \in Y$. Furthermore, we require the basic uniform ellipticity condition

$$\Re \left\langle \sum_{|\eta|=2m} \varrho_\eta(x, y) \xi^\eta \zeta, \zeta \right\rangle > 0$$

to be satisfied for $x \in \overline{\Omega}$, $y \in Y$, $0 \neq \xi \in \mathbb{R}^d$, and $0 \neq \zeta \in \mathbb{C}^D$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in \mathbb{C}^D . The Dirichlet boundary operator on $\partial\Omega$ is defined by

$$(2.4c) \quad \mathcal{B}v(x) = (v(x), \partial_\nu v(x), \dots, \partial_\nu^{m-1}v(x))$$

with ν denoting the outer normal vector to $\partial\Omega$.

The following considerations show that the initial-boundary value problem (2.4) can be cast into our analytical framework with underlying Banach space $X = L^p(\Omega)$ for $p \in (1, \infty)$. For exponents $s \in [0, 2m]$ we denote

$$W_{p,\mathcal{B}}^s(\Omega) = \{v \in W_p^s(\Omega) : \partial_\nu^j v = 0 \text{ on } \partial\Omega \text{ for } j \in \{0, 1, \dots, m-1\} \\ \text{such that } j < s - \frac{1}{p}\}.$$

If $\mu \in (0, 1)$ and $2m\mu \notin \Phi_p = \{j + \frac{1}{p} : j \in \{0, 1, \dots, m-1\}\}$ we may use that $W_{p,\mathcal{B}}^{2m\mu}(\Omega)$ can be obtained as a real or complex interpolation space

$$W_{p,\mathcal{B}}^{2m\mu}(\Omega) = \begin{cases} (L^p(\Omega), W_{p,\mathcal{B}}^{2m}(\Omega))_{\mu,p}, & 2m\mu \notin \mathbb{N}, \\ [L^p(\Omega), W_{p,\mathcal{B}}^{2m}(\Omega)]_\mu, & 2m\mu \in \mathbb{N}. \end{cases}$$

The reiteration and commutativity properties of the real and complex interpolation functors imply that the space $W_{p,\mathcal{B}}^s(\Omega)$ is $(W_{p,\mathcal{B}}^\sigma(\Omega), W_{p,\mathcal{B}}^\tau(\Omega))$ -compatible for exponents $0 \leq \sigma \leq s \leq \tau \leq 2m$ with $s, \sigma, \tau \notin \Phi_p$. Sobolev-type embedding theorems such as

$$W_p^s(\Omega) \hookrightarrow W_q^t(\Omega), \quad \frac{1}{p} \geq \frac{1}{q}, \quad s - \frac{d}{p} \geq t - \frac{d}{q}, \\ W_p^s(\Omega) \xhookrightarrow{c} W_q^t(\Omega), \quad \frac{1}{p} \geq \frac{1}{q}, \quad s - \frac{d}{p} > t - \frac{d}{q},$$

show that the set $V_p^\sigma = \{v \in W_{p,\mathcal{B}}^\sigma(\Omega) : (v, \partial_x v, \dots, \partial_x^k v)(\overline{\Omega}) \subset Y\}$ is open in $W_{p,\mathcal{B}}^\sigma(\Omega)$ if $k + \frac{d}{p} < \sigma \leq 2m$ and $\sigma \notin \Phi_p$. This ensures that $\mathcal{Q}(\cdot, v)$ is well-defined for any element $v \in V_p^\sigma$ if $k + \frac{d}{p} < \sigma \leq 2m$ and $\sigma \notin \Phi_p$.

As a consequence, the initial-boundary value problem (2.4) can be cast in the form of an abstract initial value problem (1.1) for $u(t) = U(\cdot, t)$ and $Q(v) = \mathcal{Q}(\cdot, v)$ by choosing

$$X_\mu = W_{p,\mathcal{B}}^{2m\mu}(\Omega), \quad p \in (1, \infty), \quad \mu \in [0, 1].$$

In particular, the requirements of Hypothesis 2.1 are satisfied if

$$2m\gamma, 2m\beta \notin \Phi_p, \quad k + \frac{d}{p} < 2m\gamma \leq 2m, \quad 2m\gamma \notin \Phi_p, \quad \vartheta < \frac{1}{2p}.$$

For detailed information, we refer to [1].

In the special case of a differential operator of order two, corresponding to $m = 1$ and $k \in \{0, 1\}$, we obtain the restriction $\gamma \in (\frac{d}{2p}, 1]$ if the coefficient functions ϱ_η do not involve $\partial_x v$ and $\gamma \in (\frac{1}{2} + \frac{d}{2p}, 1]$ otherwise. We note that this implies $p \geq \frac{d}{2}$ or $p \geq d$, respectively, and justifies the additional assumption $\vartheta \leq \gamma$.

In addition, Hypothesis 2.3 is fulfilled whenever the first partial derivatives of the coefficient functions ϱ_η are continuous in $x \in \overline{\Omega}$ and Lipschitz-continuous with respect to $y \in Y$.

Remark 2.5. In order to exemplify the notion of interpolation spaces occurring in the above example, we include additional details for the special choice $p = 2$, leading to

a Hilbert-space setting. In the particularly simple situation, where the spatial domain is given by the cartesian product of bounded open intervals, the representation of any element $f \in X = L^2(\Omega)$ based on a Fourier series can be employed. Assuming, for instance, that \mathcal{Q} is a differential operator of order two, subject to homogeneous Dirichlet boundary conditions, its domain is given by $D = W_{2,\mathcal{B}}^2(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$. Employing an expansion into Sine basis functions $(S_k)_{k \in \mathbb{N}}$, this corresponds to

$$D = \left\{ f = \sum_{k \in \mathbb{N}} f_k S_k \in L^2(\Omega), \sum_{k \in \mathbb{N}} |f_k|^2 \lambda_k^2 < \infty \right\},$$

where $(\lambda_k)_{k \in \mathbb{N}}$ denotes the family of associated eigenvalues of the negative Laplace operator. For $\mu \in (0, 1)$ the interpolation space $X_\mu = W_{2,\mathcal{B}}^{2\mu}(\Omega)$ comprises all functions that satisfy the condition

$$\sum_{k \in \mathbb{N}} |f_k|^2 \lambda_k^{2\mu} < \infty.$$

3. Exponential general linear methods. In this section, we introduce variable stepsize explicit exponential general linear methods for the time integration of quasi-linear parabolic problems (1.1). As indicated in section 1, the basic idea for the definition of higher-order time integration methods is to perform a linearization in each time step and to apply an explicit exponential general linear method to the resulting problem (1.2a).

Semilinear parabolic problems. The consideration of exponential general linear methods for quasi-linear parabolic problems is inspired by the work [24] on semilinear parabolic problems

$$w'(t) = A w(t) + f(w(t)), \quad t \in (0, T),$$

with solution-independent sectorial operator $A : X_1 \rightarrow X_0$ and nonlinear perturbation $f : V \subseteq X_\mu \rightarrow X_0$ defined on an intermediate space X_μ with exponent $\mu \in [0, 1)$. There, it is shown that explicit exponential general linear methods based on Runge–Kutta and Adams–Bashforth schemes possess favorable stability properties and lead to exponential q -step s -stage methods that exhibit no order reduction when applied to semilinear parabolic problems.

Variable stepsizes. When performing a time integration, it is essential to adapt the size of the time steps to the variation of the solution, in order to enhance reliability and efficiency of the numerical computations. Contrary to one-step methods such as Runge–Kutta methods, where it is straightforward to extend the formulation for constant stepsizes to variable stepsizes, the standard approaches used in the context of linear multistep methods are more involved. One possibility relies on coefficients depending on the ratios of subsequent time stepsizes, and a second possibility is to compute approximations to solution values by polynomial interpolation. Likewise, one has to face these intricacies in the context of exponential general linear methods.

Throughout, we denote by $(h_j)_{j \in \mathbb{N}}$ a sequence of positive time steps with corresponding ratios and associated grid points defined by

$$t_{j+1} = t_j + h_j, \quad \omega_{j+1} = \frac{h_{j+1}}{h_j},$$

for $j \in \mathbb{N}$, where $t_0 = 0$. We employ the following hypothesis on the stepsize ratios that it is not restrictive in practice.

Hypothesis 3.1. There exists a constant $\chi > 0$ such that the relation

$$\chi^{-1} \leq \omega_j \leq \chi$$

holds for all $j \in \mathbb{N}$.

Approximation. For given initial approximations u_0, \dots, u_{q-1} , approximation values $u_{n+1} \approx u(t_{n+1})$ to the exact solution values are defined by recurrence,

$$(3.1a) \quad \begin{aligned} U_{ni} &= e^{c_i h_n Q_n} u_n + h_n \sum_{j=1}^{i-1} a_{ij}^{(n)}(h_n Q_n) \Theta_n(U_{nj}) \\ &+ h_n \sum_{k=1}^{q-1} \tilde{a}_{ik}^{(n)}(h_n Q_n) \Theta_n(u_{n-k}), \end{aligned}$$

$$(3.1b) \quad \begin{aligned} u_{n+1} &= e^{h_n Q_n} u_n + h_n \sum_{i=1}^s b_i^{(n)}(h_n Q_n) \Theta_n(U_{ni}) \\ &+ h_n \sum_{k=1}^{q-1} \tilde{b}_k^{(n)}(h_n Q_n) \Theta_n(u_{n-k}), \end{aligned}$$

where $i \in \{1, \dots, s\}$ and $n \in \{q-1, q, \dots\}$. Here, we assume that the coefficient functions, which possibly depend on several subsequent stepsize ratios, are given as linear combinations of the exponential functions (2.3). In particular, we set

$$\tilde{a}_{1k}^{(n)} = 0$$

for $k \in \{1, \dots, q-1\}$ and $n \in \{0, 1, \dots\}$. Moreover, in order to preserve equilibrium points, we employ the standard requirement $c_1 = 0$, which implies $U_{n1} = u_n$; see [24, section 2.1].

Well-definedness of approximation. Provided that the initial approximations $u_0, \dots, u_{q-1} \in X_\beta \cap V$ lie sufficiently close to the corresponding values of the exact solution $u(t_0), \dots, u(t_{q-1}) \in X_\beta \cap V$, Theorem 4.1 stated in section 4 ensures that the exponential general linear method (3.1) is applicable as long as $t_{n+1} \leq T$; see also Remark 4.2 (i).

4. Stability results for quasi-linear problems. In this section, we analyze the stability behavior of variable stepsize explicit exponential general linear methods (3.1) for quasi-linear parabolic problems (1.1).

Related work. The stability analysis uses techniques that are closely related to our previous work [18] on a Magnus-type integrator for quasi-linear parabolic problems. In the derivation of our main result and in section 5, in order to avoid redundancies, we thus repeatedly refer to [18]. We point out that the stability result deduced in [18] is valid for constant time stepsizes and that the extension to variable stepsizes requires nontrivial changes.

Stability result. We study the case where the operator family $(Q_n)_{n \in \mathbb{N}}$ is given by $Q_n = Q(u_n) : X_1 \rightarrow X_0$ for $n \in \mathbb{N}$; see also (1.2). With regard to (3.1), we consider sequences $(v_n)_{n \in \mathbb{N}}, (w_n)_{n \in \mathbb{N}}$, defined through the recurrence formulas

$$(4.1) \quad \begin{cases} v_{n+1} = e^{h_n Q(v_n)} v_n + h_n p_{n+1}, \\ w_{n+1} = e^{h_n Q(w_n)} w_n + h_n q_{n+1}, \end{cases} \quad n \in \mathbb{N}.$$

Here, we require the initial values to satisfy $v_0, w_0 \in X_\beta \cap V$ and assume that the additional perturbations $(p_n)_{n \in \mathbb{N}}, (q_n)_{n \in \mathbb{N}}$ are bounded in X_β . Provided that the current approximation satisfies $v_n \in X_\beta \cap V$ and $h_n > 0$ is chosen sufficiently small, the relation $v_{n+1} \in X_\beta \cap V$ follows for the new approximation (see (2.2)); the analogous statement is valid for w_{n+1} .

The following result provides a bound for the difference $v_n - w_n$ with respect to the norm in X_β and thus ensures stability of (3.1) when applied to (1.1). Basic auxiliary results for its proof are collected in section 5.

THEOREM 4.1. *Provided that Hypotheses 2.1 and 3.1 are fulfilled, there exists a final time $T_1 > 0$ and a maximal time stepsize $h > 0$ such that for any stepsize sequence $(h_j)_{j \in \mathbb{N}}$ with $0 < h_j \leq h$ for $j \in \mathbb{N}$, the sequences $(v_n)_{n \in \mathbb{N}}, (w_n)_{n \in \mathbb{N}}$ given by (4.1) satisfy the bound*

$$\|v_n - w_n\|_{X_\beta} \leq C \left(\|v_0 - w_0\|_{X_\beta} + \max_{1 \leq j \leq n} \|p_j - q_j\|_{X_\beta} \right), \quad 0 \leq t_n \leq T_1,$$

with constant $C > 0$ independent of n and h_j for $j \in \mathbb{N}$.

Proof. The proof of Theorem 4.1 relies on a fixed-point iteration that is based on a global representation of the solutions in (4.1); that is, for the new iterates the relations

$$\begin{aligned} v_{\text{new}, n+1} &= \prod_{\ell=0}^{n-1} e^{h_\ell Q(v_{\text{old}, \ell})} v_0 + \sum_{k=0}^{n-1} h_k \prod_{\ell=k+1}^{n-1} e^{h_\ell Q(v_{\text{old}, \ell})} p_{k+1}, \\ w_{\text{new}, n+1} &= \prod_{\ell=0}^{n-1} e^{h_\ell Q(w_{\text{old}, \ell})} w_0 + \sum_{k=0}^{n-1} h_k \prod_{\ell=k+1}^{n-1} e^{h_\ell Q(w_{\text{old}, \ell})} q_{k+1} \end{aligned}$$

are employed. The choice of the underlying space of sequences

$$\mathcal{Z} = \left\{ (z_n)_{n \in \mathbb{N}} : \|z_0 - z^*\|_{X_\beta} \leq \varrho, \|z_n - z_m\|_{X_\vartheta} \leq L(t_n - t_m)^\alpha \right\},$$

with exponent ϑ satisfying the conditions $\gamma < \vartheta < \beta$ and $0 < \alpha < \beta - \vartheta$, requires Hölder continuity of the elements of the sequence. By Hypothesis 2.1 the difference of two elements of a sequence $(z_n)_{n \in \mathbb{N}} \in \mathcal{Z}$ can be estimated as follows:

$$\|Q(z_n) - Q(z_m)\|_{X_\vartheta \leftarrow X_{1+\vartheta}} \leq C \|z_n - z_m\|_{X_\gamma} \leq C \|z_n - z_m\|_{X_\vartheta} \leq C(t_n - t_m)^\alpha$$

with generic constant $C > 0$ and $0 \leq t_m < t_n \leq T$; this explains the connection between quasi-linear and nonautonomous linear evolution equations and in particular the requirement

$$\|A(t_n) - A(t_m)\|_{X_\vartheta \leftarrow X_{1+\vartheta}} \leq C(t_n - t_m)^\alpha$$

stated below in Hypothesis 5.4. In order to apply Banach's fixed point theorem, well-definedness and contractivity of the fixed point mapping has to be ensured; well-definedness is a consequence of Theorem 5.6, and contractivity follows by Theorem 5.7. Due to the fact that the main steps of the proof of Theorem 4.1 are in the lines of [18], we only detail the nontrivial extensions of the auxiliary results for nonautonomous linear evolution equations given in [18] to variable time stepsizes; see section 5. \square

Remark 4.2.

- (i) We note that the generally rather strong restrictions on the final time $T_1 > 0$ can be weakened by introducing exponential weights in the maximum norm. Alternatively, combining Theorem 4.1 with the convergence result given in Part II shows the validity of Theorem 4.1 on the whole interval of existence of the true solution $u : [0, T] \rightarrow X_\beta$ of (1.1). As this approach is standard, we omit further details.
- (ii) If in addition Hypothesis 2.3 is fulfilled, the above stability result remains valid whenever $Q(v_n)$ and $Q(w_n)$ in (4.1) are replaced by alternative linearizations involving the Fréchet derivatives $Q'(v_n)$ and $Q'(w_n)$.
- (iii) Provided that the time and solution dependent sectorial operator Q fulfills suitable requirements based on Hypotheses 2.1 and 2.3 as well as Hypothesis 5.4 given below, it is straightforward to extend our stability analysis to evolution equations of the form

$$u'(t) = Q(t, u(t)) u(t) + \theta(t), \quad t \in (0, T).$$

In view of numerical experiments it is convenient to admit an additional time-dependent inhomogeneity.

5. Stability results for nonautonomous linear problems. In this section, we deduce stability bounds for exponential general linear methods with variable time stepsizes applied to nonautonomous linear problems. These results are essential in view of the proof of Theorem 4.1. We recall Hypothesis 3.1 on the sequence of time stepsizes employed below.

5.1. Gronwall-type inequality. The following auxiliary result is needed in the proof of Lemma 5.3.

LEMMA 5.1. *Let $m, k \in \mathbb{N}$ as well as $\theta, \rho \in (0, 1)$, and assume that there exists a constant $\chi > 0$ such that*

$$(5.1) \quad \chi^{-1} \leq \omega_{j+1} = \frac{h_{j+1}}{h_j}$$

for $j \in \{m + 1, \dots, k - 1\}$. Then, the following bound holds:

$$\sum_{\ell=m+1}^{k-1} \frac{h_{\ell-1}}{(t_k - t_\ell)^{1-\rho} (t_\ell - t_m)^{1-\theta}} \leq \frac{(1 + \chi) B(\theta, \rho)}{(t_k - t_m)^{1-\theta-\rho}}$$

with Euler–Beta function $B(\theta, \rho)$ given by

$$B(\theta, \rho) = \int_0^1 x^{\theta-1} (1-x)^{\rho-1} dx.$$

Proof. We let $t^* \in (t_m, t_k)$ denote the point where the map

$$f(s) = \frac{1}{(t_k - s)^{1-\rho} (s - t_m)^{1-\theta}}, \quad s \in (t_m, t_k),$$

achieves its minimum and choose $j \in \{m + 1, \dots, k\}$ such that $t_{j-1} \leq t^* \leq t_j$. Consequently, the map f is monotonically decreasing in $(t_m, t_{j-1}]$ and monotonically

increasing in $[t_j, t_k]$. Straightforward estimation of the Riemann sum by the corresponding integral leads to

$$\begin{aligned} \sum_{\ell=m+1}^{j-1} \frac{h_{\ell-1}}{(t_k - t_\ell)^{1-\rho}(t_\ell - t_m)^{1-\theta}} &\leq \int_{t_m}^{t^*} \frac{1}{(t_k - s)^{1-\rho}(s - t_m)^{1-\theta}} ds \\ &\leq \frac{B(\theta, \rho)}{(t_k - t_m)^{1-\theta-\rho}}. \end{aligned}$$

For $\ell \in \{j, \dots, k\}$ we may relate $\frac{h_{\ell-1}}{(t_k - t_\ell)^{1-\rho}(t_\ell - t_m)^{1-\theta}}$ to $\frac{h_\ell}{(t_k - t_\ell)^{1-\rho}(t_\ell - t_m)^{1-\theta}}$ by employing assumption (5.1) and get

$$\begin{aligned} \sum_{\ell=j}^{k-1} \frac{h_{\ell-1}}{(t_k - t_\ell)^{1-\rho}(t_\ell - t_m)^{1-\theta}} &\leq \chi \sum_{\ell=j}^{k-1} \frac{h_\ell}{(t_k - t_\ell)^{1-\rho}(t_\ell - t_m)^{1-\theta}} \\ &\leq \chi \int_{t^*}^{t_k} \frac{1}{(t_k - s)^{1-\rho}(s - t_m)^{1-\theta}} ds \\ &\leq \chi \frac{B(\theta, \rho)}{(t_k - t_m)^{1-\theta-\rho}}, \end{aligned}$$

which implies the statement. \square

Remark 5.2. If one of the exponents arising in the statement of Lemma 5.1 satisfies $\theta \geq 1$ or $\rho \geq 1$, respectively, we instead obtain the following bound:

$$\sum_{\ell=m+1}^{k-1} \frac{h_{\ell-1}}{(t_k - t_\ell)^{1-\rho}(t_\ell - t_m)^{1-\theta}} \leq \frac{\chi B(\theta, \rho)}{(t_k - t_m)^{1-\theta-\rho}}.$$

We next provide a discrete Gronwall-type inequality adapted to the case of variable time stepsizes. We note that the constant $K > 0$ in (5.3) depends on the difference $t_n - t_m$ in a nonsingular way and in general increases exponentially with respect to $t_n - t_m$.

LEMMA 5.3. *Let $\alpha, \rho \in (0, 1)$. Moreover, let $(\Gamma_{m,n})_{0 \leq m \leq n \leq N}$ denote a sequence of nonnegative real numbers with $\Gamma_{n,n} = 0$ for $n \in \{0, \dots, N\}$ that satisfies the relation*

$$(5.2) \quad \Gamma_{m,n} \leq \frac{C_1}{(t_n - t_m)^{1-\alpha}} + C_2 \sum_{j=m+1}^{n-1} \frac{h_{j-1}}{(t_n - t_j)^{1-\rho}} \Gamma_{m,j}$$

for $0 \leq m < n \leq N$ with constants $C_1, C_2 \geq 0$. Under the additional assumption

$$\chi^{-1} \leq \omega_{j+1} = \frac{h_{j+1}}{h_j}$$

for $j \in \{0, \dots, N-1\}$, there exists a constant $K = K(\chi, C_1, C_2, \alpha, \rho, t_n - t_m) > 0$ such that the bound

$$(5.3) \quad \Gamma_{m,n} \leq \frac{K}{(t_n - t_m)^{1-\alpha}}$$

holds for $0 \leq m < n \leq N$.

Proof. Let $m, n \in \mathbb{N}$ satisfy $0 \leq m < n \leq N$. With regard to (5.2), we set

$$\gamma_{m,n} = \frac{C_1}{(t_n - t_m)^{1-\alpha}}$$

and define the operator L in the space of double sequences $\Lambda = (\Lambda_{m,n})$

$$L(\Lambda)|_{m,n} = C_2 \sum_{j=m+1}^{n-1} \frac{h_{j-1}}{(t_n - t_j)^{1-\rho}} \Lambda_{m,j}.$$

With this short notation (5.2) is rewritten as

$$\Gamma_{m,n} \leq \gamma_{m,n} + L(\Gamma)|_{m,n}.$$

Applying the operator L to this inequality yields

$$L(\Gamma)|_{m,n} \leq L(\gamma)|_{m,n} + L^2(\Gamma)|_{m,n}$$

and consequently

$$\Gamma_{m,n} \leq \gamma_{m,n} + L(\Gamma)|_{m,n} \leq \gamma_{m,n} + L(\gamma)|_{m,n} + L^2(\Gamma)|_{m,n}.$$

Repeating this process ℓ times, we get

$$(5.4) \quad \Gamma_{m,n} \leq \sum_{k=0}^{\ell-1} L^k(\gamma)|_{m,n} + L^\ell(\Gamma)|_{m,n}.$$

In the following, we employ Lemma 5.1 in order to estimate powers of the operator L . In particular, we show that the second contribution tends to zero for $\ell \rightarrow \infty$.

(i) We first study $L^\ell(\Lambda)$ for $\ell \in \{2, 3, \dots\}$. With the help of summation-by-parts and by an application of Lemma 5.1 we obtain

$$\begin{aligned} L^2(\Lambda)|_{m,n} &= C_2 \sum_{j=m+1}^{n-1} \frac{h_{j-1}}{(t_n - t_j)^{1-\rho}} \left(C_2 \sum_{p=m+1}^{j-1} \frac{h_{p-1}}{(t_j - t_p)^{1-\rho}} \Lambda_{p,m} \right) \\ &= C_2^2 \sum_{p=m+1}^{n-1} h_{p-1} \left(\sum_{j=p+1}^{n-1} \frac{h_{j-1}}{(t_n - t_j)^{1-\rho} (t_j - t_p)^{1-\rho}} \right) \Lambda_{p,m} \\ &\leq C_2^2 (1 + \chi) B(\rho, \rho) \sum_{p=m+1}^{n-1} \frac{h_{p-1}}{(t_n - t_p)^{1-2\rho}} \Lambda_{p,m}. \end{aligned}$$

Similar considerations lead to

$$L^3(\Lambda)|_{m,n} \leq C_2^3 (1 + \chi)^2 B(\rho, \rho) B(2\rho, \rho) \sum_{p=m+1}^{n-1} \frac{h_{p-1}}{(t_n - t_p)^{1-3\rho}} \Lambda_{p,m}.$$

For a compact formulation it is useful to introduce the short notation

$$c_j(\eta) = \begin{cases} (1 + \chi) B((j - 1)\rho + \eta, \rho), & j \in \{2, \dots, \nu\}, \\ \chi B((j - 1)\rho + \eta, \rho), & j \in \{\nu + 1, \nu + 2, \dots\}, \end{cases}$$

where $\nu \in \mathbb{N}$ denotes a positive integer number such that $(\nu - 1)\rho < 1 \leq \nu\rho$. We recall the basic relation between the Euler–Beta and Gamma functions

$$B((j - 1)\rho + \eta, \rho) = \frac{\Gamma((j - 1)\rho + \eta) \Gamma(\rho)}{\Gamma(j\rho + \eta)}.$$

Fundamental properties of the Gamma function imply

$$(5.5a) \quad \prod_{j=\nu+1}^{\ell} c_j(0) = \chi^{\ell-\nu} \frac{\Gamma(\rho) \Gamma(\nu\rho) \Gamma^{\ell-\nu}(\rho)}{\Gamma(\ell\rho)}.$$

Altogether, we obtain the following bound for positive integer numbers $\ell \in \mathbb{N}$:

$$(5.5b) \quad L^\ell(\Lambda)|_{m,n} \leq C_2^\ell \prod_{j=2}^{\ell} c_j(0) \sum_{p=m+1}^{n-1} \frac{h_{p-1}}{(t_n - t_p)^{1-\ell\rho}} \Lambda_{p,m}.$$

(ii) Applying (5.5) to (5.4) shows that the second contribution on the right-hand side tends to zero:

$$L^\ell(\Gamma)|_{m,n} \rightarrow 0 \quad \text{as } \ell \rightarrow \infty.$$

(iii) It remains to treat the first contribution on the right-hand side of (5.4). Similarly as before, summation-by-parts together with Lemma 5.1 yields

$$\begin{aligned} L(\gamma)|_{m,n} &= C_2 \sum_{j=m+1}^{n-1} \frac{h_{j-1}}{(t_n - t_j)^{1-\rho}} \left(\frac{C_1}{(t_j - t_m)^{1-\alpha}} \right) \\ &\leq C_1 C_2 \frac{(1 + \chi) B(\alpha, \rho)}{(t_n - t_m)^{1-\rho-\alpha}} \\ &= C_1 C_2 c_1(\alpha) \frac{1}{(t_n - t_m)^{1-\rho-\alpha}} \end{aligned}$$

and furthermore

$$\begin{aligned} L^2(\gamma)|_{m,n} &= C_2 \sum_{j=m+1}^{n-1} \frac{h_{j-1}}{(t_n - t_j)^{1-\rho}} \left(\frac{C_1 C_2 c_1(\alpha)}{(t_j - t_m)^{1-\rho-\alpha}} \right) \\ &\leq C_1 C_2^2 c_1(\alpha) c_2(\alpha) \frac{1}{(t_n - t_m)^{1-2\rho-\alpha}}. \end{aligned}$$

From these relations it is straightforward to deduce the bound

$$L^{k-1}(\gamma)|_{m,n} \leq C_1 C_2^{k-1} \prod_{i=1}^{k-1} c_i(\alpha) \frac{1}{(t_n - t_m)^{1-(k-1)\rho-\alpha}}$$

for positive integers $k \in \mathbb{N}$ such that $k \geq 2$. By means of this relation we obtain

$$\begin{aligned} \sum_{k=0}^{\ell-1} L^k(\gamma)|_{m,n} &\leq \sum_{k=0}^{\infty} L^k(\gamma)|_{m,n} \\ &= \gamma_{m,n} + \sum_{k=1}^{\infty} L(L^{k-1}(\gamma))|_{m,n} \\ &= \gamma_{m,n} + \sum_{k=1}^{\infty} C_2 \sum_{j=m+1}^{n-1} \frac{h_{j-1}}{(t_n - t_j)^{1-\rho}} L^{k-1}(\gamma)|_{m,j} \\ &= \gamma_{m,n} + \sum_{k=1}^{\infty} \sum_{j=m+1}^{n-1} \frac{h_{j-1}}{(t_n - t_j)^{1-\rho}} C_1 C_2^k \prod_{i=1}^{k-1} c_i(\alpha) \frac{1}{(t_j - t_m)^{1-(k-1)\rho-\alpha}} \\ &= \gamma_{m,n} + C_1 \sum_{j=m+1}^{n-1} \frac{h_{j-1}}{(t_n - t_j)^{1-\rho}} \frac{1}{(t_j - t_m)^{1-\alpha}} \\ &\quad \times \left(\sum_{k=1}^{\infty} C_2^k \prod_{i=1}^{k-1} c_i(\alpha) (t_j - t_m)^{(k-1)\rho} \right). \end{aligned}$$

Due to the fact that the term

$$\sum_{k=1}^{\infty} C_2^k \prod_{i=1}^{k-1} c_i(\alpha) (t_j - t_m)^{(k-1)\rho}$$

is bounded, an application of Lemma 5.1 finally proves the stated result. \square

5.2. Stability results. In this section, we study nonautonomous linear evolution equations of the form

$$u'(t) = A(t)u(t), \quad t \in (0, T).$$

In accordance with Hypothesis 2.1 we employ the following requirements on the operator family $(A(t))_{t \in [0, T]}$.

Hypothesis 5.4. Let $t \in [0, T]$. The closed linear operator $A(t) : X_1 \rightarrow X_0$ is sectorial, uniformly for $t \in [0, T]$, and the graph norm of $A(t)$ and the norm in X_1 are equivalent. Furthermore, for some exponent $\vartheta \in [0, 1)$ the interpolation space $X_{1+\vartheta}$ between X_1 and the domain of $A(t)^2$ does not depend on $t \in [0, T]$, and the mapping $A : [0, T] \rightarrow L(X_{1+\vartheta}, X_\vartheta)$ is Hölder-continuous with exponent $\alpha \in (0, 1]$, that is, the estimate

$$\|A(t) - A(s)\|_{X_\vartheta \leftarrow X_{1+\vartheta}} \leq L(t - s)^\alpha$$

is valid with a constant $L > 0$ for $s, t \in [0, T]$ such that $s < t$.

With regard to (4.1), we consider a sequence $(v_n)_{n \in \mathbb{N}}$ defined by recurrence

$$v_{n+1} = e^{h_n A(t_n)} v_n + h_n p_{n+1}, \quad n \in \mathbb{N}.$$

Provided that $v_0 \in X_\nu$ for some $\nu \in [0, 1]$ and that the perturbations $(p_n)_{n \in \mathbb{N}}$ are bounded in X_ν , the stability result given below implies that the sequence $(v_n)_{n \in \mathbb{N}}$ fulfills the relation

$$(5.6) \quad \max_{n \in \{0, \dots, N\}} \|v_n\|_{X_\nu} \leq C \left(\|v_0\|_{X_\nu} + \max_{n \in \{0, \dots, N\}} \|p_n\|_{X_\nu} \right);$$

see Theorem 5.5.

For convenience, we henceforth employ the abbreviation $\tau = (t_j)_{j \in \mathbb{N}}$ as well as $A_j = A(t_j)$ for $j \in \mathbb{N}$. Our objective is to deduce stability bounds with respect to the norm of certain interpolation spaces for

$$L_m^n(\tau) = L_m^n(\tau, A) = \begin{cases} I, & m > n, \\ \prod_{j=m}^n e^{h_j A_j}, & m \leq n, \end{cases}$$

where $m, n \in \mathbb{N}$. A first result provides the analogous statement to [18, Lemma 4.4].

THEOREM 5.5. *Let $0 \leq t_m \leq t_n < t_{n+1} \leq T$, and let $\mu, \nu \in [0, 1 + \vartheta]$ be such that $\nu - \mu \leq 1$. Provided that Hypotheses 3.1 and 5.4 are fulfilled, there exists a maximal stepsize $h > 0$ such that for any stepsize sequence $(h_j)_{j \in \mathbb{N}}$ with $0 < h_j \leq h$ for $j \in \mathbb{N}$, the bound*

$$\|L_m^n(\tau)\|_{X_\nu \leftarrow X_\mu} \leq \begin{cases} C (t_{n+1} - t_m)^{-\nu+\mu}, & \nu - \mu < 1, \\ C (1 + |\ln h|) (t_{n+1} - t_m)^{-1}, & \nu - \mu = 1, \end{cases}$$

holds with constant $C > 0$ independent of n and h_j for $j \geq 0$.

Proof. As the proof is along the lines of [18, Lemma 4.4], we focus on the essential changes due to the incorporation of variable times stepsizes. For simplicity, we henceforth set $\vartheta = 0$ and suppose $\nu - \mu < 1$. Similar arguments hold for the special case $\nu - \mu = 1$.

With regard to estimate (2.2), valid for the analytic semigroup generated by the sectorial operator A_m , it suffices to study the difference

$$\Delta_m^n = L_m^n(\tau) - e^{(t_{n+1}-t_m)A_m}.$$

By means of a telescopic identity we obtain

$$(5.7) \quad \begin{aligned} \Delta_m^n &= \sum_{j=m+1}^{n-1} \Delta_{j+1}^n \Xi_{jm} + \sum_{j=m+1}^n e^{(t_{n+1}-t_{j+1})A_m} \Xi_{jm}, \\ \Xi_{jm} &= (e^{h_j A_j} - e^{h_j A_m}) e^{(t_j-t_m)A_m}. \end{aligned}$$

Employing the integral formula of Cauchy and Hypothesis 5.4 yields the bound

$$(5.8) \quad \|\Xi_{nm}\|_{X_{\tilde{\nu}} \leftarrow X_{\tilde{\mu}}} \leq C h_n^{1-\tilde{\nu}} (t_n - t_m)^{-1+\tilde{\mu}+\alpha}$$

for exponents $\tilde{\mu}, \tilde{\nu} \in [0, 1]$. In a similar manner, we obtain

$$\begin{aligned} &\|e^{(t_{n+1}-t_{j+1})A_m} \Xi_{jm}\|_{X_\nu \leftarrow X_\mu} \\ &\leq C h_j (t_{n+1} - t_{j+1})^{-\nu} (t_j - t_m)^{-1+\mu} \|A_j - A_m\|_{X_0 \leftarrow X_1} \\ &\leq C h_j (t_{n+1} - t_{j+1})^{-\nu} (t_j - t_m)^{-1+\mu+\alpha} \end{aligned}$$

for $j \in \{m+1, \dots, n-1\}$. From Hypotheses 3.1 the bound

$$\frac{t_{j+1} - t_m}{t_j - t_m} \leq 1 + \frac{h_j}{t_j - t_m} \leq 1 + \chi$$

follows, which further ensures existence of a constant $K > 0$ such that

$$(5.9) \quad \frac{(t_j - t_m)^{-1+\mu+\alpha}}{(t_{j+1} - t_m)^{-1+\mu+\alpha}} \leq K.$$

As a consequence, we may apply Lemma 5.1. Relating the second term in Δ_m^n to a Riemann integral yields and applying Lemma 5.1 gives

$$(5.10) \quad \sum_{j=m+1}^{n-1} \left\| e^{(t_{n+1}-t_{j+1})A_m} \Xi_{jm} \right\|_{X_{\tilde{\nu}} \leftarrow X_{\tilde{\mu}}} \leq C (t_{n+1} - t_m)^{-\tilde{\nu} + \tilde{\mu} + \alpha}.$$

We first deduce an estimate for Δ_m^n as operator from X_0 to X_ν . By means of the above bounds (5.8)–(5.10), we obtain

$$\left\| \Delta_m^n \right\|_{X_\nu \leftarrow X_0} \leq C \sum_{j=m+1}^{n-1} h_j \left\| \Delta_{j+1}^n \right\|_{X_\nu \leftarrow X_0} (t_j - t_m)^{-1+\alpha} + C (t_{n+1} - t_m)^{-\nu+\alpha}.$$

Moreover, an application of Lemma 5.3 yields

$$(5.11) \quad \left\| \Delta_m^n \right\|_{X_\nu \leftarrow X_0} \leq C (t_{n+1} - t_m)^{-\nu+\alpha}.$$

Now, it is straightforward to estimate Δ_m^n as operator from X_μ to X_ν . Applying the above relations (5.8)–(5.11) together with Lemma 5.1 and the inequality (5.9), we finally obtain

$$\left\| \Delta_m^n \right\|_{X_\nu \leftarrow X_\mu} \leq C (1 + C(t_{n+1} - t_m)^\alpha) (t_{n+1} - t_m)^{-\nu+\mu+\alpha}$$

for any $0 \leq t_m \leq t_{n+1} \leq T$. The arising constant $C > 0$ in particular depends on the final time $T > 0$ and the bounds for the stepsize ratios χ . \square

The following statement is the analogue of [18, Lemma 4.5].

THEOREM 5.6. *Let $0 \leq t_m \leq t_n < t_{n+1} \leq T$ and $\nu, \nu + \eta \in [0, 1 + \vartheta]$ for some $\eta \geq 0$. Provided that Hypotheses 3.1 and 5.4 are fulfilled, there exists a maximal stepsize $h > 0$ such that for any stepsize sequence $(h_j)_{j \in \mathbb{N}}$ with $0 < h_j \leq h$ for $j \in \mathbb{N}$, the bound*

$$\left\| L_j^n(\tau) - L_j^m(\tau) \right\|_{X_\nu \leftarrow X_{\nu+\eta}} \leq C (t_{n+1} - t_{m+1})^\eta$$

is valid.

Proof. As the previous result ensures boundedness

$$\left\| L_j^m(\tau) \right\|_{X_{\nu+\eta} \leftarrow X_{\nu+\eta}} \leq C$$

it suffices to deduce the bound

$$\left\| L_{m+1}^n(\tau) - I \right\|_{X_\nu \leftarrow X_{\nu+\eta}} \leq C (t_{n+1} - t_{m+1})^\eta.$$

We recall the following relations obtained from (2.2):

$$(5.12) \quad \begin{aligned} \left\| L_j^j(\tau) - I \right\|_{X_0 \leftarrow X_0} &\leq C, & \left\| L_j^j(\tau) - I \right\|_{X_0 \leftarrow X_1} &\leq C h_j, \\ \left\| L_{m+1}^{m+1}(\tau) - I \right\|_{X_1 \leftarrow X_{\nu+\eta}} &\leq C h_{m+1}^{\nu+\eta-1}. \end{aligned}$$

With the help of the telescopic identity we obtain

$$L_{m+1}^n(\tau) - I = \sum_{j=m+1}^n (L_j^j(\tau) - I) L_{m+1}^{j-1}(\tau),$$

which by (5.12) and Theorem 5.5 yields the bound

$$\begin{aligned} \|L_{m+1}^n(\tau) - I\|_{X_0 \leftarrow X_{\nu+\eta}} &\leq \sum_{j=m+1}^n \|L_j^j(\tau) - I\|_{X_0 \leftarrow X_1} \|L_{m+1}^{j-1}(\tau)\|_{X_1 \leftarrow X_{\nu+\eta}} \\ &\leq C \sum_{j=m+1}^n h_j \|L_{m+1}^{j-1}(\tau)\|_{X_1 \leftarrow X_{\nu+\eta}} \\ &\leq C (t_{n+1} - t_{m+1})^{\nu+\eta}. \end{aligned}$$

Together with the estimate

$$\|L_{m+1}^n - I\|_{X_{\nu+\eta} \leftarrow X_{\nu+\eta}} \leq C,$$

(see also Theorem 5.5), the statement follows by interpolation. \square

A final result characterizes the dependence of $L_m^n(\tau, A)$ with respect to A .

THEOREM 5.7. *Let $0 \leq t_m \leq t_n < t_{n+1} \leq T$, and let $\mu, \nu \in [0, 1 + \vartheta]$ be such that $\nu - \mu \leq 1$. Provided that the operator families $(A(t))_{t \in [0, T]}$ and $(B(t))_{t \in [0, T]}$ satisfy the requirements of Hypotheses 5.4 and 3.1, there exists a maximal stepsize $h > 0$ such that for any stepsize sequence $(h_j)_{j \in \mathbb{N}}$ with $0 < h_j \leq h$ for $j \in \mathbb{N}$ the bound*

$$\begin{aligned} &\|L_m^n(\tau, A) - L_m^n(\tau, B)\|_{X_\nu \leftarrow X_\mu} \\ &\leq \begin{cases} C (t_{n+1} - t_m)^{-\nu+\mu} \sup_{j \in \{m, \dots, n\}} \|A_j - B_j\|_{X_\vartheta \leftarrow X_{1+\vartheta}}, & \nu - \mu < 1, \\ C (1 + |\ln h|) (t_{n+1} - t_m)^{-1} \sup_{j \in \{m, \dots, n\}} \|A_j - B_j\|_{X_\vartheta \leftarrow X_{1+\vartheta}}, & \nu - \mu = 1, \end{cases} \end{aligned}$$

holds with constant $C > 0$ independent of n and h_j for $j \geq 0$.

Proof. The statement follows by means of the stability bounds provided by Theorems 5.5 and 5.6. For details of the proof, we refer to [18, Lemma 4.6]. \square

6. Conclusions. In this first part of our work, we have studied the stability behavior of exponential general linear methods for the time discretization of autonomous quasi-linear parabolic initial-boundary value problems. The derivation of our main result is based on auxiliary stability bounds for nonautonomous linear parabolic problems, obtained under mild restrictions on the ratios of subsequent time stepsizes. Combining both results we obtain a stability bound for nonautonomous quasi-linear parabolic problems. Together with suitable local error expansions such a stability bound provides a basic ingredient for the convergence analysis of exponential general linear methods applied to quasi-linear parabolic problems, which shall be carried out in a second part of our work.

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