HIGHER-ORDER EXPONENTIAL INTEGRATORS FOR QUASI-LINEAR PARABOLIC PROBLEMS PART II. CONVERGENCE*

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Abstract. In this work, the convergence analysis of explicit exponential time integrators based on general linear methods for quasi-linear parabolic initial-boundary value problems is pursued. Compared to other types of exponential integrators encountering rather severe order reductions, in general, the considered class of exponential general linear methods provides the possibility to construct schemes that retain higher-order accuracy in time when applied to quasi-linear parabolic problems. In view of practical applications, the case of variable time stepsizes is incorporated.

The convergence analysis is based upon two fundamental ingredients. The needed stability bounds, obtained under mild restrictions on the ratios of subsequent time stepsizes, have been deduced in the recent work [5]. The core of the present work is devoted to the derivation of suitable local and global error representations. In conjunction with the stability bounds, a convergence result is established.

Key words. Quasi-linear parabolic problems, Exponential integrators, General linear methods, Variable stepsizes, Stability, Local error, Convergence.

AMS subject classifications. 35K55, 35K90, 65M12

1. Introduction. In the present work, we proceed our convergence analysis of explicit exponential integrators based on general linear methods [2] for quasi-linear parabolic problems. The considered class of time integration methods combines the benefits of exponential Runge–Kutta and exponential Adams–Bashforth methods and permits to construct explicit higher-order schemes that possess favorable stability properties for parabolic evolution equations. In view of practical applications, our investigations include the case of variable time stepsizes. In the first part [5], we have introduced the considered class of time integration methods, and we have proven stability in certain norms, under mild restrictions on the ratios of subsequent time stepsizes. This second part is devoted to the derivation of suitable local and global error representations and the resulting convergence estimates.

Quasi-linear parabolic problems. Quasi-linear parabolic initial-boundary value problems arise in the modelling of minimal surfaces and mean curvature flow, in the study of fluids in porous media and sharp fronts in polymers, and for the description of thin fluid films and diffusion processes with state-dependent diffusivity, see [5] and references given therein.

In accordance with [5], we cast a quasi-linear parabolic initial-boundary value problem into the form of an initial value problem on a Banach space $(X, \|\cdot\|_X)$

$$\begin{cases} u'(t) = \mathcal{Q}\big(u(t)\big)\, u(t)\,, & t \in (0,T)\,, \\ u(0) \;\; \text{given}\,, \end{cases} \tag{1.1}$$

and employ the basic requirement that the sectorial operator $\mathcal{Q}(v) : D(\mathcal{Q}(v)) \to X$ is defined on a domain $D = D(\mathcal{Q}(v)) \subset X$ that is independent of $v \in V$, where $V \subseteq X_\gamma$

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denotes an open subset of some intermediate space $D = X_1 \subset X_\gamma \subseteq X = X_0$. For the convenience of the reader, the precise hypotheses on (1.1) are recapitulated in the appendix, see also [1]. The analysis given in [1] (under Hypothesis A.1 with $\vartheta = 0$) ensures that (1.1) defines a semiflow in $X_\beta \cap V$ for any $\beta \in (\gamma, 1]$. In this work, we shall consider the case $\beta = 1$ and employ the fact that the exact solution fulfills $u(t) \in D \cap V$ for all $t \in [0, T]$. Besides, in view of practical applications, we may assume that Hypothesis A.1 holds for $\vartheta = 0$ as well as for some exponent $\vartheta \in (0, 1)$.

Exponential general linear methods. In the following, we recall the general format of exponential general linear methods and introduce auxiliary abbreviations. Additional details are given in [5].

Henceforth, we set $\mathbb{N} = \{n \in \mathbb{Z} : n \geq 0\}$. We denote by $(h_n)_{n\in\mathbb{N}}$ a sequence of positive time stepsizes such that the corresponding ratios fulfill the condition ($n \in \mathbb{N}$)

$$\chi^{-1} \leq \omega_{n+1} = \frac{h_{n+1}}{h_n} \leq \chi \tag{1.2}$$

with some constant $\chi > 0$. The associated time grid points are defined by ($n \in \mathbb{N}$)

$$t_0 = 0\,, \qquad t_{n+1} = t_n + h_n\,.$$

Besides, for nodes $(c_i)_{i=1}^s$ such that $c_i \in [0, 1]$ for any $i \in \{1, \dots, s\}$, we set ($n \in \mathbb{N}$, $i \in \{1, \dots, s\}$)

$$t_{ni} = t_n + c_i h_n\,.$$

As our focus is on explicit methods, we assume $c_1 = 0$.

The construction of the time-discrete solution $(u_n)_{n\in\mathbb{N}}$ relies on a suitable reformulation of the quasi-linear evolution equation in each time-step and the application of an explicit exponential general linear method to the resulting problem. More precisely, we rewrite (1.1) as follows ($n \in \mathbb{N}$, $t \in (0, T)$)

$$\begin{aligned}
u'(t) = \mathcal{Q}\big(u(t)\big)\, u(t) = \mathcal{Q}_n\, u(t) + \Theta_n\big(u(t)\big) = \mathcal{Q}_n\, u(t) + G_n(t)\,,\\
\Theta_n : D \longrightarrow X : v \longmapsto \big(\mathcal{Q}(v) - \mathcal{Q}_n\big)\, v\,,\\
G_n : [0, T] \longrightarrow X : t \longmapsto \Theta_n\big(u(t)\big) = \big(\mathcal{Q}(u(t)) - \mathcal{Q}_n\big)\, u(t)\,,
\end{aligned} \tag{1.3a}$$

where the sectorial operator $\mathcal{Q}_n : D \to X$ has to be chosen in an appropriate manner (see below). For given initial approximations $u_0, \dots, u_{q-1} \in D \cap V$, approximations to the exact solution values are defined by recurrence ($n \in \{q-1, q, \dots\}$, $i \in \{1, \dots, s\}$)

$$\begin{aligned}
U_{ni} &= \mathrm{e}^{c_i h_n \mathcal{Q}_n}\, u_n + h_n \sum_{j=1}^{i-1} a_{ij}^{(n)}(h_n \mathcal{Q}_n)\, \Theta_n(U_{nj})\\
&\quad + h_n \sum_{k=1}^{q-1} \widetilde{a}_{ik}^{(n)}(h_n \mathcal{Q}_n)\, \Theta_n(u_{n-k})\\
&\approx \widehat{U}_{ni} = u(t_{ni})\,,\\
u_{n+1} &= \mathrm{e}^{h_n \mathcal{Q}_n}\, u_n + h_n \sum_{i=1}^{s} b_i^{(n)}(h_n \mathcal{Q}_n)\, \Theta_n(U_{ni})\\
&\quad + h_n \sum_{k=1}^{q-1} \widetilde{b}_k^{(n)}(h_n \mathcal{Q}_n)\, \Theta_n(u_{n-k})\\
&\approx \widehat{u}_{n+1} = u(t_{n+1})\,.
\end{aligned} \tag{1.3b}$$

For the derivation of our convergence result, which will also ensure existence of the time-discrete solution value $u_n \in D \cap V$ as long as $t_n \leq T$, it is essential that from differences such as $\mathcal{Q}(U_{ni}) - \mathcal{Q}_n : D \to X$ we may extract a factor h_n^α for some exponent $\alpha \in (0,1)$. In conjunction with the Hölder-continuity of the time-discrete solution, this suggests the choice

$$\mathcal{Q}_n = \mathcal{Q}(u_n)\,.$$

Alternative choices involving in addition the Fréchet-derivative of $\mathcal{Q}$ are also admissable.

From a local error expansion, it becomes apparent that the coefficient functions are given as linear combinations of the exponential functions ($\ell \in \mathbb{N}$)

$$\varphi_\ell : \mathbb{C} \longrightarrow \mathbb{C} : z \longmapsto \varphi_\ell(z) = \begin{cases} \mathrm{e}^z\,, & \ell = 0\,, \\ \displaystyle\int_0^1 \mathrm{e}^{(1-\tau)z}\,\frac{\tau^{\ell-1}}{(\ell-1)!}\,\mathrm{d}\tau\,, & \ell \geq 0\,. \end{cases} \tag{1.4}$$

More precisely, we shall require that the order conditions ($n \in \mathbb{N}$, $i \in \{1,\dots,s\}$)

$$\begin{aligned} &\sum_{j=1}^{i-1} c_j^{\ell-1}\, a_{ij}^{(n)}(h_n\mathcal{Q}_n) + \sum_{k=1}^{q-1} \Big(\frac{t_{n-k}-t_n}{h_n}\Big)^{\ell-1}\, \widetilde{a}_{ik}^{(n)}(h_n\mathcal{Q}_n) \\ &\qquad = (\ell-1)!\, c_i^\ell\, \varphi_\ell(c_i h_n \mathcal{Q}_n)\,, \quad \ell \in \{1,\dots,Q\}\,, \\ &\sum_{i=1}^{s} c_i^{\ell-1}\, b_i^{(n)}(h_n\mathcal{Q}_n) + \sum_{k=1}^{q-1} \Big(\frac{t_{n-k}-t_n}{h_n}\Big)^{\ell-1}\, \widetilde{b}_k^{(n)}(h_n\mathcal{Q}_n) \\ &\qquad = (\ell-1)!\, \varphi_\ell(h_n \mathcal{Q}_n)\,, \quad \ell \in \{1,\dots,P\}\,, \end{aligned} \tag{1.5}$$

are satisfied for certain $Q, P \in \mathbb{N}$. In accordance with [6], we call $Q \in \mathbb{N}$ the stage order and $P \in \mathbb{N}$ the quadrature order of the method, see also [3]. Evidently, the following identity holds

$$\frac{t_{n-k}-t_n}{h_n} = \frac{1}{\omega_{n-k+1}\cdots\omega_n} + \dots + \frac{1}{\omega_n}\,.$$

Order of convergence for semi-linear parabolic problems. Explicit exponential general methods that satisfy these order conditions for constant time stepsizes with coefficient functions independent of $n \in \mathbb{N}$

$$\begin{aligned} h_n = h: \quad &\sum_{j=1}^{i-1} c_j^{\ell-1}\, a_{ij}(h\mathcal{Q}_n) + \sum_{k=1}^{q-1} (-k)^{\ell-1}\, \widetilde{a}_{ik}(h\mathcal{Q}_n) \\ &\qquad = (\ell-1)!\, c_i^\ell\, \varphi_\ell(c_i h \mathcal{Q}_n)\,, \quad \ell \in \{1,\dots,Q\}\,, \\ &\sum_{i=1}^{s} c_i^{\ell-1}\, b_i(h\mathcal{Q}_n) + \sum_{k=1}^{q-1} (-k)^{\ell-1}\, \widetilde{b}_k(h\mathcal{Q}_n) \\ &\qquad = (\ell-1)!\, \varphi_\ell(h \mathcal{Q}_n)\,, \quad \ell \in \{1,\dots,P\}\,, \end{aligned}$$

have been constructed in [6], see also references given therein. Furthermore, it has been shown that the order of convergence for semi-linear parabolic problems with sufficiently regular solutions is essentially $p = \min\{P, Q+1\}$. Due to the fact that the stage order of an explicit exponential one-step method is at most one, this implies that higher-order of convergence (i.e. $p \geq 3$) can only be expected for exponential general linear methods involving at least two steps (i.e. $q \geq 2$).

Convergence analysis for quasi-linear parabolic problems. In this work, we are concerned with the derivation of a convergence result for explicit exponential general linear methods of the form (1.3) applied to quasi-linear parabolic problems (1.1). We point out that, even though it is possible to sustain a general approach based on suitable local and global error representations, stability bounds, estimates for the defects, and the application of Gronwall-type inequalities, as used for instance in [6], the convergence analysis of quasi-linear parabolic problems is significantly more involved than the treatment of the semi-linear case, see also the discussion in [5]. In particular, it is essential to prove that the time-discrete solution is Hölder-continuous in a discrete sense, in analogy to the Hölder-continuity of the exact solution.

With the needed stability estimates at hand, the incorporation of variable time stepsizes only slightly increases the amount of technicalities. The issue of constructing variable stepsize exponential multi-stage multi-step methods, however, is more complex and shall be considered in future work.

Regularity requirements. As our focus is on exponential time integration methods that provide the possibility to achieve higher-order accuracy in time, contrary to exponential one-step methods, see [4, 6] and references given in [5], encountering stronger order reductions, in general, we henceforth assume that the sectorial operator $\mathcal{Q}(v) : D \to X$, $v \in V$, is sufficiently often Fréchet differentiable and that the exact solution is sufficiently regular in time. In certain situations, this regularity requirement is indeed justified.

Notation. For a family $(F_\ell)_{\ell \in \mathbb{N}}$ of non-commutative operators on a Banach space, we employ the product notation $(n, m \in \mathbb{N})$

$$\prod_{\ell=m}^{n} F_\ell = \begin{cases} F_n \cdots F_m \,, & n \geq m \,, \\ I \,, & n < m \,. \end{cases}$$

The operator norm of a linear operator F between normed spaces $(W_1, \|\cdot\|_{W_1})$ and $(W_2, \|\cdot\|_{W_1})$ is denoted by $\|F\|_{W_2 \leftarrow W_1}$. In order to simplify the notation, we do not distinguish the arising constants.

2. Convergence result for quasi-linear problems. In this section, we deduce our main result, a convergence estimate for variable stepsize explicit exponential general linear methods applied to quasi-linear parabolic problems, see Theorem 2.1. For this purpose, we first derive suitable local and global error representations.

2.1. Local error representation. In the following, we fix $n \in \{q-1, q, \dots\}$ as well as $i \in \{1, \dots, s\}$ and consider a subinterval $[t_n, t_{n+1}] \subset [0, T]$. Main tools for the derivation of suitable local error expansions reflecting the order conditions (1.5) are the notion of the defect and the linear variation-of-constants formula. In order to reveal the dependencies of the arising components on the time stepsize $h_n > 0$, we employ the symbol $\mathscr{O}(h_n^m)$. The estimation of the defects in appropriate norms is carried out below.

Defects. Replacing the time-discrete solution values by the exact solution values defines the defects of the considered exponential general linear method (1.3)

$$
\begin{aligned}
\widehat{U}_{ni} &= \mathrm{e}^{c_i h_n \mathcal{Q}_n}\, \widehat{u}_n + h_n \sum_{j=1}^{i-1} a_{ij}^{(n)}(h_n \mathcal{Q}_n)\, G_n(t_{nj}) \\
&\quad + h_n \sum_{k=1}^{q-1} \widetilde{a}_{ik}^{(n)}(h_n \mathcal{Q}_n)\, G_n(t_{n-k}) + D_{ni}\,, \\
\widehat{u}_{n+1} &= \mathrm{e}^{h_n \mathcal{Q}_n}\, \widehat{u}_n + h_n \sum_{i=1}^{s} b_i^{(n)}(h_n \mathcal{Q}_n)\, G_n(t_{ni}) \\
&\quad + h_n \sum_{k=1}^{q-1} \widetilde{b}_k^{(n)}(h_n \mathcal{Q}_n)\, G_n(t_{n-k}) + d_{n+1}\,.
\end{aligned}
\tag{2.1}
$$

Expansion of exact solution. In order to deduce a suitable expansion of the exact solution value at time $t_n + \zeta \in [0, T]$, we employ the reformulation

$$
u'(t) = \mathcal{Q}\big(u(t)\big)\, u(t) = \mathcal{Q}_n\, u(t) + G_n(t)\,,
$$

see also (1.3a), and apply the linear variation-of-constants formula

$$
u(t_n + \zeta) = e^{\zeta \mathcal{Q}_n} u(t_n) + \int_0^{\zeta} e^{(\zeta - \tau)\mathcal{Q}_n}\, G_n(t_n + \tau)\, \mathrm{d}\tau\,.
$$

Under the required regularity assumptions, we may replace $G_n(t_n + \tau)$ by its Taylor series expansion

$$
\begin{aligned}
G_n(t_n + \tau) &= \sum_{\ell=0}^{m-1} \tfrac{1}{\ell!}\, \tau^{\ell}\, G_n^{(\ell)}(t_n) + r_{n,m}(\tau)\,, \\
r_{n,m}(\tau) &= \tfrac{1}{(m-1)!} \int_0^{\tau} (\tau - \sigma)^{m-1}\, G_n^{(m)}(t_n + \sigma)\, \mathrm{d}\sigma = \mathscr{O}\big(\tau^m\big)\,,
\end{aligned}
\tag{2.2}
$$

and, by (1.4), we thus obtain the representation

$$
\begin{aligned}
u(t_n + \zeta) &= e^{\zeta \mathcal{Q}_n} u(t_n) + \sum_{\ell=0}^{m-1} \zeta^{\ell+1}\, \varphi_{\ell+1}(\zeta \mathcal{Q}_n)\, G_n^{(\ell)}(t_n) + R_{n,m}(\zeta)\,, \\
R_{n,m}(\zeta) &= \int_0^{\zeta} \mathrm{e}^{(\zeta - \tau)\mathcal{Q}_n}\, r_{n,m}(\tau)\, \mathrm{d}\tau = \mathscr{O}\big(\zeta^{m+1}\big)\,.
\end{aligned}
\tag{2.3}
$$

Expansion of defects. Setting $m = Q$, this relation for the exact solution, the Taylor series expansion (2.2), and straightforward calculations lead to the following representation

$$
\begin{aligned}
D_{ni} &= \widehat{U}_{ni} - \mathrm{e}^{c_i h_n \mathcal{Q}_n}\, \widehat{u}_n - h_n \sum_{j=1}^{i-1} a_{ij}^{(n)}(h_n \mathcal{Q}_n)\, G_n(t_{nj}) \\
&\quad - h_n \sum_{k=1}^{q-1} \widetilde{a}_{ik}^{(n)}(h_n \mathcal{Q}_n)\, G_n(t_{n-k}) \\
&= \sum_{\ell=1}^{Q} \tfrac{1}{(\ell-1)!}\, h_n^{\ell} \Bigg((\ell-1)!\, c_i^{\ell}\, \varphi_\ell(c_i h_n \mathcal{Q}_n) - \sum_{j=1}^{i-1} c_j^{\ell-1}\, a_{ij}^{(n)}(h_n \mathcal{Q}_n) \\
&\qquad\qquad - \sum_{k=1}^{q-1} \frac{(t_{n-k}-t_n)^{\ell-1}}{h_n^{\ell-1}}\, \widetilde{a}_{ik}^{(n)}(h_n \mathcal{Q}_n) \Bigg)\, G_n^{(\ell-1)}(t_n) \\
&\quad + R_{n,Q}(c_i h_n) - h_n \sum_{j=1}^{i-1} a_{ij}^{(n)}(h_n \mathcal{Q}_n)\, r_{n,Q}(c_j h_n) \\
&\quad - h_n \sum_{k=1}^{q-1} \widetilde{a}_{ik}^{(n)}(h_n \mathcal{Q}_n)\, r_{n,Q}(t_{n-k}-t_n)\,.
\end{aligned}
$$

Analogously, setting $m = P$, we obtain

$$
\begin{aligned}
d_{n+1} &= \widehat{u}_{n+1} - \mathrm{e}^{h_n \mathcal{Q}_n}\, \widehat{u}_n - h_n \sum_{i=1}^{s} b_i^{(n)}(h_n \mathcal{Q}_n)\, G_n(t_{ni}) \\
&\quad - h_n \sum_{k=1}^{q-1} \widetilde{b}_k^{(n)}(h_n \mathcal{Q}_n)\, G_n(t_{n-k}) \\
&= \sum_{\ell=1}^{P} \tfrac{1}{(\ell-1)!}\, h_n^{\ell} \Bigg((\ell-1)!\, \varphi_\ell(h_n \mathcal{Q}_n) - \sum_{i=1}^{s} c_i^{\ell-1}\, b_i^{(n)}(h_n \mathcal{Q}_n) \\
&\qquad\qquad - \sum_{k=1}^{q-1} \frac{(t_{n-k}-t_n)^{\ell-1}}{h_n^{\ell-1}}\, \widetilde{b}_k^{(n)}(h_n \mathcal{Q}_n) \Bigg)\, G_n^{(\ell-1)}(t_n) \\
&\quad + R_{n,P}(h_n) - h_n \sum_{i=1}^{s} b_i^{(n)}(h_n \mathcal{Q}_n)\, r_{n,P}(c_i h_n) \\
&\quad - h_n \sum_{k=1}^{q-1} \widetilde{b}_k^{(n)}(h_n \mathcal{Q}_n)\, r_{n,P}(t_{n-k}-t_n)\,.
\end{aligned}
$$

Altogether, due to the required validity of the order conditions (1.5) and by (1.2), we have

$$
\begin{aligned}
D_{ni} &= R_{n,Q}(c_i h_n) - h_n \sum_{j=1}^{i-1} a_{ij}^{(n)}(h_n \mathcal{Q}_n)\, r_{n,Q}(c_j h_n) \\
&\qquad - h_n \sum_{k=1}^{q-1} \widetilde{a}_{ik}^{(n)}(h_n \mathcal{Q}_n)\, r_{n,Q}(t_{n-k} - t_n) \\
&= \mathscr{O}\big(h_n^{Q+1}\big)\,, \\
d_{n+1} &= R_{n,P}(h_n) - h_n \sum_{i=1}^{s} b_i^{(n)}(h_n \mathcal{Q}_n)\, r_{n,P}(c_i h_n) \\
&\qquad - h_n \sum_{k=1}^{q-1} \widetilde{b}_k^{(n)}(h_n \mathcal{Q}_n)\, r_{n,P}(t_{n-k} - t_n) \\
&= \mathscr{O}\big(h_n^{P+1}\big)\,.
\end{aligned}
\tag{2.4}
$$

2.2. Global error representation. As before, we fix $n \in \{q-1, q, \dots\}$ and $i \in \{1, \dots, s\}$. For the following consideration, it is convenient to introduce a short notation for the approximation errors

$$
E_{ni} = U_{ni} - \widehat{U}_{ni}\,, \qquad e_n = u_n - \widehat{u}_n\,.
$$

Relations for global errors. We recall the abbreviations

$$
\Theta_n(v) = \big(\mathcal{Q}(v) - \mathcal{Q}_n\big)\, v\,, \qquad G_n(t_{ni}) = \Theta_n\big(\widehat{U}_{ni}\big) = \Big(\mathcal{Q}\big(\widehat{U}_{ni}\big) - \mathcal{Q}_n\Big)\, \widehat{U}_{ni}\,.
$$

Taking the difference between the relations in (1.3) and (2.1), we obtain

$$
\begin{aligned}
E_{ni} &= \mathrm{e}^{c_i h_n \mathcal{Q}_n}\, e_n + h_n \sum_{j=1}^{i-1} a_{ij}^{(n)}(h_n \mathcal{Q}_n) \Big(\Theta_n(U_{nj}) - \Theta_n\big(\widehat{U}_{nj}\big)\Big) \\
&\qquad + h_n \sum_{k=1}^{q-1} \widetilde{a}_{ik}^{(n)}(h_n \mathcal{Q}_n) \Big(\Theta_n(u_{n-k}) - \Theta_n\big(\widehat{u}_{n-k}\big)\Big) - D_{ni}\,, \\
e_{n+1} &= \mathrm{e}^{h_n \mathcal{Q}_n}\, e_n + h_n \sum_{i=1}^{s} b_i^{(n)}(h_n \mathcal{Q}_n) \Big(\Theta_n(U_{ni}) - \Theta_n\big(\widehat{U}_{ni}\big)\Big) \\
&\qquad + h_n \sum_{k=1}^{q-1} \widetilde{b}_k^{(n)}(h_n \mathcal{Q}_n) \Big(\Theta_n(u_{n-k}) - \Theta_n\big(\widehat{u}_{n-k}\big)\Big) - d_{n+1}\,.
\end{aligned}
\tag{2.5}
$$

For later use, employing the suggestive notation

$$
\widetilde{\mathcal{Q}}(E_{ni}) = \mathcal{Q}(U_{ni}) - \mathcal{Q}\big(\widehat{U}_{ni}\big)\,, \qquad \widetilde{\mathcal{Q}}(e_{n-k}) = \mathcal{Q}(u_{n-k}) - \mathcal{Q}\big(\widehat{u}_{n-k}\big)\,,
$$

we rewrite the arising differences such that the errors are recognised. On the one hand, by adding and substracting $\mathcal{Q}(U_{ni})\, \widehat{U}_{ni}$, we have

$$
\begin{aligned}
\Theta_n(U_{ni}) - \Theta_n\big(\widehat{U}_{ni}\big) &= \big(\mathcal{Q}(U_{ni}) - \mathcal{Q}_n\big)\, U_{ni} - \Big(\mathcal{Q}\big(\widehat{U}_{ni}\big) - \mathcal{Q}_n\Big)\, \widehat{U}_{ni} \\
&= \big(\mathcal{Q}(U_{ni}) - \mathcal{Q}_n\big)\, E_{ni} + \widetilde{\mathcal{Q}}(E_{ni})\, \widehat{U}_{ni}\,.
\end{aligned}
$$

In a similar manner, this yields

$$\Theta_n(u_{n-k}) - \Theta_n\big(\widehat{u}_{n-k}\big) = \big(\mathcal{Q}(u_{n-k}) - \mathcal{Q}_n\big)\, e_{n-k} + \widetilde{\mathcal{Q}}(e_{n-k})\, \widehat{u}_{n-k}\,.$$

As a consequence, we obtain the representations

$$\begin{aligned}
E_{ni} &= \mathrm{e}^{c_i h_n \mathcal{Q}_n}\, e_n \\
&\qquad + h_n \sum_{j=1}^{i-1} a_{ij}^{(n)}(h_n \mathcal{Q}_n)\, \Big(\big(\mathcal{Q}(U_{nj}) - \mathcal{Q}_n\big)\, E_{nj} + \widetilde{\mathcal{Q}}(E_{nj})\, \widehat{U}_{nj}\Big) \\
&\qquad + h_n \sum_{k=1}^{q-1} \widetilde{a}_{ik}^{(n)}(h_n \mathcal{Q}_n)\, \Big(\big(\mathcal{Q}(u_{n-k}) - \mathcal{Q}_n\big)\, e_{n-k} + \widetilde{\mathcal{Q}}(e_{n-k})\, \widehat{u}_{n-k}\Big) \\
&\qquad - D_{ni}\,, \\
e_{n+1} &= \mathrm{e}^{h_n \mathcal{Q}_n}\, e_n \\
&\qquad + h_n \sum_{i=1}^{s} b_{i}^{(n)}(h_n \mathcal{Q}_n)\, \Big(\big(\mathcal{Q}(U_{ni}) - \mathcal{Q}_n\big)\, E_{ni} + \widetilde{\mathcal{Q}}(E_{ni})\, \widehat{U}_{ni}\Big) \\
&\qquad + h_n \sum_{k=1}^{q-1} \widetilde{b}_{k}^{(n)}(h_n \mathcal{Q}_n)\, \Big(\big(\mathcal{Q}(u_{n-k}) - \mathcal{Q}_n\big)\, e_{n-k} + \widetilde{\mathcal{Q}}(e_{n-k})\, \widehat{u}_{n-k}\Big) \\
&\qquad - d_{n+1}\,.
\end{aligned} \tag{2.6}$$

With the help of auxiliary abbreviations for the time-discrete evolution operator and the remaining term

$$\begin{aligned}
\mathscr{E}_m^n &= \prod_{\ell=m}^{n} \mathrm{e}^{h_\ell \mathcal{Q}_\ell}\,, \\
\mathscr{R}_n &= \mathscr{R}_{n1} + \mathscr{R}_{n2}\,, \\
\mathscr{R}_{n1} &= h_n \sum_{i=1}^{s} b_{i}^{(n)}(h_n \mathcal{Q}_n)\, \big(\mathcal{Q}(U_{ni}) - \mathcal{Q}_n\big)\, E_{ni} \\
&\qquad + h_n \sum_{k=1}^{q-1} \widetilde{b}_{k}^{(n)}(h_n \mathcal{Q}_n)\, \big(\mathcal{Q}(u_{n-k}) - \mathcal{Q}_n\big)\, e_{n-k} - d_{n+1}\,, \\
\mathscr{R}_{n2} &= h_n \sum_{i=1}^{s} b_{i}^{(n)}(h_n \mathcal{Q}_n)\, \widetilde{\mathcal{Q}}(E_{ni})\, \widehat{U}_{ni} + h_n \sum_{k=1}^{q-1} \widetilde{b}_{k}^{(n)}(h_n \mathcal{Q}_n)\, \widetilde{\mathcal{Q}}(e_{n-k})\, \widehat{u}_{n-k}\,,
\end{aligned} \tag{2.7}$$

the latter relation takes the form

$$e_{n+1} = \mathscr{E}_n^n\, e_n + \mathscr{R}_n\,.$$

Resolving this recurrence finally implies

$$e_{n+1} = \mathscr{E}_{q-1}^n\, e_{q-1} + \sum_{m=q-1}^{n} \mathscr{E}_{m+1}^n\, \mathscr{R}_m\,.$$

2.3. Global error estimation. In order to deduce the desired global error estimate, we employ stability estimates for the time-discrete evolution operator and bounds for the defects. Again, we fix $n \in \{q-1, q, \dots\}$. Let $\mu, \nu \in [0,1]$ be such that $\mu \le \nu$.

Stability estimates. Relation (A.1) at once implies boundedness of the coefficient functions ($i \in \{1,\dots,s\}$, $j \in \{1,\dots,i-1\}$, $k \in \{1,\dots,q-1\}$)

$$\begin{aligned}
&\big\| h_n^{\nu-\mu}\, a_{ij}^{(n)}(h_n \mathcal{Q}_n) \big\|_{X_\nu \leftarrow X_\mu} + \big\| h_n^{\nu-\mu}\, \widetilde{a}_{ik}^{(n)}(h_n \mathcal{Q}_n) \big\|_{X_\nu \leftarrow X_\mu} \\
&\quad + \big\| h_n^{\nu-\mu}\, b_i^{(n)}(h_n \mathcal{Q}_n) \big\|_{X_\nu \leftarrow X_\mu} + \big\| h_n^{\nu-\mu}\, \widetilde{b}_k^{(n)}(h_n \mathcal{Q}_n) \big\|_{X_\nu \leftarrow X_\mu} \le K\,.
\end{aligned} \tag{2.8}$$

Moreover, from the stability analysis in [5], we obtain

$$\big\| (t_{n+1} - t_m)^{\nu-\mu}\, \mathscr{E}_m^n \big\|_{X_\nu \leftarrow X_\mu} \le C\,. \tag{2.9}$$

Estimates for the defects. By a straightforward estimation of the remainder terms in (2.2) and (2.3), we obtain

$$\begin{aligned}
\|r_{n,m}(\tau)\|_{X_\mu} &\le \tfrac{1}{m!}\, \tau^m \max_{t \in [0,T]} \big\| G_n^{(m)}(t) \big\|_{X_\mu}\,, \\
\|R_{n,m}(\tau)\|_{X_\mu} &\le \tfrac{1}{(m+1)!}\, C\, \tau^{m+1} \max_{t \in [0,T]} \big\| G_n^{(m)}(t) \big\|_{X_\mu}\,,
\end{aligned}$$

see also (A.1). The representation (2.4) for the defects together with (2.8) thus yields

$$\begin{aligned}
\|D_{ni}\|_{X_\mu} &\le \|R_{n,Q}(c_i h_n)\|_{X_\mu} + h_n \sum_{j=1}^{i-1} \big\| a_{ij}^{(n)}(h_n \mathcal{Q}_n) \big\|_{X_\mu \leftarrow X_\mu} \|r_{n,Q}(c_j h_n)\|_{X_\mu} \\
&\qquad + h_n \sum_{k=1}^{q-1} \big\| \widetilde{a}_{ik}^{(n)}(h_n \mathcal{Q}_n) \big\|_{X_\mu \leftarrow X_\mu} \|r_{n,Q}(t_{n-k} - t_n)\|_{X_\mu} \\
&\le C\, h_n^{Q+1} \max_{t \in [0,T]} \big\| G_n^{(Q)}(t) \big\|_{X_\mu}\,, \\
\|d_{n+1}\|_{X_\mu} &\le \|R_{n,P}(h_n)\|_{X_\mu} + h_n \sum_{i=1}^{s} \big\| b_i^{(n)}(h_n \mathcal{Q}_n) \big\|_{X_\mu} \|r_{n,P}(c_i h_n)\|_{X_\mu} \\
&\qquad + h_n \sum_{k=1}^{q-1} \big\| \widetilde{b}_k^{(n)}(h_n \mathcal{Q}_n) \big\|_{X_\mu} \|r_{n,P}(t_{n-k} - t_n)\|_{X_\mu} \\
&\le C\, h_n^{P+1} \max_{t \in [0,T]} \big\| G_n^{(P)}(t) \big\|_{X_\mu}\,.
\end{aligned} \tag{2.10}$$

Estimation of global error. We estimate the global error with respect to the norm of the domain D. Applying (2.9) at once yields $\|\mathscr{E}_{q-1}^n\, e_{q-1}\|_D \le C\, \|e_{q-1}\|_D$ and hence

$$\begin{aligned}
\|e_{n+1}\|_D \le C\, \|e_{q-1}\|_D & \\
&+ \sum_{m=q-1}^{n} \big\| \mathscr{E}_{m+1}^n\, \mathscr{R}_{m1} \big\|_D \\
&+ \sum_{m=q-1}^{n} \big\| \mathscr{E}_{m+1}^n\, \mathscr{R}_{m2} \big\|_D\,.
\end{aligned}$$

We point out that we need that Hypothesis A.1 holds for $\vartheta = 0$ ($v, w \in X_\gamma$, $z \in D$)

$$\big\| \big(\mathcal{Q}(v) - \mathcal{Q}(w) \big) z \big\|_X \le L\, \|v - w\|_{X_\gamma}\, \|z\|_D$$

as well as for some exponent $\vartheta \in (0,1)$ $(v,w \in X_\gamma,\, z \in X_{1+\vartheta})$

$$\big\| \big(\mathcal{Q}(v) - \mathcal{Q}(w)\big) z \big\|_{X_\vartheta} \leq L \, \|v - w\|_{X_\gamma} \, \|z\|_{X_{1+\vartheta}} \, .$$

It remains to study the decisive terms

$$\begin{aligned}
\|\mathscr{E}^n_{m+1} \mathscr{R}_{m1}\|_D &\leq h_m \sum_{i=1}^{s} \big\|\mathscr{E}^n_{m+1} \, b_i^{(m)}(h_m \mathcal{Q}_m)\big\|_{D \leftarrow X} \, \big\| \big(\mathcal{Q}(U_{mi}) - \mathcal{Q}_m\big) \, E_{mi} \big\|_X \\
&\quad + h_m \sum_{k=1}^{q-1} \big\|\mathscr{E}^n_{m+1} \, \widetilde{b}_k^{(m)}(h_m \mathcal{Q}_m)\big\|_{D \leftarrow X} \, \big\| \big(\mathcal{Q}(u_{m-k}) - \mathcal{Q}_m\big) \, e_{m-k} \big\|_X \\
&\quad + \|\mathscr{E}^n_{m+1} \, d_{m+1}\|_D \, , \\
\|\mathscr{E}^n_{m+1} \mathscr{R}_{m2}\|_D &\leq h_m \sum_{i=1}^{s} \big\|\mathscr{E}^n_{m+1} \, b_i^{(m)}(h_m \mathcal{Q}_m)\big\|_{D \leftarrow X_\vartheta} \, \big\| \widetilde{\mathcal{Q}}(E_{mi}) \, \widehat{U}_{mi} \big\|_{X_\vartheta} \\
&\quad + h_m \sum_{k=1}^{q-1} \big\|\mathscr{E}^n_{m+1} \, \widetilde{b}_k^{(m)}(h_m \mathcal{Q}_n)\big\|_{D \leftarrow X_\vartheta} \, \big\| \widetilde{\mathcal{Q}}(e_{m-k}) \, \widehat{u}_{m-k} \big\|_{X_\vartheta} \, ,
\end{aligned}$$

see also (2.7). In view of the analysis provided in [1], for initial values in D, it is reasonable to assume boundedness of the exact solution in D

$$M(\widehat{u}, D) = \max_{t \in [0,T]} \|u(t)\|_D \, ;$$

moreover, for initial values $u(0) \in X_{1+\vartheta}$, the results given in [1, Sec. 9] imply boundedness of the solution in $X_{1+\vartheta}$

$$M(\widehat{u}, X_{1+\vartheta}) = \max_{t \in [0,T]} \|u(t)\|_{X_{1+\vartheta}} \, .$$

As a consequence, we obtain the following estimates

$$\begin{aligned}
\big\| \big(\mathcal{Q}(U_{ni}) - \mathcal{Q}_n\big) \, E_{ni} \big\|_X &\leq L \, \|\mathcal{Q}(U_{ni}) - \mathcal{Q}_n\|_{X \leftarrow D} \, \|E_{ni}\|_D \, , \\
\big\| \big(\mathcal{Q}(u_{n-k}) - \mathcal{Q}_n\big) \, e_{n-k} \big\|_X &\leq L \, \|\mathcal{Q}(u_{n-k}) - \mathcal{Q}_n\|_{X \leftarrow D} \, \|e_{n-k}\|_D \, , \\
\big\| \widetilde{\mathcal{Q}}(E_{ni}) \, \widehat{U}_{ni} \big\|_{X_\vartheta} &\leq L \, M(\widehat{u}, X_{1+\vartheta}) \, \|E_{ni}\|_{X_\gamma} \, , \\
\big\| \widetilde{\mathcal{Q}}(e_{n-k}) \, \widehat{u}_{n-k} \big\|_{X_\vartheta} &\leq L \, M(\widehat{u}, X_{1+\vartheta}) \, \|e_{n-k}\|_{X_\gamma} \, .
\end{aligned}$$

Consequently, employing relation (2.8) for the coefficient functions, we have

$$\begin{aligned}
\|\mathscr{E}^n_{m+1} \mathscr{R}_{m1}\|_D &\leq C \, h_m \, (t_{n+1} - t_{m+1})^{-1} \sum_{i=1}^{s} \|\mathcal{Q}(U_{mi}) - \mathcal{Q}_m\|_{X \leftarrow D} \, \|E_{mi}\|_D \\
&\quad + C \, h_m \, (t_{n+1} - t_{m+1})^{-1} \sum_{k=1}^{q-1} \|\mathcal{Q}(u_{m-k}) - \mathcal{Q}_m\|_{X \leftarrow D} \, \|e_{m-k}\|_D \\
&\quad + \|\mathscr{E}^n_{m+1} \, d_{m+1}\|_D \, , \\
\|\mathscr{E}^n_{m+1} \mathscr{R}_{m2}\|_D &\leq C \, M(\widehat{u}, X_{1+\vartheta}) \, h_m \, (t_{n+1} - t_{m+1})^{-1+\vartheta} \sum_{i=1}^{s} \|E_{mi}\|_{X_\gamma} \\
&\quad + C \, M(\widehat{u}, X_{1+\vartheta}) \, h_m \, (t_{n+1} - t_{m+1})^{-1+\vartheta} \sum_{k=1}^{q-1} \|e_{m-k}\|_{X_\gamma} \, .
\end{aligned}$$

From this, using in addition that $\|E_{mi}\|_{X_\gamma} \le \|E_{mi}\|_D$ as well as $\|e_{m-k}\|_{X_\gamma} \le \|e_{m-k}\|_D$, we obtain the global error estimate

$$\begin{aligned}\|e_{n+1}\|_D \le C\,\|e_{q-1}\|_D \\ &+ C\sum_{i=1}^{s}\sum_{m=q-1}^{n} h_m\,(t_{n+1}-t_m)^{-1}\,\|\mathcal{Q}(U_{mi})-\mathcal{Q}_m\|_{X\leftarrow D}\,\|E_{mi}\|_D \\ &+ C\sum_{k=1}^{q-1}\sum_{m=q-1}^{n} h_m\,(t_{n+1}-t_m)^{-1}\|\mathcal{Q}(u_{m-k})-\mathcal{Q}_m\|_{X\leftarrow D}\,\|e_{m-k}\|_D \\ &+ C\,M(\widehat{u},X_{1+\vartheta})\sum_{i=1}^{s}\sum_{m=q-1}^{n} h_m\,(t_{n+1}-t_m)^{-1+\vartheta}\,\|E_{mi}\|_D \\ &+ C\,M(\widehat{u},X_{1+\vartheta})\sum_{k=1}^{q-1}\sum_{m=q-1}^{n} h_m\,(t_{n+1}-t_m)^{-1+\vartheta}\,\|e_{m-k}\|_D \\ &+ C\sum_{m=q-1}^{n}\|\mathscr{E}_{m+1}^{n}\,d_{m+1}\|_D\,.\end{aligned}$$

In a similar manner, the corresponding estimate for the internal stages follows

$$\begin{aligned}\|E_{ni}\|_D \le \|e_n\|_D \\ &+ C\sum_{j=1}^{i-1}\|\mathcal{Q}(U_{nj})-\mathcal{Q}_n\|_{X\leftarrow D}\,\|E_{nj}\|_D \\ &+ C\sum_{k=1}^{q-1}\|\mathcal{Q}(u_{n-k})-\mathcal{Q}_n\|_{X\leftarrow D}\,\|e_{n-k}\|_D \\ &+ C\,M(\widehat{u},X_{1+\vartheta})\,h_n^{\vartheta}\sum_{j=1}^{i-1}\|E_{nj}\|_D + C\,M(\widehat{u},X_{1+\vartheta})\,h_n^{\vartheta}\sum_{k=1}^{q-1}\|e_{n-k}\|_D \\ &+ \|D_{ni}\|_D\,.\end{aligned}$$

So far, we have not used the particular choice of the sectorial operator $\mathcal{Q}_n : D \to X$ defining the numerical scheme. Due to the arising strong singularity $(t_{n+1}-t_m)^{-1}$, we need to extract a certain power of the time increment from the difference $\mathcal{Q}(U_{mi})-\mathcal{Q}_m$ and $\mathcal{Q}(u_{m-k})-\mathcal{Q}_m$, respectively. More precisely, we employ the natural choice

$$\mathcal{Q}_n = \mathcal{Q}(u_n)$$

and make use of the fact that the numerical solution is Hölder-continuous for some exponent $\alpha \in (0,1)$, in analogy to the property of the exact solution, see [1] and [5]. This yields

$$\|\mathcal{Q}(U_{ni})-\mathcal{Q}_n\|_{X\leftarrow D} \le C\,(c_i h_n)^{\alpha} \le C\,(t_{n+1}-t_m)^{\alpha}$$

as well as

$$\|\mathcal{Q}(u_{n-k})-\mathcal{Q}_n\|_{X\leftarrow D} \le C\,(t_n-t_{n-k})^{\alpha} \le C\,(t_{n+1}-t_m)^{\alpha}\,,$$

with constant depending on certain ratios of subsequent time stepsizes. Altogether, we obtain the bounds

$$
\begin{aligned}
\|e_{n+1}\|_D \leq C\,\|e_{q-1}\|_D
&+ C \sum_{i=1}^{s} \sum_{m=q-1}^{n} h_m\,(t_{n+1}-t_m)^{-1+\alpha}\,\|E_{mi}\|_D \\
&+ C \sum_{k=1}^{q-1} \sum_{m=q-1}^{n} h_m\,(t_{n+1}-t_m)^{-1+\alpha}\,\|e_{m-k}\|_D \\
&+ C\,M(\widehat{u}, X_{1+\vartheta}) \sum_{i=1}^{s} \sum_{m=q-1}^{n} h_m\,(t_{n+1}-t_m)^{-1+\vartheta}\,\|E_{mi}\|_D \\
&+ C\,M(\widehat{u}, X_{1+\vartheta}) \sum_{k=1}^{q-1} \sum_{m=q-1}^{n} h_m\,(t_{n+1}-t_m)^{-1+\vartheta}\,\|e_{m-k}\|_D \\
&+ \sum_{m=q-1}^{n} \big\|\mathscr{E}_{m+1}^{n}\,d_{m+1}\big\|_D\,,
\end{aligned}
$$

$$
\begin{aligned}
\|E_{ni}\|_D \leq \|e_n\|_D + C h_n^{\alpha} \sum_{j=1}^{i-1} \|E_{nj}\|_D + C h_n^{\alpha} \sum_{k=1}^{q-1} \|e_{n-k}\|_D \\
+ C\,M(\widehat{u}, X_{1+\vartheta})\,h_n^{\vartheta} \sum_{j=1}^{i-1} \|E_{nj}\|_D + C\,M(\widehat{u}, X_{1+\vartheta})\,h_n^{\vartheta} \sum_{k=1}^{q-1} \|e_{n-k}\|_D \\
+ \big\|D_{ni}\big\|_D\,.
\end{aligned}
$$

The application of a Gronwall-type inequality, see for instance [6], thus proves our main result, Theorem 2.1, stated below.

2.4. Main result. For the sake of a compact formulation of the global error bound, we introduce the maximal time stepsize $h_{\max} = \max\{h_n : t_n \in [0,T]\}$. Due to (2.10), estimation of the arising Riemann-sum by the corresponding integral yields

$$
\sum_{m=q-1}^{n} \big\|\mathscr{E}_{m+1}^{n}\,d_{m+1}\big\|_D \leq C\,h_{\max}^{P}\,|\ln h_{\max}| \max_{t,t_n\in[0,T]} \big\|G_n^{(P)}(t)\big\|_D\,.
$$

This finally proves the following result.

THEOREM 2.1. *Assume that Hypothesis A.1 is satisfied for $\vartheta = 0$ as well as $\vartheta \in (0,1)$. Consider an exponential general linear multistep method of quadrature order $P \in \mathbb{N}$ and stage order $Q \in \mathbb{N}$ that satisfies condition* (1.2)*. Assume in addition that the quantities*

$$
\begin{aligned}
M(\widehat{u}, X_{1+\vartheta}) &= \max_{t\in[0,T]} \|u(t)\|_{X_{1+\vartheta}}\,, \\
G_n : [0,T] \to X : t &\mapsto \big(\mathcal{Q}(u(t)) - \mathcal{Q}_n\big)\,u(t)\,, \\
M(r,G,D) &= \max_{t,t_n\in[0,T]} \big\|G_n^{(r)}(t)\big\|_D\,, \qquad r = \max\{P,Q\}\,,
\end{aligned}
$$

remain bounded. Then, the global error estimate

$$
\big\|u_n - u(t_n)\big\|_D \leq C\,\big\|u_{q-1} - u(t_{q-1})\big\|_D + C\,h_{\max}^{P}\,|\ln h_{\max}| + C\,h_{\max}^{Q+1}
$$

is valid for $n \in \mathbb{N}$ such that $t_n \in [0,T]$ with $h_{\max} = \max\{h_n : t_n \in [0,T]\}$. The constant $C > 0$ in particular depends on $M(\widehat{u}, X_{1+\vartheta})$ and $M(r,G,D)$.

REMARK 2.2.

(i) Results on the differentiability of the exact solution with respect to the initial state and additional parameters are obtained in [1, Sec. 11] under stronger hypotheses regarding the differentiability of the defining operator $\mathcal{Q}$; similar arguments imply differentiability of the exact solution with respect to time and thus boundedness of $M(r,G,D)$.

(ii) Combining in a standard manner the stability result provided in [5] with our convergence result implies well-definedness of the considered exponential general linear methods, see also Theorem B.1. That is, for initial approximations $u_0, \dots, u_{q-1} \in D \cap V$ that lie sufficiently close to the corresponding values of the exact solution $u(t_0), \dots, u(t_{q-1}) \in D \cap V$, the recurrence relation (1.3) is applicable as long as $t_{n+1} \leq T$.

(iii) From this convergence result, we conclude that numerically the order

$$p = \min\{Q+1, P\}$$

will be observed. This is in accordance with the convergence result for the semi-linear parabolic case [6]. Thus, the considered class of exponential general methods provides the possibility to construct (multi-stage) multi-step schemes that retain higher-order accuracy in time when applied to quasi-linear parabolic problems with regular solutions, contrary to one-step methods such as exponential Runge–Kutta methods or exponential Magnus-type integrators, which encounter severe order reductions.

(iv) As our derivation of Theorem 2.1 relies on Gronwall-type inequalities, the effective size of the arising constant is overestimated, in general. For this reason, we merely specify its dependency on certain regularity properties of the exact solution and the related function G_n, which correspond to certain regularity and compatibility properties of the defining operator family and the initial state. In addition, the constant depends on the coefficients of the exponential general linear method and the stability constant, see Theorem B.1, and thereby is effected by the constants arising in Hypotheses A.1–A.2 and by the size of the final time $T > 0$.

(v) In situations, where G_n satisfies weaker regularity requirements, the expected order reduction can be explained by a suitable modification of our local error expansion. Besides, in order to explain a fractional order of convergence, the smoothing property of the discrete evolution operator is utilised, see for instance [4].

3. Numerical experiments. In this section, we illustrate the convergence behaviour of exponential general linear methods of orders one up to six. For studying the time discretisation error caused by higher-order methods, it is essential to ensure that the space discretisation error is sufficiently small; for this reason, we consider a test problem in a single space dimension. In addition, we focus on a linearisation about the Laplacian, since this permits to use fast Fourier transform techniques for a rapid computation of the action of the arising exponential functions on vectors.

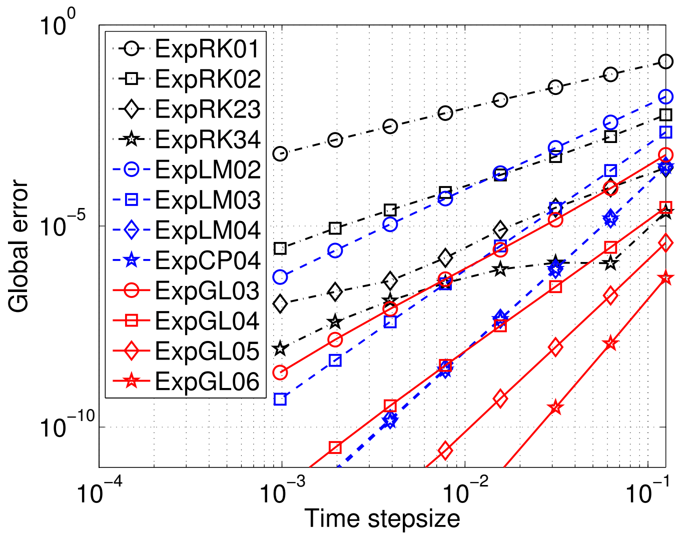

FIG. 3.1. *Numerical results obtained for the quasi-linear test problem with* $c_1 = \frac{1}{10}$, $c_2 = \frac{1}{20}$.

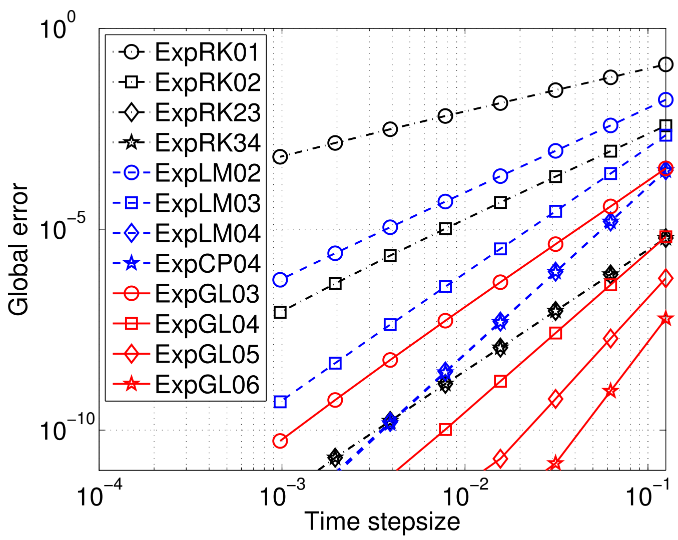

FIG. 3.2. *Numerical results for the linear test problem with* $c_1 = 0$, $c_2 = 0$.

Test problem. In the lines of [5, Ex. 2.4], we set $\Omega = (-1, 1) \subset \mathbb{R}$ and consider the initial-boundary value problem

$$\left\{ \begin{array}{l} \partial_t U(x,t) = \mathcal{Q}\big(x, U(x,t)\big)\, U(x,t) + r(x,t)\,, \qquad (x,t) \in \Omega \times (0,T)\,, \\ U(x,t) = 0\,, \qquad (x,t) \in \partial\Omega \times [0,T]\,, \\ U(x,0) = U_0(x)\,, \qquad x \in \overline{\Omega}\,, \end{array} \right. \tag{3.1a}$$

defined by the second-order differential operator

$$\begin{gathered} \alpha(z_1, z_2) = 1 + c_1 z_1^2 + c_2 z_2^2\,, \qquad c_1 = \tfrac{1}{10}\,, \quad c_2 = \tfrac{1}{20}\,, \\ \mathcal{Q}\big(v(x)\big)\, w(x) = \alpha\big(v(x), \partial_x v(x)\big)\, \partial_{xx} w(x)\,; \end{gathered} \tag{3.1b}$$

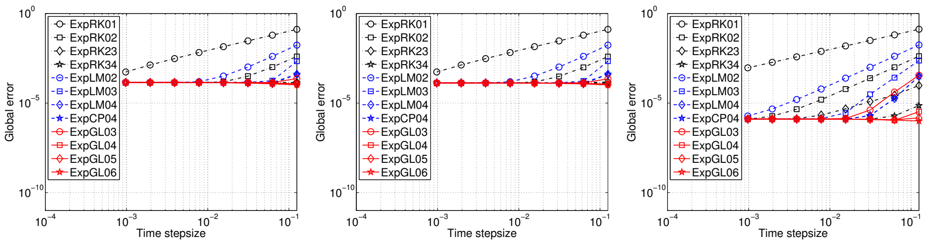

FIG. 3.3. *Numerical results obtained for the linear test problem with $c_1 = 0$, $c_2 = 0$ (left) and the quasi-linear test problem with $c_1 = \frac{1}{10}$, $c_2 = \frac{1}{20}$ (middle, right), when a smoother exact solution is prescribed ($m = 2$). For higher-order exponential general linear methods, the global error is dominated by the space discretisation error (left, middle: $M = 100$, right: $M = 1000$).*

the additional space-time-dependent inhomogeneity permits to prescribe an exact solution that in particular satisfies homogeneous Dirichlet boundary conditions. We rewrite the above initial-boundary value problem as an abstract initial value problem for $u(t) = U(\cdot, t)$

$$\begin{cases} u'(t) = \mathcal{Q}\big(u(t)\big)\, u(t) + r(t)\,, & t \in (0, T)\,, \\ u(0) \;\; \text{given}\,. \end{cases} \tag{3.1c}$$

As underlying Banach space, we consider $X = L^\kappa(\Omega)$ for some exponent $\kappa \in (1, \infty)$ and hence obtain $D = \{v \in W^2_\kappa(\Omega) : v = 0 \text{ on } \partial\Omega\}$; in the present situation, our basic assumptions are satisfied with $\vartheta < \frac{1}{2\kappa}$ and $\gamma \in (\frac{1}{2} + \frac{1}{2\kappa}, 1]$.

Exponential general linear methods. For the time discretisation of (3.1), we apply various exponential general linear methods found in literature, see [3, 6] and references given therein. In the special case of a single step, they reduce to exponential Runge–Kutta methods, and in the special case of a single stage, they reduce to exponential linear multi-step methods. For the convenience of the reader, the schemes are collected in Section C.

Global errors. In Figure 3.1, we display the global errors with respect to the norm in D, obtained for $U : \overline{\Omega} \times [0, T] \to \mathbb{R} : (x, t) \mapsto \mathrm{e}^{-t}\,(x + 1)\,(1 - x)$, a large exponent $\kappa \gg 1$ such that $\vartheta \approx 0$, $M = 100$ equidistant grid points in space, the final time $T = 1$, and time stepsizes $h = 2^j$ for $j \in \{3, \dots, 10\}$. The numerical results illustrate the favourable accuracy of higher-order exponential general linear methods. For comparison, we include the corresponding results for the linear test equation with $c_1 = c_2 = 0$. Here, it is evident that the requirements of Theorem 2.1 on the exact solution are satisfied, since $\|u(t)\|_{X_{1+\vartheta}}$ is bounded and $G_n(t) = 0$ for all $t \in [0, T]$; as expected, we thus retain the following orders of convergence with respect to $\|\cdot\|_D$

$p = 1:$	ExpRK01 (exponential Euler method),
$p = 2:$	ExpRK02, ExpLM02,
$p = 3:$	ExpRK23, ExpRK34, ExpLM03, ExpGL03,
$p = 4:$	ExpLM04, ExpCP04, ExpGL04,
$p = 5:$	ExpGL05,
$p = 6:$	ExpGL06.

The numerical observation of higher orders of convergence in time is delicate, when considering instead an exact solution, where also certain space derivatives satisfy the homogeneous Dirichlet boundary condition, for instance, of the form $U(x, t) : \overline{\Omega} \times [0, T] \to \mathbb{R} : (x, t) \mapsto \mathrm{e}^{-t}\,(x + 1)^m\,(1 - x)^m$. Indeed, from the results displayed in Figure 3.3 for $m = 2$, $M = 100$, and the refined number of space grid points $M = 1000$, we conclude that for higher-order schemes, the global errors are dominated by the space discretisation errors.

4. Conclusions. In this second part of our work, we have deduced a convergence result for exponential general linear methods applied to quasi-linear parabolic problems. The proof of the global error estimate is based on stability results that have been provided in the first part of our work. Interesting open questions that shall be investigated in the future comprise the construction of exponential general linear methods involving variable stepsizes and their realisation for an application of practical relevance.

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Appendix A. Analytical framework. In the following, we recapitulate the employed hypotheses on the quasi-linear parabolic problem (1.1) and recall fundamental bounds for the analytic semigroup generated by the sectorial operator $\mathcal{Q}(v) : D \to X$.

We consider a complex Banach space $(X, \|\cdot\|_X)$ and a dense, continuously embedded subspace $(D, \|\cdot\|_D)$. For exponents $\mu \in [0,1]$ we denote by $(X_\mu, \|\cdot\|_{X_\mu})$ interpolation spaces between $D = X_1$ and $X = X_0$, as specified in [5].

The right-hand side of the differential equation in (1.1) is defined by a family of operators $(\mathcal{Q}(v))_{v\in V}$, where $V \subseteq X_\gamma$ is an open subset of an interpolation space X_γ with exponent $\gamma \in [0,1)$. Our fundamental hypotheses on $\mathcal{Q}$ are as follows.

Hypothesis A.1. Let $v \in V$.

(i) The closed linear operator $\mathcal{Q}(v) : X_1 \to X_0$ is sectorial, uniformly for $v \in V$, that is, there exist constants $a \in \mathbb{R}$, $\phi \in (0, \frac{\pi}{2})$, and $M > 0$ such that for every element $v \in V$ and for any complex number $\lambda \in \mathbb{C}$ in the complement of the sector $S_\phi(a) = \{a\} \cup \big\{z \in \mathbb{C} : |\arg(a-z)| \leq \phi\big\}$ the resolvent estimate

$$\Big\|\big(\lambda I - \mathcal{Q}(v)\big)^{-1}\Big\|_{X\leftarrow X} \leq \frac{M}{|\lambda - a|}$$

is satisfied.

(ii) The graph norm of $\mathcal{Q}(v)$ and the norm in X_1 are equivalent, that is, the relation

$$K^{-1}\,\|x\|_{X_1} \leq \|x\|_{X_0} + \big\|\mathcal{Q}(v)\,x\big\|_{X_0} \leq K\,\|x\|_{X_1}$$

holds with a constant $K > 0$ for all elements $x \in X_1$.

(iii) For some exponent $\vartheta \in [0,\gamma]$ the interpolation space $X_{1+\vartheta}$ between X_1 and the domain of $(\mathcal{Q}(v))^2$ does not depend on $v \in V$. Moreover, the mapping $\mathcal{Q} : V \to L(X_{1+\vartheta}, X_\vartheta)$ is Lipschitz-continuous, that is, the estimate

$$\big\|\mathcal{Q}(v) - \mathcal{Q}(w)\big\|_{X_\vartheta \leftarrow X_{1+\vartheta}} \leq L \, \|v - w\|_{X_\gamma}$$

is valid with a constant $L > 0$ for all elements $v, w \in V$.

If the considered explicit exponential general linear methods are based on a linearisation involving the first derivative of $\mathcal{Q}$, we further impose the following regularity requirement with exponent $\vartheta \in [0,\gamma]$ chosen accordingly to Hypothesis A.1.

HYPOTHESIS A.2. The map $\mathcal{Q}$ belongs to $\mathscr{C}^1\big(V, L(X_{1+\vartheta}, X_\vartheta)\big)$ and its derivative $\mathcal{Q}' : V \to L(X_{1+\vartheta}, X_\vartheta)$ is Lipschitz-continuous, that is, the bound

$$\big\|\mathcal{Q}'(v) - \mathcal{Q}'(w)\big\|_{X_\vartheta \leftarrow X_{1+\vartheta}} \leq L \, \|v - w\|_{X_\gamma}$$

is valid with a constant $L > 0$ for all elements $v, w \in V$.

The analytic semigroup $\big(\mathrm{e}^{t\mathcal{Q}(v)}\big)_{t\in[0,\infty)}$ generated by the sectorial operator $\mathcal{Q}(v) : X_1 \to X_0$ is given by the integral formula of Cauchy

$$\mathrm{e}^{t\mathcal{Q}(v)} = \begin{cases} I \,, & t = 0 \,, \\ \dfrac{1}{2\pi \mathrm{i}} \displaystyle\int_\Gamma \mathrm{e}^{\lambda} \big(\lambda I - t\mathcal{Q}(v)\big)^{-1} \, \mathrm{d}\lambda \,, & t > 0 \,, \end{cases}$$

where Γ denotes a path that surrounds the spectrum of $t\mathcal{Q}(v)$. As a consequence, the estimates

$$\begin{aligned} \big\|t^{\nu-\mu} \, \mathrm{e}^{t\mathcal{Q}(v)}\big\|_{X_\nu \leftarrow X_\mu} + \big\|t^{\nu-\mu} \left(\mathrm{e}^{t\mathcal{Q}(v)} - I\right)\big\|_{X_\nu \leftarrow X_\mu} \leq K \,, \\ \big\|t^{\nu-\mu} \, \varphi_j\big(t\mathcal{Q}(v)\big)\big\|_{X_\nu \leftarrow X_\mu} \leq K \,, \end{aligned} \tag{A.1}$$

are valid for $t \in [0,T]$, $\mu, \nu \in [0,1]$ such that $\mu \leq \nu$, and $j \in \mathbb{N}$, see also (1.4).

Appendix B. Stability results for quasi-linear problems. For the convenience of the reader, we restate the main result of [5], providing stability bounds for variable stepsize explicit exponential general linear methods.

With regard to (1.3), we consider sequences $(v_n)_{n\in\mathbb{N}}, (w_n)_{n\in\mathbb{N}}$, defined through the recurrence formulas

$$\left\{ \begin{aligned} v_{n+1} &= e^{h_n \mathcal{Q}(v_n)} \, v_n + h_n \, f_{n+1} \,, \\ w_{n+1} &= e^{h_n \mathcal{Q}(w_n)} \, w_n + h_n \, g_{n+1} \,, \end{aligned} \right. \qquad n \in \mathbb{N} \,, \tag{B.1}$$

where $v_0, w_0 \in X_\beta \cap V$ and $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}}$ are assumed to be bounded in X_β.

THEOREM B.1 ([5]). *Assume that Hypothesis A.1 and condition* (1.2) *are fulfilled. Then, there exists a final time $T_1 > 0$ and a maximal time stepsize $h > 0$ such that for any stepsize sequence $(h_j)_{j\in\mathbb{N}}$ with $0 < h_j \leq h$ for $j \in \mathbb{N}$, the sequences $(v_n)_{n\in\mathbb{N}}, (w_n)_{n\in\mathbb{N}}$ given by* (B.1) *satisfy the bound*

$$\|v_n - w_n\|_{X_\beta} \leq C \left(\|v_0 - w_0\|_{X_\beta} + \max_{1\leq j\leq n} \|f_j - g_j\|_{X_\beta} \right), \qquad 0 \leq t_n \leq T_1,$$

with constant $C > 0$ independent of n and h_j for $j \in \mathbb{N}$.

Provided that also Hypothesis A.2 is satisfied, the stability result remains valid if $\mathcal{Q}(v_n), \mathcal{Q}(w_n)$ are replaced by alternative linearisations involving in addition the Fréchet derivatives $\mathcal{Q}'(v_n), \mathcal{Q}'(w_n)$.

Appendix C. Exponential general linear integrators. For the sake of completeness, we specify the exponential general linear methods employed in our numerical experiments in terms of a pseudo-code. We denote by Rn the remainder resulting from a linearisation about Q_n and by PhiQn$(\cdot, k, \cdot, \cdot)$ the k-th exponential function; for a scheme involving several steps, u and um (*minus*) correspond to the current and the previous solution values.

```
> if Integrator == 'ExpRK01'
> Ru = Rn(Qn,x,t,dt,u);
> Aux = PhiQn(Qn,1,dt,Ru);
> u = PhiQn(Qn,0,dt,u) + dt*Aux;
> end

> if Integrator == 'ExpRK02'
> c2 = 1/2;
> Ru = Rn(Qn,x,t,dt,u);
> Aux = PhiQn(Qn,1,c2*dt,c2*Ru);
> U = PhiQn(Qn,0,c2*dt,u) + dt*Aux;
> RU = Rn(Qn,x,t+c2*dt,dt,U);
> Aux = PhiQn(Qn,1,dt,Ru);
> Aux = Aux + PhiQn(Qn,2,dt,-1/c2*Ru+1/c2*RU);
> u = PhiQn(Qn,0,dt,u) + dt*Aux;
> end

> if Integrator == 'ExpRK23'
> c2 = 1/2;
> c3 = 1;
> Ru = Rn(Qn,x,t,dt,u);
> Aux = PhiQn(Qn,1,c2*dt,1/2*Ru);
> U2 = PhiQn(Qn,0,c2*dt,u) + dt*Aux;
> RU2 = Rn(Qn,x,t+c2*dt,dt,U2);
> Aux = PhiQn(Qn,1,c3*dt,-Ru+2*RU2);
> U3 = PhiQn(Qn,0,c3*dt,u) + dt*Aux;
> RU3 = Rn(Qn,x,t+c3*dt,dt,U3);
> Aux = PhiQn(Qn,1,dt,Ru);
> Aux = Aux + PhiQn(Qn,2,dt,-3*Ru+4*RU2-RU3);
> Aux = Aux + PhiQn(Qn,3,dt,4*Ru-8*RU2+4*RU3);
> u = PhiQn(Qn,0,dt,u) + dt*Aux;
> end

> if Integrator == 'ExpRK34'
> c1 = 0;
> c2 = 1/2;
> c3 = 1/2;
> c4 = 1;
> Ru = Rn(Qn,x,t,dt,u);
> Aux = 1/2*PhiQn(Qn,1,c2*dt,Ru);
> U2 = PhiQn(Qn,0,c2*dt,u) + dt*Aux;
> RU2 = Rn(Qn,x,t+c2*dt,dt,U2);
> Aux = 1/2*PhiQn(Qn,1,c2*dt,Ru);
> Aux = Aux + PhiQn(Qn,2,c2*dt,RU2-Ru);
> U3 = PhiQn(Qn,0,c3*dt,u) + dt*Aux;
> RU3 = Rn(Qn,x,t+c3*dt,dt,U3);
> Aux = PhiQn(Qn,1,c4*dt,Ru);
> Aux = Aux + PhiQn(Qn,2,c4*dt,2*(RU3-Ru));
> U4 = PhiQn(Qn,0,c4*dt,u) + dt*Aux;
> RU4 = Rn(Qn,x,t+c4*dt,dt,U4);
> Aux = PhiQn(Qn,1,dt,Ru);
> Aux = Aux + PhiQn(Qn,2,dt,-3*Ru+2*RU2+2*RU3-RU4);
> Aux = Aux + PhiQn(Qn,3,dt,4*Ru-4*RU2-4*RU3+4*RU4);
> u = PhiQn(Qn,0,dt,u) + dt*Aux;
> end
```

```
> if Integrator == 'ExpLM02' | Integrator == 'ExpGL03'
> Ru = Rn(Qn,x,t,dt,u);
> if Integrator == 'ExpLM02'
> Aux = PhiQn(Qn,1,dt,Ru);
> Aux = Aux + PhiQn(Qn,2,dt,Ru-Rum);
> end
> if Integrator == 'ExpGL03'
> c2 = 1/2;
> Aux = PhiQn(Qn,1,c2*dt,c2*Ru) + PhiQn(Qn,2,c2*dt,c2^2*(Ru - Rum));
> U = PhiQn(Qn,0,c2*dt,u) + dt*Aux;
> RU = Rn(Qn,x,t+c2*dt,dt,U);
> Aux = PhiQn(Qn,1,dt,Ru);
> Aux = Aux + PhiQn(Qn,2,dt,-(1-c2)/c2*Ru + 1/(c2*(1+c2))*RU
> - 1/(1+c2)*c2*Rum);
> Aux = Aux + PhiQn(Qn,3,dt,-2/c2*Ru + 2/(c2*(1+c2))*RU + 2/(1+c2)*Rum);
> end
> unew = PhiQn(Qn,0,dt,u) + dt*Aux;
> tm = t;
> um = u;
> Rum = Ru;
> t = t + dt;
> u = unew;
> end
```

```
> if Integrator == 'ExpLM03' | Integrator == 'ExpGL04'
> Ru = Rn(Qn,x,t,dt,u);
> Aux = PhiQn(Qn,1,dt,Ru);
> coeff2 = [3/2,-2,1/2];
> coeff3 = [1,-2,1];
> for loop = 2:3
> R = coeffloop(1)*Ru + coeffloop(2)*Rum1 + coeffloop(3)*Rum2;
> Aux = Aux + PhiQn(Qn,loop,dt,R);
> end
> if Integrator == 'ExpGL04'
> U = PhiQn(Qn,0,dt,u) + dt*Aux;
> RU = Rn(Qn,x,t+dt,dt,U);
> Aux = PhiQn(Qn,1,dt,Ru);
> coeff2 = [1/2,1/3,-1,1/6];
> coeff3 = [-2,1,1,0];
> coeff4 = [-3,1,3,-1];
> for loop = 2:4
> R = coeffloop(1)*Ru + coeffloop(2)*RU + coeffloop(3)*Rum1
> + coeffloop(4)*Rum2;
> Aux = Aux + PhiQn(Qn,loop,dt,R);
> end
> end
> unew = PhiQn(Qn,0,dt,u) + dt*Aux;
> tm2 = tm1;
> um2 = um1;
> Rum2 = Rum1;
> tm1 = t;
> um1 = u;
> Rum1 = Ru;
> t = t + dt;
> u = unew;
> end
```

```
> if Integrator == 'ExpLM04' | Integrator == 'ExpCP04'
> | Integrator == 'ExpGL05'
> Ru = Rn(Qn,x,t,dt,u);
> if Integrator == 'ExpLM04'
> Aux = PhiQn(Qn,1,dt,Ru);
> coeff2 = [11/6,-3,3/2,-1/3];
> coeff3 = [2,-5,4,-1];
> coeff4 = [1,-3,3,-1];
> for loop = 2:4
> R = coeffloop(1)*Ru + coeffloop(2)*Rum1 + coeffloop(3)*Rum2
> + coeffloop(4)*Rum3;
> Aux = Aux + PhiQn(Qn,loop,dt,R);
> end
> unew = PhiQn(Qn,0,dt,u) + dt*Aux;
> end
> if Integrator == 'ExpCP04'
> Aux = 4*PhiQn(Qn,1,4*dt,Rum3);
> coeff2 = [16/3,-24,48,-88/3];
> coeff3 = [-64,256,-320,128];
> coeff4 = [256,-768,768,-256];
> for loop = 2:4
> R = coeffloop(1)*Ru + coeffloop(2)*Rum1 + coeffloop(3)*Rum2
> + coeffloop(4)*Rum3;
> Aux = Aux + PhiQn(Qn,loop,4*dt,R);
> end
> unew = PhiQn(Qn,0,4*dt,um3) + dt*Aux;
> end
> if Integrator == 'ExpGL05'
> Aux = PhiQn(Qn,1,dt,Ru);
> coeff2 = [11/6,-3,3/2,-1/3];
> coeff3 = [2,-5,4,-1];
> coeff4 = [1,-3,3,-1];
> for loop = 2:4
> R = coeffloop(1)*Ru + coeffloop(2)*Rum1 + coeffloop(3)*Rum2
> + coeffloop(4)*Rum3;
> Aux = Aux + PhiQn(Qn,loop,dt,R);
> end
> U = PhiQn(Qn,0,dt,u) + dt*Aux;
> RU = Rn(Qn,x,t+dt,dt,U);
> Aux = PhiQn(Qn,1,dt,Ru);
> coeff2 = [5/6,1/4,-3/2,1/2,-1/12];
> coeff3 = [-5/3,11/12,1/2,1/3,-1/12];
> coeff4 = [-5,3/2,6,-3,1/2];
> coeff5 = [-4,1,6,-4,1];
> for loop = 2:5
> R = coeffloop(1)*Ru + coeffloop(2)*RU + coeffloop(3)*Rum1
> + coeffloop(4)*Rum2;
> R = R + coeffloop(5)*Rum3;
> Aux = Aux + PhiQn(Qn,loop,dt,R);
> end
> unew = PhiQn(Qn,0,dt,u) + dt*Aux;
> end
> tm3 = tm2;
> um3 = um2;
> Rum3 = Rum2;
> tm2 = tm1;
> um2 = um1;
> Rum2 = Rum1;
> tm1 = t;
> um1 = u;
> Rum1 = Ru;
> t = t + dt;
> u = unew;
> end
> end
```

```
> if Integrator == 'ExpGL06'
> Ru = Rn(Qn,x,t,dt,u);
> Aux = PhiQn(Qn,1,dt,Ru);
> coeff2 = [25/12,-4,3,-4/3,1/4];
> coeff3 = [35/12,-26/3,19/2,-14/3,11/12];
> coeff4 = [5/2,-9,12,-7,3/2];
> coeff5 = [1,-4,6,-4,1];
> for loop = 2:5
> R = coeffloop(1)*Ru + coeffloop(2)*Rum1 + coeffloop(3)*Rum2
> + coeffloop(4)*Rum3;
> R = R + coeffloop(5)*Rum4;
> Aux = Aux + PhiQn(Qn,loop,dt,R);
> end
> U = PhiQn(Qn,0,dt,u) + dt*Aux;
> RU = Rn(Qn,x,t+dt,dt,U);
> Aux = PhiQn(Qn,1,dt,Ru);
> coeff2 = [13/12,1/5,-2,1,-1/3,1/20];
> coeff3 = [-5/4,5/6,-1/3,7/6,-1/2,1/12];
> coeff4 = [-25/4,7/4,17/2,-11/2,7/4,-1/4];
> coeff5 = [-9,2,16,-14,6,-1];
> coeff6 = [-5,1,10,-10,5,-1];
> for loop = 2:6
> R = coeffloop(1)*Ru + coeffloop(2)*RU + coeffloop(3)*Rum1
> + coeffloop(4)*Rum2;
> R = R + coeffloop(5)*Rum3 + coeffloop(6)*Rum4;
> Aux = Aux + PhiQn(Qn,loop,dt,R);
> end
> unew = PhiQn(Qn,0,dt,u) + dt*Aux;
> tm4 = tm3;
> um4 = um3;
> Rum4 = Rum3;
> tm3 = tm2;
> um3 = um2;
> Rum3 = Rum2;
> tm2 = tm1;
> um2 = um1;
> Rum2 = Rum1;
> tm1 = t;
> um1 = u;
> Rum1 = Ru;
> t = t + dt;
> u = unew;
> end
```