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Fundamental models in nonlinear acoustics part I. Analytical comparison

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This work is concerned with the study of fundamental models from nonlinear acoustics. In Part I, a hierarchy of nonlinear damped wave equations arising in the description of sound propagation in thermoviscous fluids is deduced. In particular, a rigorous justification of two classical models, the Kuznetsov and Westervelt equations, retained as limiting systems for vanishing thermal conductivity and consistent initial data, is given. Numerical comparisons that confirm and complement the theoretical results are provided in Part II.

Keywords: Nonlinear acoustics; Kuznetsov equation; Westervelt equation; limiting system; energy estimates.

AMS Subject Classification: 35L72, 35L77

1. Introduction

Mathematical models in the form of damped wave equations naturally arise in the field of nonlinear acoustics, when describing the propagation of sound in thermoviscous fluids; the examination of nonlinear models is of particular importance in high-intensity ultrasonics and includes various medical and industrial applications, see Refs. 1, 3, 8 and 14 and references given therein.

Classical models. A widely-used model that neglects thermal effects is the Kuznetsov equation, see Ref. 17; if additionally local nonlinear effects are disregarded, the Westervelt equation is obtained, see Ref. 25.

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For our investigations, it is advantageous to formulate the Kuznetsov and Westervelt equations as abstract evolution equations for the space-time-dependent acoustic velocity potential ψ; moreover, with regard to a compact and unifying representation of the considered hierarchy of nonlinear damped wave equations, we introduce the auxiliary abbreviations

$$\begin{gathered}\beta_1^{(0)} = \nu\Lambda, \quad \beta_3 = c_0^2, \\ \beta_5(\sigma) = \frac{1}{c_0^2}\left(2(1-\sigma) + \frac{B}{A}\right), \quad \beta_6(\sigma) = \sigma, \quad \sigma \in \{0,1\},\end{gathered} \tag{1.1a}$$

which involve the kinematic viscosity ν, the quantity $\Lambda = \frac{\mu_B}{\mu} + \frac{4}{3}$ given by the ratio of the bulk and shear viscosities μ_B and μ, the speed of sound c_0, and the parameter of nonlinearity $\frac{B}{A}$, see Table 1. Setting $\sigma = 1$, the Kuznetsov equation reads as

$$\begin{cases} \partial_{tt}\psi(t) - \beta_1^{(0)}\,\Delta\partial_t\psi(t) - \beta_3\,\Delta\psi(t) \\ \quad + \partial_t\left(\dfrac{1}{2}\,\beta_5(\sigma)(\partial_t\psi(t))^2 + \beta_6(\sigma)\,|\nabla\psi(t)|^2\right) = 0, \quad t \in (0,T), \\ \psi(0) = \psi_0, \quad \partial_t\psi(0) = \psi_1; \end{cases} \tag{1.1b}$$

the Westervelt equation is included for $\sigma = 0$.

The additional assumption of a preferred direction of propagation leads to the Khokhlov–Zabolotskaya–Kuznetsov and the viscous Burgers equations, see Refs. 6 and 26; however, we do not consider these special cases here.

Extended models. Nonlinear damped wave equations that incorporate thermal effects and hence generalize the Kuznetsov and Westervelt equations are found in the seminal works[2,7] and the recent contributions.[5,13]

In the present work, we readdress the derivation of these extended models from the fundamental conservation laws for mass, momentum, and energy as well as an equation of state. As common, we split the basic state variables of acoustics, the mass density ϱ, the vector-valued acoustic particle velocity v, the acoustic pressure p, and the temperature T, into constant mean values and space-time-dependent fluctuations; furthermore, we employ a Helmholtz decomposition of the acoustic particle velocity and assign the irrotational part to the gradient of the acoustic velocity potential, see Table 1. According to Refs. 2 and 18, we take first- and second-order contributions with respect to the fluctuating quantities into account; denoting

$$\begin{aligned} &\beta_1^{(a)} = a\left(1 + \frac{B}{A}\right) + \nu\Lambda, \quad \beta_2^{(a)}(\sigma_0) = a\left(\nu\Lambda + a\,\frac{B}{A} + \sigma_0\,\frac{B}{A}(\nu\Lambda - a)\right), \\ &\beta_3 = c_0^2, \quad \beta_4^{(a)}(\sigma_0) = a\left(1 + \sigma_0\,\frac{B}{A}\right)c_0^2, \\ &\beta_5(\sigma) = \frac{1}{c_0^2}\left(2(1-\sigma) + \frac{B}{A}\right), \quad \beta_6(\sigma) = \sigma, \quad \sigma, \sigma_0 \in \{0,1\}, \end{aligned} \tag{1.2a}$$

Table 1. Overview of fundamental state variables with decompositions into constant mean values and space-time-dependent fluctuations, decisive physical quantities, and auxiliary abbreviations.

State variables
Mass density $\varrho = \varrho_0 + \varrho_\sim$
Vector-valued acoustic particle velocity $v = v_0 + v_\sim$, $v_0 = 0$
Associated acoustic velocity and vector potentials $v_\sim = \nabla\psi + \nabla \times S$
Acoustic pressure $p = p_0 + p_\sim$
Temperature $T = T_0 + T_\sim$
Physical quantities
Shear (or dynamic) viscosity μ
Bulk viscosity μ_B
Kinematic viscosity $\nu = \frac{\mu}{\varrho_0}$
Prandtl number Pr
Thermal conductivity $a = \frac{\nu}{\mathrm{Pr}}$
Specific heat at constant volume c_V
Specific heat at constant pressure c_p
Thermal expansion coefficient α_V
Speed of sound $c_0 = \sqrt{\frac{c_p p_0}{c_V \varrho_0}}$
Parameter of nonlinearity $\frac{B}{A}$
Auxiliary abbreviations and relations
$A = c_0^2\, \varrho_0$
$\frac{a}{c_V\, \varrho_0} = a(1 + \frac{B}{A})$
$\Lambda = \frac{\mu_B}{\mu} + \frac{4}{3}$
$\beta_1^{(a)} = a\big(1 + \frac{B}{A}\big) + \nu\Lambda$
$\beta_2^{(a)}(\sigma_0) = a\big(\nu\Lambda + a\,\frac{B}{A} + \sigma_0\,\frac{B}{A}(\nu\Lambda - a)\big)$ with $\sigma_0 \in \{0, 1\}$
$\beta_3 = c_0^2$
$\beta_4^{(a)}(\sigma_0) = a\big(1 + \sigma_0\,\frac{B}{A}\big)c_0^2$ with $\sigma_0 \in \{0, 1\}$
$\beta_5(\sigma) = \frac{1}{c_0^2}\big(2(1-\sigma) + \frac{B}{A}\big)$ with $\sigma \in \{0, 1\}$
$\beta_6(\sigma) = \sigma$ with $\sigma \in \{0, 1\}$
$\beta_0^{(a)}(\sigma_0) = \frac{\beta_2^{(a)}(\sigma_0)}{\beta_4^{(a)}(\sigma_0)} = \frac{1}{c_0^2}\big(\nu\Lambda + (1-\sigma_0)\,a\,\frac{B}{A}\big)$ with $\sigma_0 \in \{0, 1\}$
$\alpha = 1 + \beta_5(\sigma)\,\partial_t\psi$ with $\sigma \in \{0, 1\}$
$r = \beta_5(\sigma)(\partial_{tt}\psi)^2 + \beta_6(\sigma)\,\partial_{tt}\lvert\nabla\psi\rvert^2$ with $\sigma \in \{0, 1\}$

we attain the nonlinear damped wave equation

$$\begin{cases} \partial_{ttt}\psi^{(a)}(t) - \beta_1^{(a)}\,\Delta\partial_{tt}\psi^{(a)}(t) + \beta_2^{(a)}(\sigma_0)\,\Delta^2\partial_t\psi^{(a)}(t) \\ \quad - \beta_3\,\Delta\partial_t\psi^{(a)}(t) + \beta_4^{(a)}(\sigma_0)\,\Delta^2\psi^{(a)}(t) \\ \quad + \partial_{tt}\left(\dfrac{1}{2}\,\beta_5(\sigma)(\partial_t\psi^{(a)}(t))^2 + \beta_6(\sigma)\,\big|\nabla\psi^{(a)}(t)\big|^2\right) = 0, \quad t \in (0, T), \\ \psi^{(a)}(0) = \psi_0, \quad \partial_t\psi^{(a)}(0) = \psi_1, \quad \partial_{tt}\psi^{(a)}(0) = \psi_2. \end{cases} \tag{1.2b}$$

For the sake of distinctiveness, we indicate the dependence of the solution on the thermal conductivity $a > 0$; evidently, $\beta_1^{(a)} \to \beta_1^{(0)}$ and $\beta_2^{(a)}(\sigma_0) \to 0$ as well as $\beta_4^{(a)}(\sigma_0) \to 0$ if $a \to 0_+$.

The most general model studied in this work is given by (1.2) with $\sigma = 1$ and $\sigma_0 = 1$; in contrast to Ref. 4 (Eq. (1.19)) and Ref. 5 (Eq. (4)), it contains the additional term $a \, \frac{B}{A} \, c_0^2 \, \Delta^2 \psi$, which permits to decompose the differential operator comprising the linear contributions into a heat operator and a wave operator. Despite this discrepancy, we refer to (1.2) with $\sigma = 1$ and $\sigma_0 = 1$ as Blackstock–Crighton–Brunnhuber–Jordan–Kuznetsov equation or briefly as Brunnhuber–Jordan–Kuznetsov equation.

Other nonlinear damped wave equations known from the literature are embedded in our general model, see Table 2. The value $\sigma = 0$ corresponds to Westervelt-type equations, where local nonlinear effects are disregarded; the special choice $\sigma_0 = 0$ is characteristic for a monatomic gas and also referred to as Becker's assumption.

Main result. In this work, our central aim is to rigorously justify that the Kuznetsov and Westervelt equations (1.1) are retained as limiting systems of the nonlinear damped wave equation (1.2) for vanishing thermal conductivity, provided that the initial data satisfy the consistency condition

$$\psi_2 - \beta_1^{(0)} \, \Delta \psi_1 - \beta_3 \, \Delta \psi_0 + \beta_5(\sigma) \, \psi_2 \, \psi_1 + 2 \, \beta_6(\sigma) \, \nabla \psi_1 \cdot \nabla \psi_0 = 0. \tag{1.3}$$

With regard to numerical simulations included in Part II, we henceforth consider (1.1)–(1.2) on a finite time interval $[0, T]$, subject to homogeneous Dirichlet boundary conditions on a bounded space domain $\Omega \subset \mathbb{R}^d$, where $d \in \{1, 2, 3\}$; in order to realize (1.3), we prescribe ψ_0 as well as ψ_1 such that $1 + \beta_5(\sigma) \, \psi_1$ is non-degenerate and then determine ψ_2 from the relation

$$\psi_2 = (1 + \beta_5(\sigma) \, \psi_1)^{-1} (\beta_1^{(0)} \, \Delta \psi_1 + \beta_3 \, \Delta \psi_0 - 2 \, \beta_6(\sigma) \, \nabla \psi_1 \cdot \nabla \psi_0).$$

Table 2. Overview of the considered hierarchy of nonlinear damped wave equations. The Brunnhuber–Jordan–Kuznetsov equation is cast into the general formulation (1.2) with $\sigma = 1$ and $\sigma_0 = 1$, see also Table 1. The Blackstock–Crighton–Kuznetsov equation arises in situations, where the quantity $(\nu\Lambda - a) \, \frac{B}{A}$ is negligible, for instance in the description of monatomic gases; it is embedded in (1.2) for $\sigma = 1$ and $\sigma_0 = 0$. In both cases, the Kuznetsov equation results as limiting system for vanishing thermal conductivity $a \to 0_+$ and initial data satisfying the consistency condition (1.3). Westervelt-type equations do not take into account local nonlinear effects; this is reflected by the absence of the term $c_0^2 \, |\nabla \psi|^2 - (\partial_t \psi)^2$ and corresponds to the value $\sigma = 0$.

$$\begin{tikzcd}
\text{Brunnhuber–Jordan–Kuznetsov (BJK)} \arrow[r, "\sigma=0"] \arrow[d, "\sigma_0=0"] & \text{Brunnhuber–Jordan–Westervelt (BJW)} \arrow[d, "\sigma_0=0"] \\
\text{Blackstock–Crighton–Kuznetsov (BCK)} \arrow[r, "\sigma=0"] \arrow[d, "a\to 0_+"] & \text{Blackstock–Crighton–Westervelt (BCW)} \arrow[d, "a\to 0_+"] \\
\text{Kuznetsov (K)} \arrow[r, "\sigma=0"] & \text{Westervelt (W)}
\end{tikzcd}$$

Evidently, in the general model (1.2), third-order time derivatives and fourth-order space derivatives occur; on the contrary, for the reduced model (1.1), it is natural to consider a closed subspace

$$X_0 \subset H^2([0,T], H^2(\Omega))$$

as solution space. This explains that we study the associated equation

$$\begin{aligned}
&\mathcal{L}^{(a)}\psi^{(a)}(t) + \mathcal{N}(\psi^{(a)}(t), \psi^{(a)}(t)) + \mathcal{L}_0^{(a)} + \mathcal{N}_0 = 0, \quad t \in (0,T),\\
&\mathcal{L}^{(a)}\chi(t) = \partial_{tt}\chi(t) - \beta_1^{(a)}\,\Delta\partial_t\chi(t) + \beta_2^{(a)}(\sigma_0)\,\Delta^2\chi(t) - \beta_3\,\Delta\chi(t)\\
&\qquad\qquad + \beta_4^{(a)}(\sigma_0)\int_0^t \Delta^2\chi(\tau)\mathrm{d}\tau,\\
&\mathcal{N}(\phi(t),\chi(t)) = \beta_5(\sigma)\,\partial_{tt}\chi(t)\,\partial_t\phi(t) + 2\,\beta_6(\sigma)\,\nabla\partial_t\chi(t)\cdot\nabla\phi(t),\\
&\mathcal{L}_0^{(a)} = -\psi_2 + \beta_1^{(a)}\,\Delta\psi_1 - \beta_2^{(a)}(\sigma_0)\,\Delta^2\psi_0 + \beta_3\,\Delta\psi_0,\\
&\mathcal{N}_0 = -\beta_5(\sigma)\,\psi_2\,\psi_1 - 2\,\beta_6(\sigma)\,\nabla\psi_1\cdot\nabla\psi_0,
\end{aligned} \tag{1.4}$$

which follows from (1.2) by integration with respect to time; moreover, to reduce the spatial regularity requirements, we test this relation with elements in $L_1([0,T], H^1(\Omega))$ and perform integration-by-parts. Imposing appropriate consistency conditions such that the arising boundary terms vanish, we obtain the weak formulation

$$\begin{aligned}
&\int_0^T (\partial_{tt}\psi^{(a)}(t) - \psi_2 \,|\, v(t))_{L_2}\mathrm{d}t\\
&\quad + \beta_1^{(a)}\int_0^T (\nabla\partial_t\psi^{(a)}(t) - \nabla\psi_1 \,|\, \nabla v(t))_{L_2}\mathrm{d}t\\
&\quad - \beta_2^{(a)}(\sigma_0)\int_0^T (\nabla\Delta\psi^{(a)}(t) - \nabla\Delta\psi_0 \,|\, \nabla v(t))_{L_2}\mathrm{d}t\\
&\quad - \beta_3\int_0^T (\Delta\psi^{(a)}(t) - \Delta\psi_0 \,|\, v(t))_{L_2}\mathrm{d}t\\
&\quad - \beta_4^{(a)}(\sigma_0)\int_0^T\int_0^t (\nabla\Delta\psi^{(a)}(\tau) \,|\, \nabla v(t))_{L_2}\mathrm{d}\tau\mathrm{d}t\\
&\quad + \int_0^T (\mathcal{N}(\psi^{(a)}(t), \psi^{(a)}(t)) \,|\, v(t))_{L_2}\mathrm{d}t = 0, \quad v \in L_1([0,T], H^1(\Omega)).
\end{aligned}$$

Provided that the initial data fulfill suitable regularity and smallness assumptions, we show existence of a weak solution

$$\psi^{(a)} \in H^2([0,T], H^2(\Omega)) \cap W_\infty^2([0,T], H^1(\Omega)) \cap W_\infty^1([0,T], H^3(\Omega)), \tag{1.5}$$

see Proposition 3.1; as our proof relies on Schauder's fixed point theorem, it does not include uniqueness. Main tools in the derivation of Proposition 3.1 are *a priori* energy estimates, combined with an auxiliary result that ensures that the first time derivative of the solution and its inverse remain uniformly bounded. The natural approach to test (1.2) with the second time derivative of the solution and to consider the lower-order energy functional

$$\mathcal{E}_0(\psi^{(a)}(t)) = \left\|\partial_{tt}\psi^{(a)}(t)\right\|_{L_2}^2 + \beta_2^{(a)}(\sigma_0)\left\|\Delta\partial_t\psi^{(a)}(t)\right\|_{L_2}^2 + \left\|\nabla\partial_t\psi^{(a)}(t)\right\|_{L_2}^2$$

turns out to be insufficient, since higher-order space and time derivatives remain; by introducing the higher-order energy functional

$$\mathcal{E}_1(\psi^{(a)}(t)) = \left\|\nabla\partial_{tt}\psi^{(a)}(t)\right\|_{L_2}^2 + \beta_2^{(a)}(\sigma_0)\left\|\nabla\Delta\partial_t\psi^{(a)}(t)\right\|_{L_2}^2 + \left\|\Delta\partial_t\psi^{(a)}(t)\right\|_{L_2}^2,$$

we attain a bound of the form

$$\sup_{t\in[0,T]}\mathcal{E}_0(\psi^{(a)}(t)) + \sup_{t\in[0,T]}\mathcal{E}_1(\psi^{(a)}(t)) + \int_0^T \left\|\Delta\partial_{tt}\psi^{(a)}(t)\right\|_{L_2}^2 \mathrm{d}t \le C.$$

Evidently, the terms in $\mathcal{E}_1$ are associated with the Bochner–Sobolev spaces

$$W_\infty^2([0,T],H^1(\Omega)),\quad W_\infty^1([0,T],H^3(\Omega)),\quad W_\infty^1([0,T],H^2(\Omega)),$$

and hence comprise the regularity implicated by the terms in $\mathcal{E}_0$; though, for the specification of certain constants, we found it useful to maintain $\mathcal{E}_0$ and related terms. On the basis of the regularity result (1.5), we establish convergence towards the solution of the Kuznetsov equation (1.1), that is

$$\psi^{(a)} \overset{*}{\rightharpoonup} \psi \quad \text{in } H^2([0,T],H^2(\Omega)) \quad \text{as } a\to 0_+,$$

see Theorem 4.1; due to the fact that $\beta_2^{(a)}(\sigma_0)\to 0$ as $a\to 0_+$, higher spatial regularity cannot be achieved.

Methodology. As indicated before, the derivation of our main result, Theorem 4.1, and of a fundamental auxiliary result, Proposition 3.1, relies on *a priori* energy estimates and a fixed point argument to resolve the nonlinearity. In order to keep our approach applicable to nonlinear damped wave equations of a similar form, see for instance Ref. 5 (Eq. (4)), we do not exploit the factorization of the linear part into a heat and a wave operator; a mathematical analysis for the special case of a monatomic gas, where such a decomposition holds as well, is found in Ref. 4. The statement of Proposition 3.1 compares with the existence result deduced in Ref. 13; however, in Ref. 13, a different approach based on maximal parabolic regularity is used and existence as well as uniqueness is established under stronger regularity and compatibility requirements on the problem data.

Outline. Our work has the following structure. In Sec. 1.1, we collect basic notation concerning the underlying Lebesgue and Sobolev spaces. In Sec. 2, we specify the considered nonlinear damped wave equations arising in applications from nonlinear acoustics; this in particular includes the Brunnhuber–Jordan–Kuznetsov and the Kuznetsov equations. For this purpose, we review physical and mathematical principles that are relevant in the derivation of the Brunnhuber–Jordan–Kuznetsov equation and formally justify the Kuznetsov equation as limiting system for vanishing thermal conductivity and consistent initial data, see Sec. 2.1; additional details on the derivation are found in Appendix A. The considered hierarchy of nonlinear damped wave equations is introduced in Sec. 2.2.

Section 3 is devoted to the derivation of a fundamental auxiliary result that ensures existence and non-degeneracy of a weak solution to the Brunnhuber–Jordan–Kuznetsov equation and related models, see Proposition 3.1. We begin with the specification of convenient unifying representations of the different general and reduced models, see Sec. 3.1. In view of Theorem 4.1, we introduce a weak formulation of the general nonlinear damped wave equation, obtained by integration with respect to time, see Sec. 3.2. Moreover, with regard to the fixed-point argument employed in the proof of Proposition 3.1, we state a suitable modification of the general nonlinear damped wave equation; by testing with certain partial derivatives of the solution, we obtain auxiliary relations involving lower- and higher-order energy functionals. Based on these identities, we deduce *a priori* energy estimates, see Sec. 3.3. The existence result and its proof are given in Sec. 3.4.

By means of the regularity provided by Proposition 3.1, it is straightforward to derive the main result of this work in Sec. 4; Theorem 4.1 rigorously justifies the Kuznetsov and Westervelt equations as limiting systems of the general nonlinear damped wave equation for vanishing thermal conductivity and consistent initial data.

1.1. *Basic notation*

Space domain and time interval. Throughout, we consider a bounded space domain $\Omega \subset \mathbb{R}^d$ with regular boundary $\partial\Omega$ and a finite time interval $[0,T]$, see also Sec. 3.2. In Secs. 2–4, we are primarily interested in the most relevant three-dimensional case; however, with regard to numerical illustrations, we admit $d \in \{1,2,3\}$.

Euclidian norm. Let $v = (v_1,\dots,v_d)^T \in \mathbb{R}^d$ and $w = (w_1,\dots,w_d)^T \in \mathbb{R}^d$. As usual, the Euclidian inner product and the associated norm are denoted by

$$v \cdot w = \sum_{j=1}^{d} v_j\, w_j, \quad |v| = \sqrt{v \cdot v}.$$

Space derivatives. For scalar-valued and vector-valued functions

$$f : \Omega \to \mathbb{R} : x = (x_1,\dots,x_d)^T \mapsto f(x),$$
$$F : \Omega \to \mathbb{R}^d : x = (x_1,\dots,x_d)^T \mapsto F(x) = (F_1(x),\dots,F_d(x))^T,$$

we denote by $(\partial_{x_j} f)_{j=1}^d$ and $(\partial_{x_j} F_k)_{j,k=1}^d$ their spatial derivatives. Gradient, Laplacian, and divergence are defined by

$$\nabla f = (\partial_{x_1} f,\dots,\partial_{x_d} f)^T, \quad \Delta f = \sum_{j=1}^{d} \partial_{x_j}^2 f, \quad \nabla \cdot F = \sum_{j=1}^{d} \partial_{x_j} F_j.$$

Lebesgue and Sobolev spaces. For exponents $p \in [1, \infty]$ and $k \in \mathbb{N}_{\geq 1}$, we denote by $L_p(\Omega, \mathbb{R})$ and $W_p^k(\Omega, \mathbb{R})$ the standard Lebesgue and Sobolev spaces; as common, we set $H^k(\Omega, \mathbb{R}) = W_2^k(\Omega, \mathbb{R})$. In particular, the Hilbert space $L_2(\Omega, \mathbb{R})$ is endowed with inner product and associated norm given by

$$(f \,|\, g)_{L_2} = \int_\Omega f(x)\, g(x) \mathrm{d}x, \quad \big\|f\big\|_{L_2} = \sqrt{\int_\Omega (f(x))^2 \,\mathrm{d}x}, \quad f, g \in L_2(\Omega, \mathbb{R});$$

accordingly, for vector-valued functions that arise in connection with the gradient, we set

$$(F \,|\, G)_{L_2} = \int_\Omega F(x) \cdot G(x) \mathrm{d}x, \quad \big\|F\big\|_{L_2} = \sqrt{\int_\Omega \big|F(x)\big|^2 \,\mathrm{d}x}, \quad F, G \in L_2(\Omega, \mathbb{R}^d).$$

Bochner spaces. In Secs. 3 and 4, we employ reformulations of the considered nonlinear damped wave equations as abstract evolution equations on Banach spaces; for mappings that involve certain space and time derivatives of a function, we write $F(\varphi(t)) = F(\varphi, t)$ for short, see for instance (1.4). In the derivation of auxiliary estimates, we use standard notation for the norms of different Bochner–Lebesgue and Bochner–Sobolev spaces; for example, we set

$$\big\|\varphi\big\|_{L_\infty([0,T], L_\infty(\Omega))} = \operatorname*{ess\,sup}_{t \in [0,T]} \big\|\varphi(t)\big\|_{L_\infty},$$

see (3.17).

2. Fundamental Models

In this section, we introduce fundamental models arising in nonlinear acoustics, the Blackstock–Crighton–Brunnhuber–Jordan–Kuznetsov or briefly Brunnhuber–Jordan–Kuznetsov (BJK) equation, the Blackstock–Crighton–Kuznetsov (BCK) equation, the Kuznetsov (K) equation, the Blackstock–Crighton–Brunnhuber–Jordan–Westervelt of briefly Brunnhuber–Jordan–Westervelt (BJW) equation, the Blackstock–Crighton–Westervelt (BCW) equation, and the Westervelt (W) equation; these nonlinear damped wave equations form a hierarchy in the sense that some of them can be viewed as special cases of others, see Table 2. In Sec. 2.1, we specify the physical and mathematical principles employed in the derivation of the Brunnhuber–Jordan–Kuznetsov equation, which is the most general model studied in this work and provides the basis for reduced models such as the Kuznetsov and Westervelt equations. In Sec. 2.2, we review the considered nonlinear damped wave equations and put them into relation. Our collection of models is by no means complete, and we refer to Ref. 12 for recent references from the active field of modelling in nonlinear acoustics as well as to the classical works Refs. 7, 9, 11, 17, 19–21 and 24.

2.1. *Derivation of Brunnhuber–Jordan–Kuznetsov equation*

Notation. The following considerations are characteristic of three space dimensions. In order to distinguish between vector-valued and scalar-valued quantities, we meanwhile employ the notation $\mathbf{x}$ for the space variable, $\mathbf{v}$ for the vector-valued acoustic particle velocity, and $\mathbf{S}$ for the associated vector potential.

Physical quantities. The main physical quantities for the description of sound propagation in thermoviscous fluids are the mass density ϱ, the acoustic particle velocity $\mathbf{v}$, the acoustic pressure p, and the temperature T. These space- and time-dependent quantities are decomposed into constant mean values and space-time-dependent fluctuations; in the situation relevant here, the mean value of the acoustic particle velocity may be assumed to vanish. Consequently, the relations

$$\varrho(\mathbf{x},t) = \varrho_0 + \varrho_\sim(\mathbf{x},t), \quad \mathbf{v}(\mathbf{x},t) = \mathbf{v}_0 + \mathbf{v}_\sim(\mathbf{x},t) = \mathbf{v}_\sim(\mathbf{x},t),$$
$$p(\mathbf{x},t) = p_0 + p_\sim(\mathbf{x},t), \quad T(\mathbf{x},t) = T_0 + T_\sim(\mathbf{x},t),$$

are obtained.

Physical principles. A system of time-dependent nonlinear partial differential equations governing the interplay of these quantities results from the conservation laws for mass, momentum, and energy, supplemented with an equation of state. The conservation of mass is reflected by the continuity equation

$$\partial_t \varrho + \nabla\cdot(\varrho\,\mathbf{v}) = 0. \tag{2.1a}$$

The conservation of momentum corresponds to the relation

$$\partial_t(\varrho\,\mathbf{v}) + \mathbf{v}\,\nabla\cdot(\varrho\,\mathbf{v}) + \varrho(\mathbf{v}\cdot\nabla)\,\mathbf{v} + \nabla p = \mu\,\Delta\mathbf{v} + \left(\mu_B + \frac{1}{3}\,\mu\right)\nabla(\nabla\cdot\mathbf{v}), \tag{2.1b}$$

where μ and μ_B denote the shear and bulk viscosities, respectively. The relation describing the conservation of energy reads

$$\varrho\,(\partial_t E + \mathbf{v}\cdot\nabla E) + p\,\nabla\cdot\mathbf{v} = a\,\Delta T + \left(\mu_B - \frac{2}{3}\,\mu\right)(\nabla\cdot\mathbf{v})^2 + \frac{1}{2}\,\mu\,\big\|\nabla\mathbf{v} + (\nabla\mathbf{v})^T\big\|_F^2,$$

see Eq. (3c) in Ref. 2. Here, E denotes the internal energy per unit mass and $a = \frac{\nu}{\mathrm{Pr}}$ the thermal conductivity, defined by the kinematic viscosity $\nu = \frac{\mu}{\varrho_0}$ and the Prandtl number Pr; the subscript F indicates that the Frobenius norm is used. Rewriting the left-hand side of this equation by means of the specific heat at constant volume and pressure, c_V and c_p, respectively, as well as the thermal expansion coefficient α_V, the conservation of energy is given by

$$\begin{aligned}\varrho&\left(c_V\,\partial_t T + c_V\,\mathbf{v}\cdot\nabla T + \frac{c_p - c_V}{\alpha_V}\,\nabla\cdot\mathbf{v}\right)\\ &= a\,\Delta T + \left(\mu_B - \frac{2}{3}\,\mu\right)(\nabla\cdot\mathbf{v})^2 + \frac{1}{2}\,\mu\,\big\|\nabla\mathbf{v} + (\nabla\mathbf{v})^T\big\|_F^2,\end{aligned} \tag{2.1c}$$

see Eq. (3c′) in Ref. 2. The heuristic equation of state for the acoustic pressure in dependence of mass density and temperature is approximated by the first terms of a Taylor-like expansion

$$p_\sim \approx A\,\frac{\varrho_\sim}{\varrho_0} + \frac{B}{2}\left(\frac{\varrho_\sim}{\varrho_0}\right)^2 + \hat{A}\,\frac{T_\sim}{T_0} \tag{2.2}$$

involving certain positive coefficients $A, B, \hat{A} > 0$, see Eq. (5d) in Ref. 2 and also Table 1.

Helmholtz decomposition. A Helmholtz decomposition of the acoustic particle velocity into an irrotational and a solenoidal part

$$\mathbf{v}_\sim = \nabla\psi + \nabla \times \mathbf{S} \tag{2.3}$$

leads to a reformulation of the conservation laws (2.1) in terms of the acoustic velocity potential ψ and the vector potential $\mathbf{S}$. We note that some authors use instead the relation $\mathbf{v}_\sim = -\nabla\psi + \nabla \times \mathbf{S}$ which explains a differing sign in the resulting nonlinear damped wave equations.

Derivation of reduced models. In order to derive reduced models from (2.1)–(2.2), three categories of contributions are distinguished. First, terms that are linear with respect to the fluctuating quantities and not related to dissipative effects are taken into account (first-order contributions). Second, quadratic terms with respect to fluctuations and dissipative linear terms are included (second-order contributions). All remaining terms are considered to be higher-order contributions. Due to the fact that the conservation laws contain at least first-order space or time derivatives, zero-order terms with respect to the fluctuating quantities do not play a role further on. This classification and the so-called *substitution corollary*, which allows to replace any quantity in a second-order or higher-order term by its first-order approximation, was introduced by Lighthill in Ref. 18 and described by Blackstock in Ref. 2.

Linear wave equation. A natural approach for the derivation of a single higher-order partial differential equation is to combine the equations for conservation of mass and momentum. Subtracting the time derivative of (2.1a) from the divergence of (2.1b) and assuming interchangeability of space and time differentiation, the term $\partial_t \nabla \cdot (\varrho\,\mathbf{v}) = \nabla \cdot \partial_t(\varrho\,\mathbf{v})$ cancels

$$\nabla \cdot (\mathbf{v}\,\nabla \cdot (\varrho\,\mathbf{v}) + \varrho(\mathbf{v} \cdot \nabla)\,\mathbf{v}) + \Delta p - \partial_{tt}\varrho = \mu\,\Lambda\,\Delta(\nabla \cdot \mathbf{v});$$

here, we set $\Lambda = \frac{\mu_B}{\mu} + \frac{4}{3}$. Retaining only the first-order contribution $\Delta p_\sim - \partial_{tt}\varrho_\sim$ and replacing (2.2) by the first-order approximation $\varrho_\sim \approx \frac{\varrho_0}{A}\,p_\sim$, where $A = c_0^2\,\varrho_0$ and $c_0 = \sqrt{\frac{c_p\,p_0}{c_V\,\varrho_0}}$ denotes the speed of sound, yields a linear wave equation for the acoustic pressure

$$\partial_{tt}p_\sim - c_0^2\,\Delta p_\sim = 0.$$

Nonlinear damped wave equation (Brunnhuber–Jordan–Kuznetsov equation). If additionally all second-order contributions are taken into account in (2.1) and (2.2), a more involved procedure for eliminating $\varrho_\sim$, $p_\sim$, and $T_\sim$ leads to a nonlinear damped wave equation for the acoustic velocity potential

$$\begin{aligned}\partial_{ttt}\psi &- \left(a\left(1+\frac{B}{A}\right)+\nu\Lambda\right)\Delta\partial_{tt}\psi + a\left(1+\frac{B}{A}\right)\nu\Lambda\,\Delta^2\partial_t\psi - c_0^2\,\Delta\partial_t\psi \\ &+ a\left(1+\frac{B}{A}\right)c_0^2\,\Delta^2\psi + \partial_{tt}\left(\frac{1}{2c_0^2}\,\frac{B}{A}(\partial_t\psi)^2 + |\nabla\psi|^2\right) = 0; \qquad (2.4\text{a})\end{aligned}$$

details of the derivation are included in Appendix A. As this equation coincides with Eq. (1.19) in Ref. 4 and Eq. (4) in Ref. 5, aside from the extension of the term $a\,c_0^2\,\Delta^2\psi$ to $a(1+\frac{B}{A})\,c_0^2\,\Delta^2\psi$, we refer to it as Brunnhuber–Jordan–Kuznetsov equation. We point out that the differential operator defining the linear contributions is given by the composition of a heat operator and a wave operator

$$\begin{aligned}&\left(\partial_t - a\left(1+\frac{B}{A}\right)\Delta\right)(\partial_{tt}\psi - \nu\Lambda\,\Delta\partial_t\psi - c_0^2\,\Delta\psi) \\ &\quad + \partial_{tt}\left(\frac{1}{2c_0^2}\,\frac{B}{A}(\partial_t\psi)^2 + |\nabla\psi|^2\right) = 0, \qquad (2.4\text{b})\end{aligned}$$

see also Eq. (1) in Ref. 4 and Eq. (1) in Ref. 5; due to the fact that relation (2.1c) reflecting energy conservation involves the heat operator $\partial_t - a\,\Delta$, its appearance is quite intuitive. Our analysis, however, does not exploit the fact that the general model is factorizable and thus also applies to Eq. (1.19) in Ref. 4 and Eq. (4) in Ref. 5. A significant discrepancy of (2.4) compared to the model obtained by Blackstock, see Eq. (7) in Ref. 2, is the presence of the term comprising $\Delta^2\partial_t\psi$, which is essential for proving well-posedness, see Ref. 13.

Limiting model (Kuznetsov equation). In situations where temperature constraints are insignificant, the Kuznetsov (K) equation

$$\partial_{tt}\psi - \nu\Lambda\,\Delta\partial_t\psi - c_0^2\,\Delta\psi + \partial_t\left(\frac{1}{2c_0^2}\,\frac{B}{A}(\partial_t\psi)^2 + |\nabla\psi|^2\right) = 0, \qquad (2.5)$$

see Ref. 17, results from (2.4) by considering the formal limit $a = \frac{\nu}{\mathrm{Pr}} \to 0_+$ (but not necessarily $\nu \to 0_+$). More precisely, setting

$$F(\psi) = \partial_{tt}\psi - \nu\Lambda\,\Delta\partial_t\psi - c_0^2\,\Delta\psi + \partial_t\left(\frac{1}{2c_0^2}\,\frac{B}{A}(\partial_t\psi)^2 + |\nabla\psi|^2\right),$$

it is evident that any solution to (2.5) satisfies $F(\psi) = 0$ and in particular fulfills $\partial_t F(\psi) = 0$, which corresponds to (2.4) with $a = 0$; on the other hand, integration of the condition $\partial_t F(\psi) = 0$ with respect to time implies that any solution to (2.4) with $a = 0$ solves (2.5), provided that the prescribed initial data satisfy a consistency condition such that $F(\psi(\cdot,0)) = 0$. A rigorous justification of this limiting process is given in Sec. 4.

2.2. *Hierarchy of nonlinear damped wave equations*

We next introduce the considered hierarchy of nonlinear damped wave equations, see also Table 2; we distinguish equations of Kuznetsov and Westervelt type, respectively.

Equations of Kuznetsov type

(1) For convenience, we restate the Brunnhuber–Jordan–Kuznetsov equation (2.4) in elaborate and factorized form

$$\begin{aligned}&\partial_{ttt}\psi - \left(a\left(1+\frac{B}{A}\right)+\nu\Lambda\right)\Delta\partial_{tt}\psi\\&\quad + a\left(1+\frac{B}{A}\right)\nu\Lambda\,\Delta^2\partial_t\psi - c_0^2\,\Delta\partial_t\psi + a\left(1+\frac{B}{A}\right)c_0^2\,\Delta^2\psi\\&\quad + \partial_{tt}\left(\frac{1}{2c_0^2}\,\frac{B}{A}(\partial_t\psi)^2+|\nabla\psi|^2\right)=0, \qquad \text{(BJK)}\\&\left(\partial_t - a\left(1+\frac{B}{A}\right)\Delta\right)(\partial_{tt}\psi-\nu\Lambda\,\Delta\partial_t\psi-c_0^2\,\Delta\psi)\\&\quad + \partial_{tt}\left(\frac{1}{2c_0^2}\,\frac{B}{A}(\partial_t\psi)^2+|\nabla\psi|^2\right)=0,\end{aligned}$$

see also Eq. (1.19) in Ref. 4 and Eq. (4) in Ref. 5.

(2) In the special case of a monatomic gas, where the identity $\Lambda\,\mathrm{Pr}=1$ holds, or, more generally, when $a(\Lambda\,\mathrm{Pr}-1)\,\frac{B}{A}=(\nu\Lambda-a)\,\frac{B}{A}$ is negligible, i.e. $\nu\Lambda\frac{B}{A}\approx a\,\frac{B}{A}$, the contribution involving $\Delta^2\partial_t\psi$ formally reduces to

$$a\left(1+\frac{B}{A}\right)\nu\Lambda\,\Delta^2\partial_t\psi \approx a\left(\nu\Lambda + a\,\frac{B}{A}\right)\Delta^2\partial_t\psi;$$

if we replace in addition the term $a\,(1+\frac{B}{A})\,c_0^2\,\Delta^2\psi$ by $a\,c_0^2\,\Delta^2\psi$, we retain the factorizable reduced model

$$\begin{aligned}&\partial_{ttt}\psi - \left(a\left(1+\frac{B}{A}\right)+\nu\Lambda\right)\Delta\partial_{tt}\psi + a\left(\nu\Lambda + a\,\frac{B}{A}\right)\Delta^2\partial_t\psi\\&\quad - c_0^2\,\Delta\partial_t\psi + a\,c_0^2\,\Delta^2\psi + \partial_{tt}\left(\frac{1}{2c_0^2}\,\frac{B}{A}(\partial_t\psi)^2+|\nabla\psi|^2\right)=0, \qquad \text{(BCK)}\\&(\partial_t - a\,\Delta)\left(\partial_{tt}\psi - \left(\nu\Lambda + a\,\frac{B}{A}\right)\Delta\partial_t\psi - c_0^2\,\Delta\psi\right)\\&\quad + \partial_{tt}\left(\frac{1}{2c_0^2}\,\frac{B}{A}(\partial_t\psi)^2+|\nabla\psi|^2\right)=0,\end{aligned}$$

which we refer to as Blackstock–Crighton–Kuznetsov equation, see also Eq. (1) in Ref. 4 and Eq. (1) in Ref. 5.

(3) As shown in Sec. 4, the Kuznetsov equation

$$\partial_{tt}\psi - \nu\Lambda\,\Delta\partial_t\psi - c_0^2\,\Delta\psi + \partial_t\left(\frac{1}{2c_0^2}\,\frac{B}{A}(\partial_t\psi)^2 + |\nabla\psi|^2\right) = 0, \tag{K}$$

see also Eq. (3) in Refs. 4 and 17, is obtained from (BJK) and (BCK) in the limit $a \to 0_+$; for this reduced model, the orders of the arising space and time derivatives are significantly lowered.

Equations of Westervelt type

(1) In certain situations, local nonlinear effects reflected by $|\nabla\psi|^2 - \frac{1}{c_0^2}(\partial_t\psi)^2$ are negligible and thus the nonlinearity can be replaced by

$$\frac{1}{2c_0^2}\,\frac{B}{A}(\partial_t\psi)^2 + |\nabla\psi|^2 \approx \frac{1}{2c_0^2}\,\left(2 + \frac{B}{A}\right)(\partial_t\psi)^2;$$

in accordance with our derivation of the Brunnhuber–Jordan–Kuznetsov equation, we keep the term $a(1+\frac{B}{A})\,c_0^2\,\Delta^2\psi$. Altogether, this yields the nonlinear damped wave equation

$$\begin{aligned}\partial_{ttt}\psi &- \left(a\left(1+\frac{B}{A}\right)+\nu\Lambda\right)\Delta\partial_{tt}\psi + a\left(1+\frac{B}{A}\right)\nu\Lambda\,\Delta^2\partial_t\psi\\ &- c_0^2\,\Delta\partial_t\psi + a\left(1+\frac{B}{A}\right)c_0^2\,\Delta^2\psi + \frac{1}{2c_0^2}\left(2+\frac{B}{A}\right)\partial_{tt}(\partial_t\psi)^2 = 0,\end{aligned} \tag{BJW}$$

which we refer to as Brunnhuber–Jordan–Westervelt equation; as in (BJK), the linear contributions are given by the composition of a wave and a heat operator.

(2) In analogy to (BCK), the Blackstock–Crighton–Westervelt equation

$$\begin{aligned}\partial_{ttt}\psi &- \left(a\left(1+\frac{B}{A}\right)+\nu\Lambda\right)\Delta\partial_{tt}\psi + a\left(\nu\Lambda + a\,\frac{B}{A}\right)\Delta^2\partial_t\psi\\ &- c_0^2\,\Delta\partial_t\psi + a\,c_0^2\,\Delta^2\psi + \frac{1}{2c_0^2}\left(2+\frac{B}{A}\right)\partial_{tt}(\partial_t\psi)^2 = 0\end{aligned} \tag{BCW}$$

is retained as a reduced model from (BJW), see also Eq. (2) in Ref. 4.

(3) The Westervelt equation is given by

$$\partial_{tt}\psi - \nu\Lambda\,\Delta\partial_t\psi - c_0^2\,\Delta\psi + \frac{1}{2c_0^2}\left(2+\frac{B}{A}\right)\partial_t(\partial_t\psi)^2 = 0, \tag{W}$$

see also Eq. (4) in Refs. 4 and 25; as justified in Sec. 4, it results as limiting model from (BJK) for vanishing thermal conductivity and negligible local nonlinear effects.

3. Auxiliary Results

In this section, we state unifying representations of the nonlinear damped wave equations studied in this work. Furthermore, we deduce reformulations of the

Brunnhuber–Jordan–Kuznetsov equation and *a priori* energy estimates that are needed in Sec. 4.

3.1. *Unifying representations*

Abbreviations. In view of a unifying representation, it is convenient to introduce switching variables $\sigma_0, \sigma \in \{0, 1\}$ and abbreviations for the arising non-negative coefficients

$$\begin{aligned}
\beta_1^{(a)} &= a\left(1 + \frac{B}{A}\right) + \nu\Lambda > 0,\\
\beta_2^{(a)}(\sigma_0) &= a\left(\nu\Lambda + a\,\frac{B}{A} + \sigma_0\,\frac{B}{A}(\nu\Lambda - a)\right) > 0,\\
\beta_3 &= c_0^2 > 0, \quad \beta_4^{(a)}(\sigma_0) = a\left(1 + \sigma_0\,\frac{B}{A}\right) c_0^2 > 0,\\
\beta_5(\sigma) &= \frac{1}{c_0^2}\left(2(1-\sigma) + \frac{B}{A}\right) > 0, \quad \beta_6(\sigma) = \sigma \geq 0;
\end{aligned} \tag{3.1a}$$

we recall that the quantities $a, \frac{B}{A}, \nu\Lambda, c_0^2 > 0$ are strictly positive. Besides, we set

$$\beta_0^{(a)}(\sigma_0) = \frac{\beta_2^{(a)}(\sigma_0)}{\beta_4^{(a)}(\sigma_0)} = \frac{1}{c_0^2}\left(\nu\Lambda + (1-\sigma_0)\,a\,\frac{B}{A}\right) > 0. \tag{3.1b}$$

Evidently, these definitions imply the relations

$$\begin{aligned}
&\beta_0^{(a)}(1) = \frac{1}{c_0^2}\,\nu\Lambda, \quad \beta_0^{(a)}(0) = \frac{1}{c_0^2}\left(\nu\Lambda + a\,\frac{B}{A}\right),\\
&\beta_2^{(a)}(1) = a\left(1 + \frac{B}{A}\right)\nu\Lambda, \quad \beta_2^{(a)}(0) = a\left(\nu\Lambda + a\,\frac{B}{A}\right),\\
&\beta_4^{(a)}(1) = a\left(1 + \frac{B}{A}\right)c_0^2, \quad \beta_4^{(a)}(0) = a c_0^2,\\
&\beta_5(1) = \frac{1}{c_0^2}\,\frac{B}{A}, \quad \beta_5(0) = \frac{1}{c_0^2}\left(2 + \frac{B}{A}\right),\\
&\beta_6(1) = 1, \quad \beta_6(0) = 0;
\end{aligned} \tag{3.1c}$$

in the limit $a \to 0_+$, the following values are obtained

$$\beta_0^{(0)}(\sigma_0) = \frac{1}{c_0^2}\,\nu\Lambda, \quad \beta_1^{(0)} = \nu\Lambda, \quad \beta_2^{(0)}(\sigma_0) = 0, \quad \beta_4^{(0)}(\sigma_0) = 0. \tag{3.1d}$$

With regard to the statement of Proposition 3.1 and Theorem 4.1, we introduce uniform lower and upper bounds for coefficients involving $a > 0$; that is, we

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denote

$$\begin{aligned}
&\underline{\beta}_0 = \frac{1}{c_0^2}\,\nu\Lambda, \quad \overline{\beta}_0(\sigma_0) = \frac{1}{c_0^2}\left(\nu\Lambda + (1-\sigma_0)\,\overline{a}\,\frac{B}{A}\right),\\
&\underline{\beta}_1 = \nu\Lambda, \quad \overline{\beta}_1 = \overline{a}\left(1 + \frac{B}{A}\right) + \nu\Lambda,\\
&\overline{\beta}_2(\sigma_0) = \overline{a}\left(\nu\Lambda + \overline{a}\,\frac{B}{A} + \sigma_0\,\frac{B}{A}(\nu\Lambda - \overline{a})\right),\\
&\overline{\beta}_4(\sigma_0) \leq \overline{a}\left(1 + \sigma_0\,\frac{B}{A}\right)c_0^2, \quad a \in (0,\overline{a}].
\end{aligned} \tag{3.1e}$$

Unifying representations. Employing a compact formulation as abstract evolution equation, the Brunnhuber–Jordan–Kuznetsov equation takes the following form with $\sigma_0 = \sigma = 1$

$$\begin{aligned}
&\partial_{ttt}\psi(t) - \beta_1^{(a)}\,\Delta\partial_{tt}\psi(t) + \beta_2^{(a)}(\sigma_0)\,\Delta^2\partial_t\psi(t) - \beta_3\,\Delta\partial_t\psi(t)\\
&\quad + \beta_4^{(a)}(\sigma_0)\,\Delta^2\psi(t) + \partial_{tt}\left(\frac{1}{2}\,\beta_5(\sigma)(\partial_t\psi(t))^2 + \beta_6(\sigma)\,|\nabla\psi(t)|^2\right) = 0,
\end{aligned} \tag{3.1f}$$

see (BJK); the equations (BCK), (BJW), and (BCW) are included as special cases, see Table 2. Moreover, the Kuznetsov and Westervelt equations rewrite as

$$\partial_{tt}\psi(t) - \beta_1^{(0)}\,\Delta\partial_t\psi(t) - \beta_3\,\Delta\psi(t) + \partial_t\left(\frac{1}{2}\,\beta_5(\sigma)(\partial_t\psi(t))^2 + \beta_6(\sigma)\,|\nabla\psi(t)|^2\right) = 0, \tag{3.2}$$

when setting $\sigma = 1$ or $\sigma = 0$, respectively.

3.2. *Reformulations*

With regard to the proof of Theorem 4.1, we next state a weak formulation of the general nonlinear damped wave equation (3.1), obtained by integration with respect to time; moreover, in view of the proof of Proposition 3.1, we introduce a reformulation of the general equation that presupposes non-degeneracy of the first time derivative of the solution. Accordingly, in formulas (3.3)–(3.6), we denote by ψ a solution to (3.1).

Initial and boundary conditions. Throughout, we study the general nonlinear damped wave equation (3.1) on a finite time interval $[0,T]$. When performing integration-by-parts, we need the boundary of the space domain to be sufficiently smooth, namely $\partial\Omega \in C^4$. In order to avoid the presence of additional boundary terms in (3.9) and (3.11), we impose homogeneous Dirichlet conditions on the following space and time derivatives of the solution

$$\partial_{tt}\psi(t)|_{\partial\Omega} = 0, \quad \Delta\partial_t\psi(t)|_{\partial\Omega} = 0, \quad \Delta\psi(t)|_{\partial\Omega} = 0, \tag{3.3a}$$

$$\partial_{ttt}\psi(t)|_{\partial\Omega} = 0, \quad \Delta\partial_{tt}\psi(t)|_{\partial\Omega} = 0; \tag{3.3b}$$

in Ref. 13, due to the fact that the proofs rely on maximal parabolic regularity and do not employ energy estimates, the first condition in (3.3b) does not occur. Moreover, we suppose that the initial conditions

$$\psi(0) = \psi_0, \quad \partial_t \psi(0) = \psi_1, \quad \partial_{tt} \psi(0) = \psi_2, \tag{3.4}$$

are fulfilled; the needed regularity, compatibility, and smallness requirements on ψ_0, ψ_1, and ψ_2 are specified in Proposition 3.1.

Reformulation by integration. With regard to (3.2), assuming interchangeability of space and time differentiation, we set

$$\begin{aligned} F(\psi(t)) &= \partial_{tt}\psi(t) - \beta_1^{(0)}\, \Delta\partial_t\psi(t) - \beta_3\, \Delta\psi(t) \\ &\quad + \beta_5(\sigma)\, \partial_{tt}\psi(t)\, \partial_t\psi(t) + 2\, \beta_6(\sigma)\, \nabla\partial_t\psi(t) \cdot \nabla\psi(t); \end{aligned} \tag{3.5a}$$

straightforward differentiation shows that its time derivative is given by

$$\begin{aligned} \partial_t F(\psi(t)) &= \partial_{ttt}\psi(t) - \beta_1^{(0)}\, \Delta\partial_{tt}\psi(t) - \beta_3\, \Delta\partial_t\psi(t) \\ &\quad + \beta_5(\sigma)\, \partial_{ttt}\psi(t)\, \partial_t\psi(t) + \beta_5(\sigma)(\partial_{tt}\psi(t))^2 \\ &\quad + 2\, \beta_6(\sigma)\, \nabla\partial_{tt}\psi(t) \cdot \nabla\psi(t) + 2\, \beta_6(\sigma)\, \big|\nabla\partial_t\psi(t)\big|^2 \end{aligned}$$

and that (3.1) rewrites as

$$\partial_t F(\psi(t)) = (\beta_1^{(a)} - \beta_1^{(0)})\, \Delta\partial_{tt}\psi(t) - \beta_2^{(a)}(\sigma_0)\, \Delta^2\partial_t\psi(t) - \beta_4^{(a)}(\sigma_0)\, \Delta^2\psi(t).$$

Provided that the prescribed initial data are sufficiently regular and satisfy the consistency condition

$$\psi_2 - \beta_1^{(0)}\, \Delta\psi_1 - \beta_3\, \Delta\psi_0 + \beta_5(\sigma)\, \psi_2\, \psi_1 + 2\, \beta_6(\sigma)\, \nabla\psi_1 \cdot \nabla\psi_0 = 0 \tag{3.5b}$$

such that $F(\psi(0)) = 0$, integration with respect to time implies

$$\begin{aligned} F(\psi(t)) &= (\beta_1^{(a)} - \beta_1^{(0)})(\Delta\partial_t\psi(t) - \Delta\psi_1) \\ &\quad - \beta_2^{(a)}(\sigma_0)(\Delta^2\psi(t) - \Delta^2\psi_0) - \beta_4^{(a)}(\sigma_0) \int_0^t \Delta^2\psi(\tau)\mathrm{d}\tau. \end{aligned} \tag{3.5c}$$

Reformulation by differentiation. A reformulation of (3.1) is obtained by straightforward differentiation of the nonlinear term; suppressing for the sake of notational simplicity the dependence on ψ and $\sigma \in \{0, 1\}$, we set

$$\begin{aligned} \alpha(t) &= 1 + \beta_5(\sigma)\, \partial_t\psi(t), \\ r(t) &= \beta_5(\sigma)(\partial_{tt}\psi(t))^2 + \beta_6(\sigma)\, \partial_{tt}\big|\nabla\psi(t)\big|^2 \\ &= \beta_5(\sigma)(\partial_{tt}\psi(t))^2 + 2\, \beta_6(\sigma)\, \partial_t(\nabla\partial_t\psi(t) \cdot \nabla\psi(t)) \\ &= \beta_5(\sigma)(\partial_{tt}\psi(t))^2 + 2\, \beta_6(\sigma)\, \nabla\partial_{tt}\psi(t) \cdot \nabla\psi(t) + 2\, \beta_6(\sigma)\, \big|\nabla\partial_t\psi(t)\big|^2, \end{aligned} \tag{3.6a}$$

and, as a consequence, we obtain the relation

$$\begin{aligned}\alpha(t)\,\partial_{ttt}\psi(t) - \beta_1^{(a)}\,\Delta\partial_{tt}\psi(t) + \beta_2^{(a)}(\sigma_0)\,\Delta^2\partial_t\psi(t) - \beta_3\,\Delta\partial_t\psi(t)&\\ + \beta_4^{(a)}(\sigma_0)\,\Delta^2\psi(t) + r(t) = 0;&\qquad(3.6\text{b})\end{aligned}$$

provided that non-degeneracy of $\alpha(t)$ is ensured, this further yields

$$\begin{aligned}\partial_{ttt}\psi(t) - \beta_1^{(a)}\,\frac{1}{\alpha(t)}\,\Delta\partial_{tt}\psi(t) + \beta_2^{(a)}(\sigma_0)\,\frac{1}{\alpha(t)}\,\Delta^2\partial_t\psi(t) - \beta_3\,\frac{1}{\alpha(t)}\,\Delta\partial_t\psi(t)&\\ + \beta_4^{(a)}(\sigma_0)\,\frac{1}{\alpha(t)}\,\Delta^2\psi(t) + \frac{1}{\alpha(t)}\,r(t) = 0.&\qquad(3.6\text{c})\end{aligned}$$

Fixed-point argument. Our approach for the derivation of *a priori* energy estimates uses a fixed-point argument based on a suitable modification of (3.6); that is, we consider two functions ϕ and ψ that satisfy the initial conditions

$$\phi(0) = \psi(0) = \psi_0, \quad \partial_t\phi(0) = \partial_t\psi(0) = \psi_1, \quad \partial_{tt}\phi(0) = \partial_{tt}\psi(0) = \psi_2, \qquad (3.7)$$

and replace α and r in relations (3.6b) and (3.6c) by

$$\begin{aligned}&\alpha^{(\phi)}(t) = 1 + \beta_5(\sigma)\,\partial_t\phi(t),\\ &r^{(\phi)}(t) = \beta_5(\sigma)\,\partial_{tt}\psi(t)\,\partial_{tt}\phi(t) + 2\,\beta_6(\sigma)\,\nabla\partial_{tt}\psi(t)\cdot\nabla\phi(t)\\ &\qquad\quad + 2\,\beta_6(\sigma)\,\nabla\partial_t\psi(t)\cdot\nabla\partial_t\phi(t).\end{aligned}\qquad(3.8)$$

First energy identity. Our starting point is (3.6b) with α and r substituted by $\alpha^{(\phi)}$ and $r^{(\phi)}$; testing with $\partial_{tt}\psi(t)$ yields

$$\begin{aligned}&(\alpha^{(\phi)}(t)\,\partial_{ttt}\psi(t)\,|\,\partial_{tt}\psi(t))_{L_2} - \beta_1^{(a)}\,(\Delta\partial_{tt}\psi(t)\,|\,\partial_{tt}\psi(t))_{L_2}\\ &\quad + \beta_2^{(a)}(\sigma_0)\,(\Delta^2\partial_t\psi(t)\,|\,\partial_{tt}\psi(t))_{L_2} - \beta_3\,(\Delta\partial_t\psi(t)\,|\,\partial_{tt}\psi(t))_{L_2}\\ &\quad + \beta_4^{(a)}(\sigma_0)\,(\Delta^2\psi(t)\,|\,\partial_{tt}\psi(t))_{L_2} + (r^{(\phi)}(t)\,|\,\partial_{tt}\psi(t))_{L_2} = 0.\end{aligned}$$

In order to rewrite this relation as the time derivative of a function plus additional terms, we apply the identity

$$\begin{aligned}(\alpha^{(\phi)}(t)\,\partial_{ttt}\psi(t)\,|\,\partial_{tt}\psi(t))_{L_2} &= \frac{1}{2}\,\partial_t\left\|\sqrt{\alpha^{(\phi)}(t)}\,\partial_{tt}\psi(t)\right\|_{L_2}^2\\ &\quad - \frac{1}{2}\,(\partial_t\alpha^{(\phi)}(t)\,\partial_{tt}\psi(t)\,|\,\partial_{tt}\psi(t))_{L_2};\end{aligned}$$

under assumption (3.3a), integration-by-parts implies

$$\begin{aligned}&(\Delta\partial_{tt}\psi(t)\,|\,\partial_{tt}\psi(t))_{L_2} = -\big\|\nabla\partial_{tt}\psi(t)\big\|_{L_2}^2,\\ &(\Delta^2\partial_t\psi(t)\,|\,\partial_{tt}\psi(t))_{L_2} = (\Delta\partial_t\psi(t)\,|\,\Delta\partial_{tt}\psi(t))_{L_2} = \frac{1}{2}\,\partial_t\big\|\Delta\partial_t\psi(t)\big\|_{L_2}^2,\\ &(\Delta\partial_t\psi(t)\,|\,\partial_{tt}\psi(t))_{L_2} = -(\nabla\partial_t\psi(t)\,|\,\nabla\partial_{tt}\psi(t))_{L_2} = -\frac{1}{2}\,\partial_t\big\|\nabla\partial_t\psi(t)\big\|_{L_2}^2\end{aligned}$$

$$\begin{aligned}(\Delta^2\psi(t)\,|\,\partial_{tt}\psi(t))_{L_2} &= (\Delta\psi(t)\,|\,\Delta\partial_{tt}\psi(t))_{L_2}\\ &= \partial_t(\Delta\partial_t\psi(t)\,|\,\Delta\psi(t))_{L_2} - \big\|\Delta\partial_t\psi(t)\big\|_{L_2}^2.\end{aligned} \tag{3.9}$$

As a consequence, we have

$$\begin{aligned}&\frac{1}{2}\,\partial_t\left\|\sqrt{\alpha^{(\phi)}(t)}\,\partial_{tt}\psi(t)\right\|_{L_2}^2 + \beta_1^{(a)}\,\big\|\nabla\partial_{tt}\psi(t)\big\|_{L_2}^2 + \frac{\beta_2^{(a)}(\sigma_0)}{2}\,\partial_t\big\|\Delta\partial_t\psi(t)\big\|_{L_2}^2\\ &\quad+\frac{\beta_3}{2}\,\partial_t\big\|\nabla\partial_t\psi(t)\big\|_{L_2}^2 + \beta_4^{(a)}(\sigma_0)\,\partial_t(\Delta\partial_t\psi(t)\,|\,\Delta\psi(t))_{L_2} - \beta_4^{(a)}(\sigma_0)\,\big\|\Delta\partial_t\psi(t)\big\|_{L_2}^2\\ &\quad+\left(r^{(\phi)}(t) - \frac{1}{2}\,\partial_t\alpha^{(\phi)}(t)\,\partial_{tt}\psi(t)\,\middle|\,\partial_{tt}\psi(t)\right)_{L_2} = 0;\end{aligned}$$

by means of the abbreviation

$$\begin{aligned}\widetilde{E}_0(\phi(t),\psi(t)) &= \frac{1}{2}\,\left\|\sqrt{\alpha^{(\phi)}(t)}\,\partial_{tt}\psi(t)\right\|_{L_2}^2 + \frac{\beta_2^{(a)}(\sigma_0)}{2}\,\big\|\Delta\partial_t\psi(t)\big\|_{L_2}^2\\ &\quad+\frac{\beta_3}{2}\,\big\|\nabla\partial_t\psi(t)\big\|_{L_2}^2,\end{aligned} \tag{3.10a}$$

the following relation results:

$$\begin{aligned}&\partial_t\widetilde{E}_0(\phi(t),\psi(t)) + \beta_1^{(a)}\,\big\|\nabla\partial_{tt}\psi(t)\big\|_{L_2}^2\\ &\quad= -\beta_4^{(a)}(\sigma_0)\,\partial_t(\Delta\partial_t\psi(t)\,|\,\Delta\psi(t))_{L_2} + \beta_4^{(a)}(\sigma_0)\,\big\|\Delta\partial_t\psi(t)\big\|_{L_2}^2\\ &\qquad-\left(r^{(\phi)}(t) - \frac{1}{2}\,\partial_t\alpha^{(\phi)}(t)\,\partial_{tt}\psi(t)\,\middle|\,\partial_{tt}\psi(t)\right)_{L_2}.\end{aligned}$$

Integration with respect to time finally yields

$$\begin{aligned}&\widetilde{E}_0(\phi(t),\psi(t)) + \beta_1^{(a)}\int_0^t\big\|\nabla\partial_{tt}\psi(\tau)\big\|_{L_2}^2\,\mathrm{d}\tau\\ &\quad= \widetilde{E}_0(\psi_0,\psi_0) + \beta_4^{(a)}(\sigma_0)\,(\Delta\psi_1\,|\,\Delta\psi_0)_{L_2} - \beta_4^{(a)}(\sigma_0)\,(\Delta\partial_t\psi(t)\,|\,\Delta\psi(t))_{L_2}\\ &\qquad+\beta_4^{(a)}(\sigma_0)\int_0^t\big\|\Delta\partial_t\psi(\tau)\big\|_{L_2}^2\,\mathrm{d}\tau\\ &\qquad-\int_0^t\left(r^{(\phi)}(\tau) - \frac{1}{2}\,\partial_t\alpha^{(\phi)}(\tau)\,\partial_{tt}\psi(\tau)\,\middle|\,\partial_{tt}\psi(\tau)\right)_{L_2}\mathrm{d}\tau;\end{aligned} \tag{3.10b}$$

note that we here set

$$\widetilde{E}_0(\psi_0,\psi_0) = \frac{1}{2}\,\left\|\sqrt{1+\beta_5(\sigma)\,\psi_1}\,\psi_2\right\|_{L_2}^2 + \frac{\beta_2^{(a)}(\sigma_0)}{2}\,\big\|\Delta\psi_1\big\|_{L_2}^2 + \frac{\beta_3}{2}\,\big\|\nabla\psi_1\big\|_{L_2}^2. \tag{3.10c}$$

Second energy identity. On the other hand, we substitute α and r in (3.6c) by $\alpha^{(\phi)}$ and $r^{(\phi)}$; by testing with $\Delta\partial_{tt}\psi(t)$, we obtain

$$\begin{aligned}
&(\partial_{ttt}\psi(t)\,|\,\Delta\partial_{tt}\psi(t))_{L_2} - \beta_1^{(a)}\left\|\frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\Delta\partial_{tt}\psi(t)\right\|_{L_2}^2\\
&\quad + \beta_2^{(a)}(\sigma_0)\left(\frac{1}{\alpha^{(\phi)}(t)}\,\Delta^2\partial_t\psi(t)\,\middle|\,\Delta\partial_{tt}\psi(t)\right)_{L_2} - \beta_3\left(\frac{1}{\alpha^{(\phi)}(t)}\,\Delta\partial_t\psi(t)\,\middle|\,\Delta\partial_{tt}\psi(t)\right)_{L_2}\\
&\quad + \beta_4^{(a)}(\sigma_0)\left(\frac{1}{\alpha^{(\phi)}(t)}\,\Delta^2\psi(t)\,\middle|\,\Delta\partial_{tt}\psi(t)\right)_{L_2} + \left(\frac{1}{\alpha^{(\phi)}(t)}\,r^{(\phi)}(t)\,\middle|\,\Delta\partial_{tt}\psi(t)\right)_{L_2} = 0.
\end{aligned}$$

Similar to before, we employ integration-by-parts under assumption (3.3b) and replace the arising space and time derivatives of $\frac{1}{\alpha^{(\phi)}}$ by

$$\nabla\frac{1}{\alpha^{(\phi)}(t)} = -\beta_5(\sigma)\,\frac{1}{(\alpha^{(\phi)}(t))^2}\,\nabla\partial_t\phi(t), \quad \partial_t\frac{1}{\alpha^{(\phi)}(t)} = -\beta_5(\sigma)\,\frac{1}{(\alpha^{(\phi)}(t))^2}\,\partial_{tt}\phi(t);$$

this yields the identities

$$\begin{aligned}
&(\partial_{ttt}\psi(t)\,|\,\Delta\partial_{tt}\psi(t))_{L_2} = -(\nabla\partial_{ttt}\psi(t)\,|\,\nabla\partial_{tt}\psi(t))_{L_2} = -\frac{1}{2}\,\partial_t\big\|\nabla\partial_{tt}\psi(t)\big\|_{L_2}^2,\\
&\left(\frac{1}{\alpha^{(\phi)}(t)}\,\Delta^2\partial_t\psi(t)\,\middle|\,\Delta\partial_{tt}\psi(t)\right)_{L_2}\\
&\qquad = -\left(\nabla\Delta\partial_t\psi(t)\,\middle|\,\nabla\left(\frac{1}{\alpha^{(\phi)}(t)}\,\Delta\partial_{tt}\psi(t)\right)\right)_{L_2}\\
&\qquad = -\left(\nabla\Delta\partial_t\psi(t)\,\middle|\,\nabla\frac{1}{\alpha^{(\phi)}(t)}\,\Delta\partial_{tt}\psi(t)\right)_{L_2}\\
&\qquad\quad - \left(\nabla\Delta\partial_t\psi(t)\,\middle|\,\frac{1}{\alpha^{(\phi)}(t)}\,\nabla\Delta\partial_{tt}\psi(t)\right)_{L_2}\\
&\qquad = -\left(\nabla\frac{1}{\alpha^{(\phi)}(t)}\,\middle|\,\Delta\partial_{tt}\psi(t)\,\nabla\Delta\partial_t\psi(t)\right)_{L_2}\\
&\qquad\quad - \frac{1}{2}\,\partial_t\left\|\frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\nabla\Delta\partial_t\psi(t)\right\|_{L_2}^2 + \frac{1}{2}\left(\partial_t\frac{1}{\alpha^{(\phi)}(t)}\,\middle|\,\big|\nabla\Delta\partial_t\psi(t)\big|^2\right)_{L_2}\\
&\qquad = -\frac{1}{2}\,\partial_t\left\|\frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\nabla\Delta\partial_t\psi(t)\right\|_{L_2}^2\\
&\qquad\quad + \beta_5(\sigma)\left(\frac{1}{(\alpha^{(\phi)}(t))^2}\,\middle|\,\Delta\partial_{tt}\psi(t)\,\nabla\Delta\partial_t\psi(t)\cdot\nabla\partial_t\phi(t)\right)_{L_2}\\
&\qquad\quad - \frac{\beta_5(\sigma)}{2}\left(\frac{1}{(\alpha^{(\phi)}(t))^2}\,\middle|\,\partial_{tt}\phi(t)\,\big|\nabla\Delta\partial_t\psi(t)\big|^2\right)_{L_2},
\end{aligned} \tag{3.11}$$

as well as

$$\begin{aligned}
&\left(\frac{1}{\alpha^{(\phi)}(t)}\,\Delta\partial_t\psi(t)\,\middle|\,\Delta\partial_{tt}\psi(t)\right)_{L_2}\\
&\quad= \frac{1}{2}\,\partial_t\left\|\frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\Delta\partial_t\psi(t)\right\|_{L_2}^2 - \frac{1}{2}\,\left(\partial_t\frac{1}{\alpha^{(\phi)}(t)}\,\middle|\,(\Delta\partial_t\psi(t))^2\right)_{L_2}\\
&\quad= \frac{1}{2}\,\partial_t\left\|\frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\Delta\partial_t\psi(t)\right\|_{L_2}^2 + \frac{\beta_5(\sigma)}{2}\,\left(\frac{1}{(\alpha^{(\phi)}(t))^2}\,\middle|\,\partial_{tt}\phi(t)(\Delta\partial_t\psi(t))^2\right)_{L_2};
\end{aligned} \tag{3.12}$$

furthermore, we make use of the relation

$$\begin{aligned}
&\left(\frac{1}{\alpha^{(\phi)}(t)}\,\Delta^2\psi(t)\,\middle|\,\Delta\partial_{tt}\psi(t)\right)_{L_2}\\
&\quad= -\left(\nabla\Delta\psi(t)\,\middle|\,\nabla\left(\frac{1}{\alpha^{(\phi)}(t)}\,\Delta\partial_{tt}\psi(t)\right)\right)_{L_2}\\
&\quad= -\left(\nabla\Delta\psi(t)\,\middle|\,\nabla\frac{1}{\alpha^{(\phi)}(t)}\,\Delta\partial_{tt}\psi(t)\right)_{L_2} - \left(\nabla\Delta\psi(t)\,\middle|\,\frac{1}{\alpha^{(\phi)}(t)}\,\nabla\Delta\partial_{tt}\psi(t)\right)_{L_2}\\
&\quad= -\left(\nabla\frac{1}{\alpha^{(\phi)}(t)}\,\middle|\,\Delta\partial_{tt}\psi(t)\,\nabla\Delta\psi(t)\right)_{L_2} - \partial_t\left(\frac{1}{\alpha^{(\phi)}(t)}\,\middle|\,\nabla\Delta\partial_t\psi(t)\cdot\nabla\Delta\psi(t)\right)_{L_2}\\
&\qquad+ \left(\partial_t\frac{1}{\alpha^{(\phi)}(t)}\,\middle|\,\nabla\Delta\partial_t\psi(t)\cdot\nabla\Delta\psi(t)\right)_{L_2} + \left\|\frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\nabla\Delta\partial_t\psi(t)\right\|_{L_2}^2\\
&\quad= -\partial_t\left(\frac{1}{\alpha^{(\phi)}(t)}\,\middle|\,\nabla\Delta\partial_t\psi(t)\cdot\nabla\Delta\psi(t)\right)_{L_2}\\
&\qquad+ \beta_5(\sigma)\,\left(\frac{1}{(\alpha^{(\phi)}(t))^2}\,\middle|\,\Delta\partial_{tt}\psi(t)\,\nabla\partial_t\phi(t)\cdot\nabla\Delta\psi(t)\right)_{L_2}\\
&\qquad- \beta_5(\sigma)\,\left(\frac{1}{(\alpha^{(\phi)}(t))^2}\,\middle|\,\partial_{tt}\phi(t)\,\nabla\Delta\partial_t\psi(t)\cdot\nabla\Delta\psi(t)\right)_{L_2}\\
&\qquad+ \left\|\frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\nabla\Delta\partial_t\psi(t)\right\|_{L_2}^2.
\end{aligned}$$

With the help of the abbreviation

$$\begin{aligned}
\widetilde{E}_1(\phi(t),\psi(t)) &= \frac{1}{2}\,\big\|\nabla\partial_{tt}\psi(t)\big\|_{L_2}^2 + \frac{\beta_2^{(a)}(\sigma_0)}{2}\,\left\|\frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\nabla\Delta\partial_t\psi(t)\right\|_{L_2}^2\\
&\quad+ \frac{\beta_3}{2}\,\left\|\frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\Delta\partial_t\psi(t)\right\|_{L_2}^2,
\end{aligned} \tag{3.13a}$$

we thus obtain

$$
\begin{aligned}
&\partial_t \widetilde{E}_1(\phi(t),\psi(t)) + \beta_1^{(a)} \left\| \frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\, \Delta\partial_{tt}\psi(t) \right\|_{L_2}^2 \\
&\quad = -\beta_4^{(a)}(\sigma_0)\, \partial_t \left(\frac{1}{\alpha^{(\phi)}(t)} \,\middle|\, \nabla\Delta\partial_t\psi(t)\cdot\nabla\Delta\psi(t) \right)_{L_2} \\
&\qquad + \beta_4^{(a)}(\sigma_0) \left\| \frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\, \nabla\Delta\partial_t\psi(t) \right\|_{L_2}^2 \\
&\qquad + \left(\frac{1}{\alpha^{(\phi)}(t)}\, r^{(\phi)}(t) \,\middle|\, \Delta\partial_{tt}\psi(t) \right)_{L_2} \\
&\qquad + \beta_2^{(a)}(\sigma_0)\,\beta_5(\sigma) \left(\frac{1}{(\alpha^{(\phi)}(t))^2} \,\middle|\, \Delta\partial_{tt}\psi(t)\, \nabla\Delta\partial_t\psi(t)\cdot\nabla\partial_t\phi(t) \right)_{L_2} \\
&\qquad - \frac{\beta_2^{(a)}(\sigma_0)\,\beta_5(\sigma)}{2} \left(\frac{1}{(\alpha^{(\phi)}(t))^2} \,\middle|\, \partial_{tt}\phi(t)\, \left|\nabla\Delta\partial_t\psi(t)\right|^2 \right)_{L_2} \\
&\qquad - \frac{\beta_3\,\beta_5(\sigma)}{2} \left(\frac{1}{(\alpha^{(\phi)}(t))^2} \,\middle|\, \partial_{tt}\phi(t)(\Delta\partial_t\psi(t))^2 \right)_{L_2} \\
&\qquad + \beta_4^{(a)}(\sigma_0)\,\beta_5(\sigma) \left(\frac{1}{(\alpha^{(\phi)}(t))^2} \,\middle|\, \Delta\partial_{tt}\psi(t)\, \nabla\partial_t\phi(t)\cdot\nabla\Delta\psi(t) \right)_{L_2} \\
&\qquad - \beta_4^{(a)}(\sigma_0)\,\beta_5(\sigma) \left(\frac{1}{(\alpha^{(\phi)}(t))^2} \,\middle|\, \partial_{tt}\phi(t)\, \nabla\Delta\partial_t\psi(t)\cdot\nabla\Delta\psi(t) \right)_{L_2}.
\end{aligned}
$$

Performing integration with respect to time, finally leads to

$$
\begin{aligned}
&\widetilde{E}_1(\phi(t),\psi(t)) + \beta_1^{(a)} \int_0^t \left\| \frac{1}{\sqrt{\alpha^{(\phi)}(\tau)}}\, \Delta\partial_{tt}\psi(\tau) \right\|_{L_2}^2 \mathrm{d}\tau \\
&\quad = \widetilde{E}_1(\psi_0,\psi_0) + \beta_4^{(a)}(\sigma_0) \left(\frac{1}{\alpha^{(\phi)}(0)} \,\middle|\, \nabla\Delta\psi_1\cdot\nabla\Delta\psi_0 \right)_{L_2} \\
&\qquad - \beta_4^{(a)}(\sigma_0) \left(\frac{1}{\alpha^{(\phi)}(t)} \,\middle|\, \nabla\Delta\partial_t\psi(t)\cdot\nabla\Delta\psi(t) \right)_{L_2} \\
&\qquad + \beta_4^{(a)}(\sigma_0) \int_0^t \left\| \frac{1}{\sqrt{\alpha^{(\phi)}(\tau)}}\, \nabla\Delta\partial_t\psi(\tau) \right\|_{L_2}^2 \mathrm{d}\tau \\
&\qquad + \int_0^t \left(\frac{1}{\alpha^{(\phi)}(\tau)}\, r^{(\phi)}(\tau) \,\middle|\, \Delta\partial_{tt}\psi(\tau) \right)_{L_2} \mathrm{d}\tau \\
&\qquad + \beta_2^{(a)}(\sigma_0)\,\beta_5(\sigma) \int_0^t \left(\frac{1}{(\alpha^{(\phi)}(\tau))^2} \,\middle|\, \Delta\partial_{tt}\psi(\tau)\, \nabla\Delta\partial_t\psi(\tau)\cdot\nabla\partial_t\phi(\tau) \right)_{L_2} \mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\beta_2^{(a)}(\sigma_0)\beta_5(\sigma)}{2}\int_0^t\left(\frac{1}{(\alpha^{(\phi)}(\tau))^2}\,\middle|\,\partial_{tt}\phi(\tau)\left|\nabla\Delta\partial_t\psi(\tau)\right|^2\right)_{L_2}\mathrm{d}\tau\\
&-\frac{\beta_3\beta_5(\sigma)}{2}\int_0^t\left(\frac{1}{(\alpha^{(\phi)}(\tau))^2}\,\middle|\,\partial_{tt}\phi(\tau)(\Delta\partial_t\psi(\tau))^2\right)_{L_2}\mathrm{d}\tau\\
&+\beta_4^{(a)}(\sigma_0)\,\beta_5(\sigma)\int_0^t\left(\frac{1}{(\alpha^{(\phi)}(\tau))^2}\,\middle|\,\Delta\partial_{tt}\psi(\tau)\,\nabla\partial_t\phi(\tau)\cdot\nabla\Delta\psi(\tau)\right)_{L_2}\mathrm{d}\tau\\
&-\beta_4^{(a)}(\sigma_0)\,\beta_5(\sigma)\int_0^t\left(\frac{1}{(\alpha^{(\phi)}(\tau))^2}\,\middle|\,\partial_{tt}\phi(\tau)\,\nabla\Delta\partial_t\psi(\tau)\cdot\nabla\Delta\psi(\tau)\right)_{L_2}\mathrm{d}\tau;
\end{aligned}
\tag{3.13b}
$$

similar to before, we here set

$$
\begin{aligned}
\widetilde{E}_1(\psi_0,\psi_0) &= \frac{1}{2}\left\|\nabla\psi_2\right\|_{L_2}^2 + \frac{\beta_2^{(a)}(\sigma_0)}{2}\left\|\frac{1}{\sqrt{1+\beta_5(\sigma)\,\psi_1}}\,\nabla\Delta\psi_1\right\|_{L_2}^2\\
&\quad + \frac{\beta_3}{2}\left\|\frac{1}{\sqrt{1+\beta_5(\sigma)\,\psi_1}}\,\Delta\psi_1\right\|_{L_2}^2.
\end{aligned}
\tag{3.13c}
$$

3.3. *Energy estimates*

Objective. In the following, we deduce *a priori* estimates for the energy functionals:

$$
\begin{aligned}
E_0(\phi(t),\psi(t)) &= \frac{1}{2}\left\|\sqrt{\alpha^{(\phi)}(t)}\,\partial_{tt}\psi(t)\right\|_{L_2}^2 + \frac{\beta_2^{(a)}(\sigma_0)}{4}\left\|\Delta\partial_t\psi(t)\right\|_{L_2}^2\\
&\quad + \frac{\beta_3}{2}\left\|\nabla\partial_t\psi(t)\right\|_{L_2}^2,\\
E_1(\phi(t),\psi(t)) &= \frac{1}{2}\left\|\nabla\partial_{tt}\psi(t)\right\|_{L_2}^2 + \frac{\beta_2^{(a)}(\sigma_0)}{4}\left\|\frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\nabla\Delta\partial_t\psi(t)\right\|_{L_2}^2\\
&\quad + \frac{\beta_3}{2}\left\|\frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\Delta\partial_t\psi(t)\right\|_{L_2}^2,
\end{aligned}
\tag{3.14a}
$$

on bounded time intervals $[0,T]$; we recall that $\alpha^{(\phi)} = 1+\beta_5(\sigma)\,\partial_t\phi$ and note that the values at the initial time are given by

$$
\begin{aligned}
E_0(\psi_0,\psi_0) &= \frac{1}{2}\left\|\sqrt{1+\beta_5(\sigma)\,\psi_1}\,\psi_2\right\|_{L_2}^2 + \frac{\beta_2^{(a)}(\sigma_0)}{4}\left\|\Delta\psi_1\right\|_{L_2}^2\\
&\quad + \frac{\beta_3}{2}\left\|\nabla\psi_1\right\|_{L_2}^2,
\end{aligned}
$$

$$E_1(\psi_0, \psi_0) = \frac{1}{2}\,\big\|\nabla \psi_2\big\|_{L_2}^2 + \frac{\beta_2^{(a)}(\sigma_0)}{4}\,\left\|\frac{1}{\sqrt{1+\beta_5(\sigma)\,\psi_1}}\,\nabla\Delta\psi_1\right\|_{L_2}^2 + \frac{\beta_3}{2}\,\left\|\frac{1}{\sqrt{1+\beta_5(\sigma)\,\psi_1}}\,\Delta\psi_1\right\|_{L_2}^2, \tag{3.14b}$$

see (3.7). In order to keep the formulas short, we introduce auxiliary abbreviations for the basic components

$$\begin{aligned}
E_{01}(\phi(t),\psi(t)) &= \left\|\sqrt{\alpha^{(\phi)}(t)}\,\partial_{tt}\psi(t)\right\|_{L_2}^2,\\
E_{02}(\phi(t),\psi(t)) &= \big\|\Delta\partial_t\psi(t)\big\|_{L_2}^2,\\
E_{03}(\phi(t),\psi(t)) &= \big\|\nabla\partial_t\psi(t)\big\|_{L_2}^2,\\
E_0(\phi(t),\psi(t)) &= \frac{1}{2}\,E_{01}(\phi(t),\psi(t)) + \frac{\beta_2^{(a)}(\sigma_0)}{4}\,E_{02}(\phi(t),\psi(t))\\
&\quad + \frac{\beta_3}{2}\,E_{03}(\phi(t),\psi(t)),\\
E_{11}(\phi(t),\psi(t)) &= \big\|\nabla\partial_{tt}\psi(t)\big\|_{L_2}^2,\\
E_{12}(\phi(t),\psi(t)) &= \left\|\frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\nabla\Delta\partial_t\psi(t)\right\|_{L_2}^2,\\
E_{13}(\phi(t),\psi(t)) &= \left\|\frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\Delta\partial_t\psi(t)\right\|_{L_2}^2,\\
E_1(\phi(t),\psi(t)) &= \frac{1}{2}\,E_{11}(\phi(t),\psi(t)) + \frac{\beta_2^{(a)}(\sigma_0)}{4}\,E_{12}(\phi(t),\psi(t))\\
&\quad + \frac{\beta_3}{2}\,E_{13}(\phi(t),\psi(t));
\end{aligned} \tag{3.14c}$$

we in particular apply the relations

$$\begin{aligned}
&E_{01}(\phi(t),\psi(t)) \le 2\,E_0(\phi(t),\psi(t)), \quad E_{03}(\phi(t),\psi(t)) \le \frac{2}{\beta_3}\,E_0(\phi(t),\psi(t)),\\
&E_{11}(\phi(t),\psi(t)) \le 2\,E_1(\phi(t),\psi(t)), \quad E_{13}(\phi(t),\psi(t)) \le \frac{2}{\beta_3}\,E_1(\phi(t),\psi(t)).
\end{aligned} \tag{3.14d}$$

Moreover, we denote

$$\begin{aligned} E_{20}(\phi(t),\psi(t)) &= \left\| \frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\Delta\partial_{tt}\psi(t)\right\|_{L_2}^2, \\ E_2(\phi(t),\psi(t)) &= \frac{1}{4}\,\widetilde{E}_2(\phi(t),\psi(t)) = \frac{\beta_1^{(a)}}{4}\,E_{20}(\phi(t),\psi(t)). \end{aligned} \tag{3.14e}$$

Our essential premise in the proof of Proposition 3.1 is boundedness of the energy functionals by positive constants $\overline{E}_0, \overline{E}_1, \overline{E}_2 > 0$, when inserting ϕ twice

$$\begin{gathered} \sup_{t\in[0,T]} E_0(\phi(t),\phi(t)) \le \overline{E}_0, \qquad \sup_{t\in[0,T]} E_1(\phi(t),\phi(t)) \le \overline{E}_1, \\ \int_0^T E_2(\phi(t),\phi(t))\,\mathrm{d}t \le \overline{E}_2; \end{gathered} \tag{3.14f}$$

evidently, this yields the relations

$$\begin{aligned} &\sup_{t\in[0,T]} E_{01}(\phi(t),\phi(t)) \le 2\,\overline{E}_0, \qquad \sup_{t\in[0,T]} E_{03}(\phi(t),\phi(t)) \le \frac{2}{\beta_3}\,\overline{E}_0, \\ &\sup_{t\in[0,T]} E_{11}(\phi(t),\phi(t)) \le 2\,\overline{E}_1, \qquad \sup_{t\in[0,T]} E_{13}(\phi(t),\phi(t)) \le \frac{2}{\beta_3}\,\overline{E}_1. \end{aligned} \tag{3.14g}$$

We note that $\beta_2^{(a)}(\sigma_0) \to 0$ if $a \to 0_+$; for this reason, E_{02} will be related to E_{13}, employing uniform boundedness of $\alpha^{(\phi)}$ from above and below.

Basic auxiliary estimates. Considering in the first instance regular bounded spatial domains $\Omega \subset \mathbb{R}^3$, we exploit the Poincaré–Friedrichs inequality, the continuous embeddings $H^1(\Omega) \hookrightarrow L_6(\Omega)$ as well as $H^2(\Omega) \hookrightarrow L_\infty(\Omega)$, and assume elliptic regularity; the application of Höder's inequality with exponent $p = 3$ and conjugate exponent $p^* = \frac{p}{p-1} = \frac{3}{2}$ also shows $H^1(\Omega) \hookrightarrow L_6(\Omega) \hookrightarrow L_4(\Omega)$, since

$$\|f\|_{L_4}^4 = \int_\Omega (f(x))^4\,\mathrm{d}x \le \left(\int_\Omega 1\,\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_\Omega (f(x))^{4p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}} = |\Omega|^{\frac{1}{3}}\,\|f\|_{L_6}^4.$$

To summarize, we apply the estimates

$$\begin{aligned} &\|f\|_{H^1} \le C_{\mathrm{PF}}\,\|\nabla f\|_{L_2}, \quad f \in H_0^1(\Omega), \\ &\|f\|_{L_4} \le C_{L_4\hookleftarrow H^1}\,\|f\|_{H^1}, \quad \|f\|_{L_6} \le C_{L_6\hookleftarrow H^1}\,\|f\|_{H^1}, \quad f \in H^1(\Omega), \\ &\|f\|_{L_\infty} \le C_{L_\infty\hookleftarrow H^2}\,\|f\|_{H^2}, \quad f \in H^2(\Omega), \\ &\|f\|_{H^2} \le C_\Delta\,\|\Delta f\|_{L_2}, \quad f \in H^2(\Omega)\cap H_0^1(\Omega); \end{aligned} \tag{3.15}$$

in all cases, the arising constant depends on the space domain.

Gronwall's inequality. We use that a non-negative function $f\colon[0,T]\to\mathbb{R}_{\geq 0}$ that solves an integral equation of the form

$$f(t)=f(0)+\gamma^2\int_0^t f(\tau)\mathrm{d}\tau+\int_0^t g(\tau)\mathrm{d}\tau,$$

where $\gamma>0$ and $g\colon[0,T]\to\mathbb{R}_{\geq 0}$, satisfies the relation

$$f(t)=\mathrm{e}^{\gamma^2 t}f(0)+\int_0^t \mathrm{e}^{\gamma^2(t-\tau)}g(\tau)\mathrm{d}\tau\leq \mathrm{e}^{\gamma^2 t}\left(f(0)+\int_0^t g(\tau)\mathrm{d}\tau\right).$$

Setting $f(t)=\|\varphi(t)\|_{L_2}^2$ and applying Cauchy's inequality as well as Young's inequality with weight $\gamma=\frac{1}{\sqrt{T}}$, this in particular implies

$$\begin{aligned}\big\|\varphi(t)\big\|_{L_2}^2&=\big\|\varphi(0)\big\|_{L_2}^2+2\int_0^t(\partial_t\varphi(\tau)\,|\,\varphi(\tau))_{L_2}\,\mathrm{d}\tau\\&\leq\big\|\varphi(0)\big\|_{L_2}^2+2\int_0^t\big\|\partial_t\varphi(\tau)\big\|_{L_2}\big\|\varphi(\tau)\big\|_{L_2}\mathrm{d}\tau\\&\leq\big\|\varphi(0)\big\|_{L_2}^2+\frac{1}{T}\int_0^t\big\|\varphi(\tau)\big\|_{L_2}^2\mathrm{d}\tau+T\int_0^t\big\|\partial_t\varphi(\tau)\big\|_{L_2}^2\mathrm{d}\tau\\&\leq 3\left(\big\|\varphi(0)\big\|_{L_2}^2+T\int_0^t\big\|\partial_t\varphi(\tau)\big\|_{L_2}^2\mathrm{d}\tau\right).\end{aligned}\tag{3.16}$$

Auxiliary estimates ensuring non-degeneracy. We first prove that the time-dependent function $\alpha^{(\phi)}=1+\beta_5(\sigma)\,\partial_t\phi$ defined in (3.6) is uniformly bounded from below and above

$$0<\underline{\alpha}=\frac{1}{2}\leq\big\|\alpha^{(\phi)}\big\|_{L_\infty([0,T],L_\infty(\Omega))}\leq\overline{\alpha}=\frac{3}{2},\tag{3.17a}$$

provided that the upper bound for the higher-order energy functional on the considered time interval $[0,T]$ satisfies the smallness requirement

$$C_0\,\overline{E}_1\leq\frac{1}{12},\quad C_0=\frac{(C_\Delta C_{L_\infty\hookleftarrow H^2}\beta_5(\sigma))^2}{\beta_3},\tag{3.17b}$$

see also (3.1), (3.14) and (3.15); we point out that the arising constant $C_0>0$ does not depend on $a>0$. With regard to the relation

$$\big|1-\big\|\alpha^{(\phi)}(t)-1\big\|_{L_\infty}\big|\leq\big\|\alpha^{(\phi)}(t)\big\|_{L_\infty}\leq 1+\big\|\alpha^{(\phi)}(t)-1\big\|_{L_\infty}$$

obtained by triangular inequalities, it remains to show boundedness of $\|\alpha^{(\phi)}(t)-1\|_{L_\infty}$ for any $t\in[0,T]$. By means of (3.15), we have

$$\begin{aligned}\big\|\alpha^{(\phi)}(t)-1\big\|_{L_\infty}&=\beta_5(\sigma)\big\|\partial_t\phi(t)\big\|_{L_\infty}\leq C_{L_\infty\hookleftarrow H^2}\,\beta_5(\sigma)\big\|\partial_t\phi(t)\big\|_{H^2}\\&\leq C_\Delta\,C_{L_\infty\hookleftarrow H^2}\,\beta_5(\sigma)\big\|\Delta\partial_t\phi(t)\big\|_{L_2}\\&\leq C_\Delta\,C_{L_\infty\hookleftarrow H^2}\,\beta_5(\sigma)\left\|\sqrt{\alpha^{(\phi)}(t)}\right\|_{L_\infty}\left\|\frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\Delta\partial_t\phi(t)\right\|_{L_2}\end{aligned}$$

$$\leq \sqrt{C_0}\,\sqrt{\beta_3\,E_{13}(\phi(t),\phi(t))}\,\sqrt{\big\|\alpha^{(\phi)}(t)\big\|_{L_\infty}}$$
$$\leq \sqrt{2\,C_0\,\overline{E}_1}\,\sqrt{1+\big\|\alpha^{(\phi)}(t)-1\big\|_{L_\infty}},$$

see also (3.14). Due to the smallness requirement $C_0\,\overline{E}_1 \leq \frac{1}{12}$, the positive solution to this inequality satisfies

$$\eta \leq \sqrt{2\,C_0\,\overline{E}_1}\,\sqrt{1+\eta}, \quad \eta^2 - 2\,C_0\,\overline{E}_1\,\eta - 2\,C_0\,\overline{E}_1 \leq 0,$$

$$(\eta - C_0\,\overline{E}_1)^2 \leq (2 + C_0\,\overline{E}_1)C_0\,\overline{E}_1, \quad 0 \leq \eta \leq C_0\,\overline{E}_1 + \sqrt{(2 + C_0\,\overline{E}_1)C_0\,\overline{E}_1} \leq \frac{1}{2};$$

this implies the stated relation, since

$$\frac{1}{2} \leq \big|1 - \big\|\alpha^{(\phi)}(t)-1\big\|_{L_\infty}\big| \leq \big\|\alpha^{(\phi)}(t)\big\|_{L_\infty} \leq 1 + \big\|\alpha^{(\phi)}(t)-1\big\|_{L_\infty} \leq \frac{3}{2},$$

and in particular ensures non-degeneracy

$$0 < \frac{1}{\overline{\alpha}} = \frac{2}{3} \leq \left\|\frac{1}{\alpha^{(\phi)}}\right\|_{L_\infty([0,T],L_\infty(\Omega))} \leq \frac{1}{\underline{\alpha}} = 2. \tag{3.17c}$$

Auxiliary estimate for nonlinearity. We next deduce an auxiliary estimate for the nonlinearity

$$r^{(\phi)} = \beta_5(\sigma)\,\partial_{tt}\psi\,\partial_{tt}\phi + 2\,\beta_6(\sigma)\,\nabla\partial_{tt}\psi\cdot\nabla\phi + 2\,\beta_6(\sigma)\,\nabla\partial_t\psi\cdot\nabla\partial_t\phi,$$

see (3.8) and recall (3.14). The estimation of the first term uses Cauchy's inequality and relation (3.15); that is, we have

$$\begin{aligned}
\|\partial_{tt}\psi(t)\,\partial_{tt}\phi(t)\|_{L_2}^2 &\leq \big\|\partial_{tt}\psi(t)\big\|_{L_4}^2\,\big\|\partial_{tt}\phi(t)\big\|_{L_4}^2 \\
&\leq C_{L_4\hookleftarrow H^1}^4\,\big\|\partial_{tt}\psi(t)\big\|_{H^1}^2\,\big\|\partial_{tt}\phi(t)\big\|_{H^1}^2 \\
&\leq C_{\mathrm{PF}}^4\,C_{L_4\hookleftarrow H^1}^4\,\big\|\nabla\partial_{tt}\psi(t)\big\|_{L_2}^2\,\big\|\nabla\partial_{tt}\phi(t)\big\|_{L_2}^2 \\
&\leq C_{\mathrm{PF}}^4\,C_{L_4\hookleftarrow H^1}^4\,E_{11}(\phi(t),\psi(t))\,E_{11}(\phi(t),\phi(t)) \\
&\leq 4\,C_{\mathrm{PF}}^4\,C_{L_4\hookleftarrow H^1}^4\,\overline{E}_1\,E_1(\phi(t),\psi(t)).
\end{aligned}$$

For the third term, we apply the same arguments and use boundedness of $\alpha^{(\phi)}$ by $\overline{\alpha} = \frac{3}{2}$, see (3.17), to obtain

$$\begin{aligned}
\|\nabla\partial_t\psi(t)\cdot\nabla\partial_t\phi(t)\|_{L_2}^2 &\leq \big\|\nabla\partial_t\psi(t)\big\|_{L_4}^2\,\big\|\nabla\partial_t\phi(t)\big\|_{L_4}^2 \\
&\leq C_{L_4\hookleftarrow H^1}^4\,\big\|\nabla\partial_t\psi(t)\big\|_{H^1}^2\,\big\|\nabla\partial_t\phi(t)\big\|_{H^1}^2 \\
&\leq C_{\mathrm{PF}}^4\,C_{L_4\hookleftarrow H^1}^4\,\big\|\Delta\partial_t\psi(t)\big\|_{L_2}^2\,\big\|\Delta\partial_t\phi(t)\big\|_{L_2}^2
\end{aligned}$$

$$
\begin{aligned}
&\le C_{\mathrm{PF}}^4\, C_{L_4\hookleftarrow H^1}^4 \,\left\|\alpha^{(\phi)}(t)\right\|_{L_\infty}^2 \left\|\frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\Delta\partial_t\psi(t)\right\|_{L_2}^2 \\
&\quad\times \left\|\frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\Delta\partial_t\phi(t)\right\|_{L_2}^2 \\
&\le C_{\mathrm{PF}}^4\, C_{L_4\hookleftarrow H^1}^4\, \overline{\alpha}^2\, E_{13}(\phi(t),\psi(t))\, E_{13}(\phi(t),\phi(t)) \\
&\le \frac{9\, C_{\mathrm{PF}}^4 C_{L_4\hookleftarrow H^1}^4}{\beta_3^2}\, \overline{E}_1\, E_1(\phi(t),\psi(t)).
\end{aligned}
$$

For the second term, we in addition employ Gronwall's inequality, see (3.16) with $\varphi = \Delta\phi$; this yields

$$
\begin{aligned}
&\left\|\nabla\partial_{tt}\psi(t)\cdot\nabla\phi(t)\right\|_{L_2}^2 \\
&\quad\le \left\|\nabla\partial_{tt}\psi(t)\right\|_{L_4}^2 \left\|\nabla\phi(t)\right\|_{L_4}^2 \le C_{L_4\hookleftarrow H^1}^4 \left\|\nabla\partial_{tt}\psi(t)\right\|_{H^1}^2 \left\|\nabla\phi(t)\right\|_{H^1}^2 \\
&\quad\le C_{\mathrm{PF}}^4\, C_{L_4\hookleftarrow H^1}^4 \left\|\Delta\partial_{tt}\psi(t)\right\|_{L_2}^2 \left\|\Delta\phi(t)\right\|_{L_2}^2 \\
&\quad\le C_{\mathrm{PF}}^4\, C_{L_4\hookleftarrow H^1}^4 \left\|\Delta\partial_{tt}\psi(t)\right\|_{L_2}^2 \left(3\,\|\Delta\psi_0\|_{L_2}^2 + 3\,T\int_0^t \left\|\Delta\partial_t\phi(\tau)\right\|_{L_2}^2 \mathrm{d}\tau\right) \\
&\quad\le 3\, C_{\mathrm{PF}}^4\, C_{L_4\hookleftarrow H^1}^4\, \overline{\alpha} \left\|\frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\Delta\partial_{tt}\psi(t)\right\|_{L_2}^2 \\
&\qquad\times \left(\left\|\Delta\psi_0\right\|_{L_2}^2 + \overline{\alpha}\,T\int_0^t \left\|\frac{1}{\sqrt{\alpha^{(\phi)}(t)}}\,\Delta\partial_t\phi(\tau)\right\|_{L_2}^2 \mathrm{d}\tau\right) \\
&\quad\le 3\, C_{\mathrm{PF}}^4\, C_{L_4\hookleftarrow H^1}^4\, \overline{\alpha}\, E_{20}(\phi(t),\psi(t)) \left(\left\|\Delta\psi_0\right\|_{L_2}^2 + \overline{\alpha}\,T^2 \sup_{t\in[0,T]} E_{13}(\phi(t),\phi(t))\right) \\
&\quad\le \frac{9\, C_{\mathrm{PF}}^4 C_{L_4\hookleftarrow H^1}^4}{2\,\beta_1^{(a)}}\, \widetilde{E}_2(\phi(t),\psi(t)) \left(\left\|\Delta\psi_0\right\|_{L_2}^2 + \frac{3}{\beta_3}\,T^2\,\overline{E}_1\right).
\end{aligned}
$$

By the elementary inequality $(a_1 + a_2 + a_3)^2 \le 3(a_1^2 + a_2^2 + a_3^2)$, valid for positive real numbers $a_1, a_2, a_3 > 0$, the estimate follows.

$$
\begin{aligned}
\int_0^t \left\|r^{(\phi)}(\tau)\right\|_{L_2}^2 \mathrm{d}\tau &\le 3\int_0^t \Big((\beta_5(\sigma))^2\, \left\|\partial_{tt}\psi(\tau)\,\partial_{tt}\phi(\tau)\right\|_{L_2}^2 \\
&\quad + 4(\beta_6(\sigma))^2\, \left\|\nabla\partial_t\psi(\tau)\,\nabla\partial_t\phi(\tau)\right\|_{L_2}^2 \\
&\quad + 4(\beta_6(\sigma))^2 \left\|\nabla\partial_{tt}\psi(\tau)\cdot\nabla\phi(\tau)\right\|_{L_2}^2\Big)\mathrm{d}\tau \\
&\le 12\, C_{\mathrm{PF}}^4\, C_{L_4\hookleftarrow H^1}^4 \left((\beta_5(\sigma))^2 + \frac{9(\beta_6(\sigma))^2}{\beta_3^2}\right)
\end{aligned}
$$

$$\times\,\overline{E}_1 \int_0^t E_1(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau + \frac{54\,C_{\mathrm{PF}}^4 C_{L_4\hookleftarrow H^1}^4 (\beta_6(\sigma))^2}{\beta_1^{(a)}}$$
$$\times\left(\|\Delta\psi_0\|_{L_2}^2 + \frac{3}{\beta_3}\,T^2\,\overline{E}_1\right)\int_0^t \widetilde{E}_2(\phi(\tau),\psi(\tau))\mathrm{d}\tau$$

In order to deduce a bound that holds uniformly for $a \in (0,\overline{a}]$, we use (3.1e); denoting

$$\begin{aligned} C_1 &= 12\,C_{\mathrm{PF}}^4\,C_{L_4\hookleftarrow H^1}^4\left((\beta_5(\sigma))^2 + \frac{9(\beta_6(\sigma))^2}{\beta_3^2}\right),\\ C_2 &= \frac{54\,C_{\mathrm{PF}}^4 C_{L_4\hookleftarrow H^1}^4 (\beta_6(\sigma))^2}{\underline{\beta}_1}, \quad C_3 = \frac{3}{\beta_3}, \end{aligned} \tag{3.18a}$$

we arrive at the auxiliary estimate

$$\begin{aligned} \int_0^t \left\|r^{(\phi)}(\tau)\right\|_{L_2}^2 \mathrm{d}\tau &\le C_1\,\overline{E}_1 \int_0^t E_1(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau \\ &\quad + C_2\Big(\left\|\Delta\psi_0\right\|_{L_2}^2 + C_3\,T^2\,\overline{E}_1\Big)\int_0^t \widetilde{E}_2(\phi(\tau),\psi(\tau))\mathrm{d}\tau \\ &\le C_1\,\overline{E}_1 \int_0^t E_1(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau \\ &\quad + 4\,C_2\Big(\left\|\Delta\psi_0\right\|_{L_2}^2 + C_3\,T^2\,\overline{E}_1\Big)\int_0^t E_2(\phi(\tau),\psi(\tau))\mathrm{d}\tau. \end{aligned} \tag{3.18b}$$

First energy estimate. Our starting point is (3.10), which we restate for convenience

$$\begin{aligned} &\widetilde{E}_0(\phi(t),\psi(t)) + \beta_1^{(a)} \int_0^t E_{11}(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau \\ &\quad = \widetilde{E}_0(\psi_0,\psi_0) + \beta_4^{(a)}(\sigma_0)\,(\Delta\psi_1\,|\,\Delta\psi_0)_{L_2} - \beta_4^{(a)}(\sigma_0)\,(\Delta\partial_t\psi(t)\,|\,\Delta\psi(t))_{L_2} \\ &\qquad + \beta_4^{(a)}(\sigma_0)\int_0^t \left\|\Delta\partial_t\psi(\tau)\right\|_{L_2}^2 \mathrm{d}\tau \\ &\qquad + \frac{1}{2}\int_0^t (\partial_t\alpha^{(\phi)}(\tau)\,\partial_{tt}\psi(\tau)\,|\,\partial_{tt}\psi(\tau))_{L_2}\,\mathrm{d}\tau - \int_0^t (r^{(\phi)}(\tau)\,|\,\partial_{tt}\psi(\tau))_{L_2}\,\mathrm{d}\tau, \end{aligned}$$

see also (3.14). In order to suitably estimate and absorb the terms arising on the right-hand side, we proceed as follows.

(i) By means of Cauchy's inequality and Young's inequality, we have

$$\begin{aligned} \beta_4^{(a)}(\sigma_0)\left|(\Delta\psi_1\,|\,\Delta\psi_0)_{L_2}\right| &\le \beta_4^{(a)}(\sigma_0)\left\|\Delta\psi_1\right\|_{L_2}\left\|\Delta\psi_0\right\|_{L_2} \\ &\le \frac{\beta_4^{(a)}(\sigma_0)}{2}\left\|\Delta\psi_1\right\|_{L_2}^2 + \frac{\beta_4^{(a)}(\sigma_0)}{2}\left\|\Delta\psi_0\right\|_{L_2}^2. \end{aligned}$$

(ii) In a similar manner, incorporating an additional weight $\gamma_1 > 0$, we obtain

$$\begin{aligned} \beta_4^{(a)}(\sigma_0)\big|(\Delta\partial_t\psi(t)|\Delta\psi(t))_{L_2}\big| &\leq \beta_4^{(a)}(\sigma_0)\,\big\|\Delta\partial_t\psi(t)\big\|_{L_2}\big\|\Delta\psi(t)\big\|_{L_2} \\ &\leq \frac{\gamma_1^2\beta_4^{(a)}(\sigma_0)}{2}\big\|\Delta\partial_t\psi(t)\big\|_{L_2}^2 + \frac{\beta_4^{(a)}(\sigma_0)}{2\gamma_1^2}\big\|\Delta\psi(t)\big\|_{L_2}^2; \end{aligned}$$

with regard to the relation $\beta_2^{(a)}(\sigma_0) = \beta_0^{(a)}(\sigma_0)\,\beta_4^{(a)}(\sigma_0)$, we set $\gamma_1^2 = \frac{\beta_0^{(a)}(\sigma_0)}{2}$ such that

$$\beta_4^{(a)}(\sigma_0)\big|(\Delta\partial_t\psi(t)\,|\,\Delta\psi(t))_{L_2}\big| \leq \frac{\beta_2^{(a)}(\sigma_0)}{4}\big\|\Delta\partial_t\psi(t)\big\|_{L_2}^2 + \frac{\beta_4^{(a)}(\sigma_0)}{\beta_0^{(a)}(\sigma_0)}\big\|\Delta\psi(t)\big\|_{L_2}^2.$$

This permits to absorb the first term involving $\|\Delta\partial_t\psi(t)\|_{L_2}^2$ and explains the definition of the energy functional

$$\begin{aligned} E_0(\phi(t),\psi(t)) &= \widetilde{E}_0(\phi(t),\psi(t)) - \frac{\beta_2^{(a)}(\sigma_0)}{4}\,\big\|\Delta\partial_t\psi(t)\big\|_{L_2}^2 \\ &= \frac{1}{2}\,\Big\|\sqrt{\alpha^{(\phi)}(t)}\,\partial_{tt}\psi(t)\Big\|_{L_2}^2 + \frac{\beta_2^{(a)}(\sigma_0)}{4}\,\big\|\Delta\partial_t\psi(t)\big\|_{L_2}^2 \\ &\quad + \frac{\beta_3}{2}\,\big\|\nabla\partial_t\psi(t)\big\|_{L_2}^2; \end{aligned}$$

for the second term, we apply Gronwall's inequality, see (3.16) with $\varphi = \Delta\psi$, which yields

$$\big\|\Delta\psi(t)\big\|_{L_2}^2 \leq 3\,\big\|\Delta\psi_0\big\|_{L_2}^2 + 3\,T\int_0^t \big\|\Delta\partial_t\psi(\tau)\big\|_{L_2}^2\,\mathrm{d}\tau\,.$$

(iii) Again by Cauchy's inequality, we have

$$\big((\partial_{tt}\psi(\tau))^2\,|\,\partial_{tt}\phi(\tau)\big)_{L_2} \leq \big\|\partial_{tt}\psi(\tau)\big\|_{L_4}^2\,\big\|\partial_{tt}\phi(\tau)\big\|_{L_2};$$

relation (3.15) and the uniform bound $\frac{1}{\underline{\alpha}} = 2$, see (3.17), imply

$$\begin{aligned} &\frac{1}{2}\int_0^t (\partial_t\alpha^{(\phi)}(\tau)\,\partial_{tt}\psi(\tau)\,|\,\partial_{tt}\psi(\tau))_{L_2}\,\mathrm{d}\tau \\ &\quad\leq \frac{\beta_5(\sigma)}{2}\int_0^t \big\|\partial_{tt}\psi(\tau)\big\|_{L_4}^2\,\big\|\partial_{tt}\phi(\tau)\big\|_{L_2}\,\mathrm{d}\tau \\ &\quad\leq \frac{C_{\mathrm{PF}}^2 C_{L_4\hookleftarrow H^1}^2\beta_5(\sigma)}{2}\,\sqrt{\frac{1}{\underline{\alpha}}}\int_0^t \big\|\nabla\partial_{tt}\psi(\tau)\big\|_{L_2}^2\,\Big\|\sqrt{\alpha^{(\phi)}(\tau)}\,\partial_{tt}\phi(\tau)\Big\|_{L_2}\,\mathrm{d}\tau \\ &\quad\leq \frac{C_{\mathrm{PF}}^2 C_{L_4\hookleftarrow H^1}^2\beta_5(\sigma)}{2}\,\sqrt{\frac{1}{\underline{\alpha}}}\int_0^t \sqrt{E_{01}(\phi(\tau),\phi(\tau))}\,E_{11}(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau \\ &\quad\leq C_{\mathrm{PF}}^2\,C_{L_4\hookleftarrow H^1}^2\,\beta_5(\sigma)\,\sqrt{\overline{E}_0}\int_0^t E_{11}(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau. \end{aligned}$$

Provided that the smallness requirement

$$\frac{C_{\mathrm{PF}}^2 C_{L_4 \hookrightarrow H^1}^2 \beta_5(\sigma)}{\underline{\beta}_1} \sqrt{\overline{E}_0} \leq \frac{1}{2} \tag{3.19}$$

is satisfied, see (3.1e), the resulting term

$$\frac{1}{2} \int_0^t (\partial_t \alpha^{(\phi)}(\tau)\, \partial_{tt}\psi(\tau) \,|\, \partial_{tt}\psi(\tau))_{L_2}\, \mathrm{d}\tau \leq \frac{\beta_1^{(a)}}{2} \int_0^t E_{11}(\phi(\tau), \psi(\tau))\, \mathrm{d}\tau$$

can be absorbed by the corresponding term arising on the left-hand side.

(iv) Cauchy's inequality and Young's inequality with weight $\gamma_2 > 0$ as well as (3.15) yield

$$\begin{aligned} \big|(r^{(\phi)}(\tau) \,|\, \partial_{tt}\psi(\tau))_{L_2}\big| &\leq \big\|r^{(\phi)}(\tau)\big\|_{L_2} \big\|\partial_{tt}\psi(\tau)\big\|_{L_2} \\ &\leq \frac{1}{2\gamma_2^2} \big\|r^{(\phi)}(\tau)\big\|_{L_2}^2 + \frac{\gamma_2^2}{2} \big\|\partial_{tt}\psi(\tau)\big\|_{L_2}^2 \\ &\leq \frac{1}{2\gamma_2^2} \big\|r^{(\phi)}(\tau)\big\|_{L_2}^2 + \frac{\gamma_2^2}{2} \big\|\partial_{tt}\psi(\tau)\big\|_{H^1}^2 \\ &\leq \frac{1}{2\gamma_2^2} \big\|r^{(\phi)}(\tau)\big\|_{L_2}^2 + \frac{C_{\mathrm{PF}}^2 \gamma_2^2}{2} E_{11}(\phi(\tau), \psi(\tau)); \end{aligned}$$

with the special choice $\gamma_2^2 = \frac{\beta_1^{(a)}}{2\, C_{\mathrm{PF}}^2}$ such that $\frac{C_{\mathrm{PF}}^2 \gamma_2^2}{2} = \frac{\beta_1^{(a)}}{4}$ the second term arising on the right-hand side of

$$\begin{aligned} \int_0^t \big|(r^{(\phi)}(\tau) \,|\, \partial_{tt}\psi(\tau))_{L_2}\big|\, \mathrm{d}\tau &\leq \frac{C_{\mathrm{PF}}^2}{\beta_1^{(a)}} \int_0^t \big\|r^{(\phi)}(\tau)\big\|_{L_2}^2\, \mathrm{d}\tau \\ &\quad + \frac{\beta_1^{(a)}}{4} \int_0^t E_{11}(\phi(\tau), \psi(\tau))\, \mathrm{d}\tau \end{aligned}$$

can be absorbed.

The above considerations imply the estimate

$$\begin{aligned} &E_0(\phi(t), \psi(t)) + \frac{\beta_1^{(a)}}{4} \int_0^t E_{11}(\phi(\tau), \psi(\tau))\, \mathrm{d}\tau \\ &\quad \leq \widetilde{E}_0(\psi_0, \psi_0) + \frac{\beta_4^{(a)}(\sigma_0)}{2} \|\Delta\psi_1\|_{L_2}^2 + \beta_4^{(a)}(\sigma_0) \left(\frac{1}{2} + \frac{3}{\beta_0^{(a)}(\sigma_0)}\right) \|\Delta\psi_0\|_{L_2}^2 \\ &\qquad + \beta_4^{(a)}(\sigma_0) \left(1 + \frac{3\,T}{\beta_0^{(a)}(\sigma_0)}\right) \int_0^t \big\|\Delta\partial_t\psi(\tau)\big\|_{L_2}^2\, \mathrm{d}\tau + \frac{C_{\mathrm{PF}}^2}{\beta_1^{(a)}} \int_0^t \big\|r^{(\phi)}(\tau)\big\|_{L_2}^2\, \mathrm{d}\tau; \end{aligned}$$

together with (3.17) providing the uniform bound $\overline{\alpha} = \frac{3}{2}$ and (3.18), this yields

$$
\begin{aligned}
&E_0(\phi(t),\psi(t)) + \frac{\beta_1^{(a)}}{4}\int_0^t E_{11}(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau \\
&\quad\le \widetilde{E}_0(\psi_0,\psi_0) + \frac{\beta_4^{(a)}(\sigma_0)}{\beta_3}\,\overline{\alpha}\,E_1(\psi_0,\psi_0) + \beta_4^{(a)}(\sigma_0)\left(\frac{1}{2} + \frac{3}{\beta_0^{(a)}(\sigma_0)}\right)\|\Delta\psi_0\|_{L_2}^2 \\
&\qquad + \left(\frac{2\,\beta_4^{(a)}(\sigma_0)}{\beta_3}\left(1 + \frac{3\,T}{\beta_0^{(a)}(\sigma_0)}\right)\overline{\alpha} + \frac{C_{\mathrm{PF}}^2 C_1}{\beta_1^{(a)}}\,\overline{E}_1\right)\int_0^t E_1(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau \\
&\qquad + \frac{4\,C_{\mathrm{PF}}^2 C_2}{\beta_1^{(a)}}\left(\|\Delta\psi_0\|_{L_2}^2 + C_3\,T^2\,\overline{E}_1\right)\int_0^t E_2(\phi(\tau),\psi(\tau))\mathrm{d}\tau \\
&\quad\le \widetilde{E}_0(\psi_0,\psi_0) + \frac{3\,\beta_4^{(a)}(\sigma_0)}{2\,\beta_3}\,E_1(\psi_0,\psi_0) + \beta_4^{(a)}(\sigma_0)\left(\frac{1}{2} + \frac{3}{\beta_0^{(a)}(\sigma_0)}\right)\|\Delta\psi_0\|_{L_2}^2 \\
&\qquad + \left(\frac{3\,\beta_4^{(a)}(\sigma_0)}{\beta_3}\left(1 + \frac{3\,T}{\beta_0^{(a)}(\sigma_0)}\right) + \frac{C_{\mathrm{PF}}^2 C_1}{\beta_1^{(a)}}\,\overline{E}_1\right)\int_0^t E_1(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau \\
&\qquad + \frac{4C_{\mathrm{PF}}^2 C_2}{\beta_1^{(a)}}\left(\|\Delta\psi_0\|_{L_2}^2 + C_3\,T^2\,\overline{E}_1\right)\int_0^t E_2(\phi(\tau),\psi(\tau))\mathrm{d}\tau.
\end{aligned}
$$

Employing again (3.15) and (3.17), we obtain $\widetilde{E}_0(\psi_0,\psi_0) \le 4\,C_{\mathrm{PF}}^2 E_1(\psi_0,\psi_0)$; with the help of the bounds collected in (3.1e), which hold uniformly for $a \in (0,\bar{a}]$, we finally arrive at the relation

$$
\begin{aligned}
&E_0(\phi(t),\psi(t)) \\
&\quad\le \Phi_0\left(E_1(\psi_0,\psi_0) + \|\Delta\psi_0\|_{L_2}^2 + (1+\overline{E}_1)\int_0^t E_1(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau\right. \\
&\qquad\left. + \left(\|\Delta\psi_0\|_{L_2}^2 + \overline{E}_1\right)\int_0^t E_2(\phi(\tau),\psi(\tau))\mathrm{d}\tau\right), \\
&\Phi_0 = \Phi_0(T) \\
&\quad= \max\left\{4\,C_{\mathrm{PF}}^2 + \frac{3\,\overline{\beta}_4(\sigma_0)}{2\,\beta_3}, \overline{\beta}_4(\sigma_0)\left(\frac{1}{2} + \frac{3}{\underline{\beta}_0}\right), \frac{3\,\overline{\beta}_4(\sigma_0)}{\beta_3}\left(1 + \frac{3\,T}{\underline{\beta}_0}\right),\right. \\
&\qquad\qquad\left.\frac{C_{\mathrm{PF}}^2 C_1}{\underline{\beta}_1}, \frac{4\,C_{\mathrm{PF}}^2 C_2}{\underline{\beta}_1}\,\max\{1, C_3\,T^2\}\right\},
\end{aligned}
\tag{3.20}
$$

see also (3.18a). Due to the appearance of $E_1(\phi(t),\psi(t))$ and $E_2(\phi(\tau),\psi(\tau))$ on the right-hand side, further considerations are needed.

Second energy estimate. In order to deduce a suitable *a priori* estimate for the higher-order energy functional, our starting point is

$$
\begin{aligned}
&\widetilde{E}_1(\phi(t),\psi(t)) + \int_0^t \widetilde{E}_2(\phi(\tau),\psi(\tau))\mathrm{d}\tau \\
&\quad = \widetilde{E}_1(\psi_0,\psi_0) + \beta_4(\sigma_0)\left(\frac{1}{\alpha^{(\phi)}(0)}\,\middle|\,\nabla\Delta\psi_1\cdot\nabla\Delta\psi_0\right)_{L_2} \\
&\qquad - \beta_4^{(a)}(\sigma_0)\left(\frac{1}{\alpha^{(\phi)}(t)}\,\middle|\,\nabla\Delta\partial_t\psi(t)\cdot\nabla\Delta\psi(t)\right)_{L_2} \\
&\qquad + \beta_4^{(a)}(\sigma_0)\int_0^t E_{12}(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau \\
&\qquad + \int_0^t \left(\frac{1}{\alpha^{(\phi)}(\tau)}\,r^{(\phi)}(\tau)\,\middle|\,\Delta\partial_{tt}\psi(\tau)\right)_{L_2}\mathrm{d}\tau + R(t),
\end{aligned}
$$

where we employ the convenient abbreviation

$$
\begin{aligned}
R(t) &= \beta_2^{(a)}(\sigma_0)\,\beta_5(\sigma)\int_0^t \left(\frac{1}{(\alpha^{(\phi)}(\tau))^2}\,\middle|\,\Delta\partial_{tt}\psi(\tau)\,\nabla\Delta\partial_t\psi(\tau)\cdot\nabla\partial_t\phi(\tau)\right)_{L_2}\mathrm{d}\tau \\
&\quad - \frac{\beta_2^{(a)}(\sigma_0)\beta_5(\sigma)}{2}\int_0^t \left(\frac{1}{(\alpha^{(\phi)}(\tau))^2}\,\middle|\,\partial_{tt}\phi(\tau)\left|\nabla\Delta\partial_t\psi(\tau)\right|^2\right)_{L_2}\mathrm{d}\tau \\
&\quad - \frac{\beta_3\beta_5(\sigma)}{2}\int_0^t \left(\frac{1}{(\alpha^{(\phi)}(\tau))^2}\,\middle|\,\partial_{tt}\phi(\tau)(\Delta\partial_t\psi(\tau))^2\right)_{L_2}\mathrm{d}\tau \\
&\quad + \beta_4^{(a)}(\sigma_0)\,\beta_5(\sigma)\int_0^t \left(\frac{1}{(\alpha^{(\phi)}(\tau))^2}\,\middle|\,\Delta\partial_{tt}\psi(\tau)\,\nabla\partial_t\phi(\tau)\cdot\nabla\Delta\psi(\tau)\right)_{L_2}\mathrm{d}\tau \\
&\quad - \beta_4^{(a)}(\sigma_0)\,\beta_5(\sigma)\int_0^t \left(\frac{1}{(\alpha^{(\phi)}(\tau))^2}\,\middle|\,\partial_{tt}\phi(\tau)\,\nabla\Delta\partial_t\psi(\tau)\cdot\nabla\Delta\psi(\tau)\right)_{L_2}\mathrm{d}\tau,
\end{aligned}
$$

see also (3.13); similar arguments to before permit to estimate and absorb the arising terms.

(i) The application of Cauchy's inequality, Young's inequality with

$$
\gamma_1^2 = \frac{\beta_2^{(a)}(\sigma_0)}{2\,\beta_4^{(a)}(\sigma_0)} = \frac{\beta_0^{(a)}(\sigma_0)}{2},
$$

and the uniform bound $\frac{1}{\underline{\alpha}} = 2$, see (3.17), yields

$$
\begin{aligned}
&\beta_4^{(a)}(\sigma_0)\left|\left(\frac{1}{\alpha^{(\phi)}(0)}\,\middle|\,\nabla\Delta\psi_1\cdot\nabla\Delta\psi_0\right)_{L_2}\right| \\
&\quad \le \beta_4^{(a)}(\sigma_0)\left\|\frac{1}{\sqrt{\alpha^{(\phi)}(0)}}\,\nabla\Delta\psi_1\right\|_{L_2}\left\|\frac{1}{\sqrt{\alpha^{(\phi)}(0)}}\,\nabla\Delta\psi_0\right\|_{L_2}
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{\gamma_1^2 \beta_4^{(a)}(\sigma_0)}{2} \left\| \frac{1}{\sqrt{\alpha^{(\phi)}(0)}} \nabla\Delta\psi_1 \right\|_{L_2}^2 + \frac{\beta_4^{(a)}(\sigma_0)}{2\,\gamma_1^2} \left\| \frac{1}{\sqrt{\alpha^{(\phi)}(0)}} \nabla\Delta\psi_0 \right\|_{L_2}^2 \\
&\le \frac{\beta_2^{(a)}(\sigma_0)}{4} E_{12}(\psi_0,\psi_0) + \frac{\beta_4^{(a)}(\sigma_0)}{\beta_0^{(a)}(\sigma_0)} \frac{1}{\underline{\alpha}} \|\nabla\Delta\psi_0\|_{L_2}^2 \\
&\le E_1(\psi_0,\psi_0) + \frac{2\,\beta_4^{(a)}(\sigma_0)}{\beta_0^{(a)}(\sigma_0)} \|\nabla\Delta\psi_0\|_{L_2}^2.
\end{aligned}
$$

(ii) Using in addition Gronwall's inequality, see (3.16) with $\varphi = \nabla\Delta\psi$, and the uniform bound $\overline{\alpha} = \frac{3}{2}$, see again (3.17), we obtain

$$
\begin{aligned}
&\beta_4^{(a)}(\sigma_0) \left| \left(\frac{1}{\alpha^{(\phi)}(t)} \middle| \nabla\Delta\partial_t\psi(t) \cdot \nabla\Delta\psi(t) \right)_{L_2} \right| \\
&\quad\le \beta_4^{(a)}(\sigma_0) \left\| \frac{1}{\sqrt{\alpha^{(\phi)}(t)}} \nabla\Delta\partial_t\psi(t) \right\|_{L_2} \left\| \frac{1}{\sqrt{\alpha^{(\phi)}(t)}} \nabla\Delta\psi(t) \right\|_{L_2} \\
&\quad\le \frac{\gamma_1^2 \beta_4^{(a)}(\sigma_0)}{2} \left\| \frac{1}{\sqrt{\alpha^{(\phi)}(t)}} \nabla\Delta\partial_t\psi(t) \right\|_{L_2}^2 \\
&\qquad + \frac{\beta_4^{(a)}(\sigma_0)}{2\,\gamma_1^2} \left\| \frac{1}{\sqrt{\alpha^{(\phi)}(t)}} \nabla\Delta\psi(t) \right\|_{L_2}^2 \\
&\quad\le \frac{\beta_2^{(a)}(\sigma_0)}{4} E_{12}(\phi(t),\psi(t)) + \frac{\beta_2^{(a)}(\sigma_0)}{(\beta_0^{(a)}(\sigma_0))^2} \frac{1}{\underline{\alpha}} \|\nabla\Delta\psi(t)\|_{L_2}^2 \\
&\quad\le \frac{\beta_2^{(a)}(\sigma_0)}{4} E_{12}(\phi(t),\psi(t)) \\
&\qquad + \frac{6\,\beta_2^{(a)}(\sigma_0)}{(\beta_0^{(a)}(\sigma_0))^2} \left(\|\nabla\Delta\psi_0\|_{L_2}^2 + T\int_0^t \|\nabla\Delta\partial_t\psi(\tau)\|_{L_2}^2 \,\mathrm{d}\tau \right) \\
&\quad\le \frac{\beta_2^{(a)}(\sigma_0)}{4} E_{12}(\phi(t),\psi(t)) \\
&\qquad + \frac{6\,\beta_2^{(a)}(\sigma_0)}{(\beta_0^{(a)}(\sigma_0))^2} \left(\|\nabla\Delta\psi_0\|_{L_2}^2 + \overline{\alpha}\,T\int_0^t E_{12}(\phi(\tau),\psi(\tau)) \,\mathrm{d}\tau \right) \\
&\quad\le \frac{\beta_2^{(a)}(\sigma_0)}{4} E_{12}(\phi(t),\psi(t)) \\
&\qquad + \frac{6\,\beta_2^{(a)}(\sigma_0)}{(\beta_0^{(a)}(\sigma_0))^2} \|\nabla\Delta\psi_0\|_{L_2}^2 + \frac{9\,\beta_2^{(a)}(\sigma_0)}{(\beta_0^{(a)}(\sigma_0))^2}\, T\int_0^t E_{12}(\phi(\tau),\psi(\tau)) \,\mathrm{d}\tau
\end{aligned}
$$

$$\leq \frac{\beta_2^{(a)}(\sigma_0)}{4}\, E_{12}(\phi(t),\psi(t))$$
$$+ \frac{6\,\beta_4^{(a)}(\sigma_0)}{\beta_0^{(a)}(\sigma_0)}\, \|\nabla\Delta\psi_0\|_{L_2}^2 + \frac{36}{(\beta_0^{(a)}(\sigma_0))^2}\, T \int_0^t E_1(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau;$$

this shows that the first term on the right-hand side can be absorbed to yield $E_1(\phi(t),\psi(t))$ on the left-hand side, which explains the definition of the energy functional

$$E_1(\phi(t),\psi(t)) = \widetilde{E}_1(\phi(t),\psi(t)) - \frac{\beta_2^{(a)}(\sigma_0)}{4}\, E_{12}(\phi(t),\psi(t)).$$

(iii) Recalling once more the abbreviation $\beta_0^{(a)}(\sigma_0) = \frac{\beta_2^{(a)}(\sigma_0)}{\beta_4^{(a)}(\sigma_0)}$, the bound

$$\beta_4^{(a)}(\sigma_0) \int_0^t E_{12}(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau \leq \frac{4}{\beta_0^{(a)}(\sigma_0)} \int_0^t E_1(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau$$

is obvious.

(iv) By Cauchy's inequality, Young's inequality with weight $\gamma_2^2 = \beta_1^{(a)}$, and the upper bound $\frac{1}{\underline{\alpha}} = 2$, we have

$$\int_0^t \left(\frac{1}{\alpha^{(\phi)}(\tau)}\, r^{(\phi)}(\tau) \,\middle|\, \Delta\partial_{tt}\psi(\tau) \right)_{L_2} \mathrm{d}\tau$$
$$\leq \int_0^t \left\| \frac{1}{\sqrt{\alpha^{(\phi)}(\tau)}}\, r^{(\phi)}(\tau) \right\|_{L_2} \left\| \frac{1}{\sqrt{\alpha^{(\phi)}(\tau)}}\, \Delta\partial_{tt}\psi(\tau) \right\|_{L_2} \mathrm{d}\tau$$
$$\leq \frac{1}{2\,\gamma_2^2} \int_0^t \left\| \frac{1}{\sqrt{\alpha^{(\phi)}(\tau)}}\, r^{(\phi)}(\tau) \right\|_{L_2}^2 \mathrm{d}\tau$$
$$+ \frac{\gamma_2^2}{2} \int_0^t \left\| \frac{1}{\sqrt{\alpha^{(\phi)}(\tau)}}\, \Delta\partial_{tt}\psi(\tau) \right\|_{L_2}^2 \mathrm{d}\tau$$
$$\leq \frac{1}{\beta_1^{(a)}} \int_0^t \|r^{(\phi)}(\tau)\|_{L_2}^2\,\mathrm{d}\tau + \frac{1}{2} \int_0^t \widetilde{E}_2(\phi(\tau),\psi(\tau))\mathrm{d}\tau;$$

together with estimate (3.18) for the nonlinearity, this implies

$$\int_0^t \left(\frac{1}{\alpha^{(\phi)}(\tau)}\, r^{(\phi)}(\tau) \,\middle|\, \Delta\partial_{tt}\psi(\tau) \right)_{L_2} \mathrm{d}\tau$$
$$\leq \frac{C_1}{\beta_1^{(a)}}\, \overline{E}_1 \int_0^t E_1(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau$$
$$+ \left(\frac{1}{2} + \frac{C_2}{\beta_1^{(a)}} \left(\|\Delta\psi_0\|_{L_2}^2 + C_3\, T^2\, \overline{E}_1 \right) \right) \int_0^t \widetilde{E}_2(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau.$$

Under the additional smallness requirement

$$\frac{C_2}{\underline{\beta}_1}\Big(\big\|\Delta\psi_0\big\|_{L_2}^2 + C_3\,T^2\,\overline{E}_1\Big) \leq \frac{1}{4}, \tag{3.21}$$

we obtain the relation

$$\begin{aligned}
&\int_0^t \left(\frac{1}{\alpha^{(\phi)}(\tau)}\, r^{(\phi)}(\tau)\,\middle|\,\Delta\partial_{tt}\psi(\tau)\right)_{L_2} \mathrm{d}\tau \\
&\quad \leq \frac{C_1}{\beta_1^{(a)}}\,\overline{E}_1 \int_0^t E_1(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau + \frac{3}{4}\int_0^t \widetilde{E}_2(\phi(\tau),\psi(\tau))\mathrm{d}\tau;
\end{aligned}$$

thus, the second term involving $\widetilde{E}_2$ can be absorbed into the left-hand side and yields the integral over E_2.

As an intermediate result, we attain a bound of the form

$$\begin{aligned}
&E_1(\phi(t),\psi(t)) + \int_0^t E_2(\phi(\tau),\psi(\tau))\mathrm{d}\tau \\
&\quad \leq \Phi_1\left(E_1(\psi_0,\psi_0) + \big\|\nabla\Delta\psi_0\big\|_{L_2}^2 + \int_0^t E_1(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau\right) + \big|R(t)\big|, \qquad (3.22)\\
&\Phi_1 = \Phi_1(T,\overline{E}_1) = \max\left\{1, \frac{8\overline{\beta}_4(\sigma_0)}{\underline{\beta}_0}, \frac{36}{\underline{\beta}_0^2}\,T + \frac{4}{\underline{\beta}_0} + \frac{C_1}{\underline{\beta}_1}\overline{E}_1\right\}.
\end{aligned}$$

The remaining terms are estimated with the help of Cauchy's inequality and (3.15), that is, we use that a product of functions satisfies the relation

$$\begin{aligned}
\big|(\varphi_1(\tau)\,\varphi_2(\tau)\,|\,\varphi_3(\tau))_{L_2}\big| &\leq \big\|\varphi_1(\tau)\,\varphi_2(\tau)\big\|_{L_2}\big\|\varphi_3(\tau)\big\|_{L_2} \\
&\leq \big\|\varphi_1(\tau)\big\|_{L_\infty}\big\|\varphi_2(\tau)\big\|_{L_2}\big\|\varphi_3(\tau)\big\|_{L_2} \\
&\leq C_{L_\infty\hookrightarrow H^2}\big\|\varphi_1(\tau)\big\|_{H^2}\big\|\varphi_2(\tau)\big\|_{L_2}\big\|\varphi_3(\tau)\big\|_{L_2} \\
&\leq C_\Delta\,C_{L_\infty\hookrightarrow H^2}\big\|\Delta\varphi_1(\tau)\big\|_{L_2}\big\|\varphi_2(\tau)\big\|_{L_2}\big\|\varphi_3(\tau)\big\|_{L_2}.
\end{aligned}$$

As a consequence, by (3.14), inserting again $\frac{1}{\underline{\alpha}} = 2$, we obtain

$$\begin{aligned}
\big|R(t)\big| &\leq 2\,\beta_2^{(a)}(\sigma_0)\,\beta_5(\sigma)\int_0^t \sqrt{E_{20}(\phi(\tau),\psi(\tau))}\,\sqrt{E_{12}(\phi(\tau),\psi(\tau))}\,\big\|\nabla\partial_t\phi(\tau)\big\|_{L_\infty}\mathrm{d}\tau \\
&\quad + \beta_2^{(a)}(\sigma_0)\,\beta_5(\sigma)\int_0^t \big\|\partial_{tt}\phi(\tau)\big\|_{L_\infty} E_{12}(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau \\
&\quad + \beta_3\,\beta_5(\sigma)\int_0^t \big\|\partial_{tt}\phi(\tau)\big\|_{L_\infty} E_{13}(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau
\end{aligned}$$

$$+ 2\sqrt{2}\, \beta_4^{(a)}(\sigma_0)\, \beta_5(\sigma) \int_0^t \sqrt{E_{20}(\phi(\tau), \psi(\tau))}\, \big\|\nabla \partial_t \phi(\tau)\big\|_{L_\infty} \big\|\nabla \Delta \psi(\tau)\big\|_{L_2}\, \mathrm{d}\tau$$

$$+ 2\sqrt{2}\, \beta_4^{(a)}(\sigma_0)\, \beta_5(\sigma) \int_0^t \big\|\partial_{tt} \phi(\tau)\big\|_{L_\infty} \sqrt{E_{12}(\phi(\tau), \psi(\tau))}\, \big\|\nabla \Delta \psi(\tau)\big\|_{L_2}\, \mathrm{d}\tau.$$

Recalling the upper bound $\overline{\alpha} = \frac{3}{2}$, we employ the estimates

$$\begin{aligned} \big\|\nabla \partial_t \phi(\tau)\big\|_{L_\infty} &\leq C_\Delta\, C_{L_\infty \hookleftarrow H^2} \big\|\nabla \Delta \partial_t \phi(\tau)\big\|_{L_2} \\ &\leq \sqrt{\frac{3}{2}}\, C_\Delta\, C_{L_\infty \hookleftarrow H^2} \sqrt{E_{12}(\phi(\tau), \phi(\tau))}, \\ \big\|\partial_{tt} \phi(\tau)\big\|_{L_\infty} &\leq C_\Delta\, C_{L_\infty \hookleftarrow H^2} \big\|\Delta \partial_{tt} \phi(\tau)\big\|_{L_2} \\ &\leq \sqrt{\frac{3}{2}}\, C_\Delta\, C_{L_\infty \hookleftarrow H^2} \sqrt{E_{20}(\phi(\tau), \phi(\tau))}; \end{aligned}$$

moreover, the application of Gronwall's inequality, see (3.16) with $\varphi = \nabla \Delta \psi$, and the elementary relation $\sqrt{x^2 + y^2} \leq x + y$, valid for positive real numbers $x, y > 0$, implies

$$\begin{aligned} \big\|\nabla \Delta \psi(\tau)\big\|_{L_2}^2 &\leq 3 \big\|\nabla \Delta \psi_0\big\|_{L_2}^2 + 3\, T \int_0^\tau \big\|\nabla \Delta \partial_t \psi(\widetilde{\tau})\big\|_{L_2}^2 \, \mathrm{d}\widetilde{\tau} \\ &\leq 3 \big\|\nabla \Delta \psi_0\big\|_{L_2}^2 + 3\, T\, \overline{\alpha} \int_0^\tau E_{12}(\phi(\widetilde{\tau}), \psi(\widetilde{\tau}))\, \mathrm{d}\widetilde{\tau}, \\ \big\|\nabla \Delta \psi(\tau)\big\|_{L_2} &\leq \sqrt{3}\, \big\|\nabla \Delta \psi_0\big\|_{L_2} + \frac{3}{\sqrt{2}} \sqrt{T} \sqrt{\int_0^\tau E_{12}(\phi(\widetilde{\tau}), \psi(\widetilde{\tau}))\, \mathrm{d}\widetilde{\tau}}. \end{aligned}$$

Introducing the auxiliary abbreviations

$$\begin{aligned} R_1(t) &= \int_0^t E_1(\phi(\tau), \psi(\tau)) \sqrt{E_2(\phi(\tau), \phi(\tau))}\, \mathrm{d}\tau, \\ R_2(t) &= \int_0^t \sqrt{E_1(\phi(\tau), \psi(\tau))} \sqrt{\int_0^\tau E_1(\phi(\widetilde{\tau}), \psi(\widetilde{\tau}))\, \mathrm{d}\widetilde{\tau}} \sqrt{E_2(\phi(\tau), \phi(\tau))}\, \mathrm{d}\tau, \end{aligned}$$

as well as the constant

$$C_4 = C_\Delta\, C_{L_\infty \hookleftarrow H^2} \frac{\beta_5(\sigma)}{\sqrt{\underline{\beta}_1}} \max\left\{ 8\sqrt{6}, \frac{24\sqrt{\beta_2(\sigma_0)}}{\underline{\beta}_0}, \frac{24\sqrt{6}}{\underline{\beta}_0} \sqrt{T} \right\}, \qquad (3.23)$$

this leads to the relation

$$\begin{aligned}\big|R(t)\big| \le C_4 \bigg(&\int_0^t \sqrt{E_1(\phi(\tau),\psi(\tau))}\,\sqrt{E_2(\phi(\tau),\psi(\tau))}\,\sqrt{E_1(\phi(\tau),\phi(\tau))}\,\mathrm{d}\tau\\ &+ \big\|\nabla\Delta\psi_0\big\|_{L_2}\int_0^t \sqrt{E_2(\phi(\tau),\psi(\tau))}\,\sqrt{E_1(\phi(\tau),\phi(\tau))}\,\mathrm{d}\tau\\ &+ \big\|\nabla\Delta\psi_0\big\|_{L_2}\int_0^t \sqrt{E_1(\phi(\tau),\psi(\tau))}\,\sqrt{E_2(\phi(\tau),\phi(\tau))}\,\mathrm{d}\tau\\ &+ \int_0^t \sqrt{E_2(\phi(\tau),\psi(\tau))}\,\sqrt{\int_0^\tau E_1(\phi(\widetilde{\tau}),\psi(\widetilde{\tau}))\,\mathrm{d}\widetilde{\tau}}\,\sqrt{E_1(\phi(\tau),\phi(\tau))}\,\mathrm{d}\tau\\ &+ R_1(t) + R_2(t)\bigg).\end{aligned}$$

We next make use of the fundamental assumption

$$\sup_{t\in[0,T]} E_1(\phi(t),\phi(t)) \le \overline{E}_1, \qquad \int_0^T E_2(\phi(t),\phi(t))\,\mathrm{d}t \le \overline{E}_2,$$

see also (3.14). Replacing the interval of integration $[0,\tau]$ by $[0,t]$ and applying Cauchy's inequality, yields

$$\begin{aligned}R_2(t) &\le \sqrt{\int_0^t E_1(\phi(\widetilde{\tau}),\psi(\widetilde{\tau}))\,\mathrm{d}\widetilde{\tau}}\int_0^t \sqrt{E_1(\phi(\tau),\psi(\tau))}\,\sqrt{E_2(\phi(\tau),\phi(\tau))}\,\mathrm{d}\tau\\ &\le \sqrt{\int_0^t E_1(\phi(\widetilde{\tau}),\psi(\widetilde{\tau}))\,\mathrm{d}\widetilde{\tau}}\,\sqrt{\int_0^t E_1(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau}\,\sqrt{\int_0^t E_2(\phi(\tau),\phi(\tau))\,\mathrm{d}\tau}\\ &\le \sqrt{\overline{E}_2}\int_0^t E_1(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau\,;\end{aligned}$$

together with Young's inequality, this shows

$$\begin{aligned}\big|R(t)\big| \le C_4 \bigg(&\frac{1}{2}\sqrt{\overline{E}_1}\int_0^t \Big(E_1(\phi(\tau),\psi(\tau)) + E_2(\phi(\tau),\psi(\tau))\Big)\mathrm{d}\tau\\ &+ \frac{1}{2}\big\|\nabla\Delta\psi_0\big\|_{L_2}\bigg(T\,\overline{E}_1 + \int_0^t E_2(\phi(\tau),\psi(\tau))\mathrm{d}\tau\bigg)\\ &+ \frac{1}{2}\big\|\nabla\Delta\psi_0\big\|_{L_2}\bigg(\overline{E}_2 + \int_0^t E_1(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau\bigg)\\ &+ \frac{1}{2}\sqrt{\overline{E}_1}\bigg(\int_0^t E_2(\phi(\tau),\psi(\tau))\mathrm{d}\tau + T\int_0^t E_1(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau\bigg)\\ &+ R_1(t) + R_2(t)\bigg)\end{aligned}$$

$$\leq C_4 \left(\frac{1}{2} \left\|\nabla\Delta\psi_0\right\|_{L_2} (T\,\overline{E}_1 + \overline{E}_2) \right.$$

$$+ \left(\frac{1}{2} \left\|\nabla\Delta\psi_0\right\|_{L_2} + \frac{1}{2}(1+T)\sqrt{\overline{E}_1} + \sqrt{\overline{E}_2} \right) \int_0^t E_1(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau$$

$$+ \left(\frac{1}{2} \left\|\nabla\Delta\psi_0\right\|_{L_2} + \sqrt{\overline{E}_1} \right) \int_0^t E_2(\phi(\tau),\psi(\tau))\mathrm{d}\tau$$

$$\left. + \int_0^t E_1(\phi(\tau),\psi(\tau))\sqrt{E_2(\phi(\tau),\phi(\tau))}\,\mathrm{d}\tau \right).$$

Under the smallness requirement

$$C_4 \left(\frac{1}{2} \left\|\nabla\Delta\psi_0\right\|_{L_2} + \sqrt{\overline{E}_1} \right) \leq \frac{1}{2}, \tag{3.24}$$

the third term can be absorbed and we have

$$\begin{aligned} & E_1(\phi(t),\psi(t)) + \frac{1}{2}\int_0^t E_2(\phi(\tau),\psi(\tau))\mathrm{d}\tau \\ & \quad \leq \Phi_2 \left(E_1(\psi_0,\psi_0) + \left\|\nabla\Delta\psi_0\right\|_{L_2} \right. \\ & \qquad \left. + \int_0^t \left(1 + \sqrt{E_2(\phi(\tau),\phi(\tau))}\right) E_1(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau \right), \\ & \Phi_2 = \Phi_2(T, \left\|\nabla\Delta\psi_0\right\|_{L_2}, \overline{E}_1, \overline{E}_2) \\ & \quad = \max\left\{ C_4, \Phi_1 \left\|\nabla\Delta\psi_0\right\|_{L_2} + \frac{C_4}{2}(T\,\overline{E}_1 + \overline{E}_2), \right. \\ & \qquad \left. \Phi_1 + C_4 \left(\frac{1}{2}\left\|\nabla\Delta\psi_0\right\|_{L_2} + \frac{1}{2}(1+T)\sqrt{\overline{E}_1} + \sqrt{\overline{E}_2} \right) \right\}, \end{aligned} \tag{3.25}$$

see (3.22) and (3.23). Combining this with estimate (3.20) for the lower-order energy functional

$$\frac{1}{4\,\Phi_0(\|\Delta\psi_0\|_{L_2}^2 + \overline{E}_1)}\,E_0(\phi(t),\psi(t))$$

$$\leq \frac{1}{4(\|\Delta\psi_0\|_{L_2}^2 + \overline{E}_1)}\left(E_1(\psi_0,\psi_0) + \left\|\Delta\psi_0\right\|_{L_2}^2 \right)$$

$$+ \frac{1}{4(\|\Delta\psi_0\|_{L_2}^2 + \overline{E}_1)}(1 + \overline{E}_1)\int_0^t E_1(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau$$

$$+ \frac{1}{4}\int_0^t E_2(\phi(\tau),\psi(\tau))\,\mathrm{d}\tau$$

$$\leq \frac{1}{4(\|\Delta\psi_0\|_{L_2}^2 + \overline{E}_1)} \Big(E_1(\psi_0, \psi_0) + C_{\mathrm{PF}} \left\|\Delta\psi_0\right\|_{L_2} \left\|\nabla\Delta\psi_0\right\|_{L_2} \Big)$$
$$+ \frac{1}{4(\|\Delta\psi_0\|_{L_2}^2 + \overline{E}_1)} (1 + \overline{E}_1) \int_0^t E_1(\phi(\tau), \psi(\tau))\, \mathrm{d}\tau$$
$$+ \frac{1}{4} \int_0^t E_2(\phi(\tau), \psi(\tau))\mathrm{d}\tau,$$

see (3.15), yields

$$\frac{1}{4\,\Phi_0(\|\Delta\psi_0\|_{L_2}^2 + \overline{E}_1)}\, E_0(\phi(t), \psi(t)) + E_1(\phi(t), \psi(t)) + \frac{1}{4} \int_0^t E_2(\phi(\tau), \psi(\tau))\mathrm{d}\tau$$
$$\leq \left(\Phi_2 + \frac{1}{4(\|\Delta\psi_0\|_{L_2}^2 + \overline{E}_1)} \right) E_1(\psi_0, \psi_0)$$
$$+ \left(\Phi_2 + \frac{C_{\mathrm{PF}}}{4(\|\Delta\psi_0\|_{L_2}^2 + \overline{E}_1)} \left\|\Delta\psi_0\right\|_{L_2} \right) \left\|\nabla\Delta\psi_0\right\|_{L_2}$$
$$+ \left(\Phi_2 + \frac{1}{4(\|\Delta\psi_0\|_{L_2}^2 + \overline{E}_1)} (1 + \overline{E}_1) \right) \int_0^t E_1(\phi(\tau), \psi(\tau))\, \mathrm{d}\tau$$
$$+ \Phi_2 \int_0^t \sqrt{E_2(\phi(\tau), \phi(\tau))}\, E_1(\phi(\tau), \psi(\tau))\, \mathrm{d}\tau.$$

Altogether, we obtain the relation

$$\begin{aligned} &E_0(\phi(t), \psi(t)) + E_1(\phi(t), \psi(t)) + \int_0^t E_2(\phi(\tau), \psi(\tau))\mathrm{d}\tau \\ &\quad \leq \Phi_3 \bigg(E_1(\psi_0, \psi_0) + \left\|\nabla\Delta\psi_0\right\|_{L_2} \\ &\qquad + \int_0^t \Big(1 + \sqrt{E_2(\phi(\tau), \phi(\tau))} \Big) E_1(\phi(\tau), \psi(\tau))\, \mathrm{d}\tau \bigg), \\ &\Phi_3 = \Phi_3(T, \left\|\Delta\psi_0\right\|_{L_2}, \left\|\nabla\Delta\psi_0\right\|_{L_2}, \overline{E}_1, \overline{E}_2) \\ &\quad = 4\max\big\{1, \Phi_0(\left\|\Delta\psi_0\right\|_{L_2}^2 + \overline{E}_1)\big\} \\ &\qquad \times \max\bigg\{ 1, \Phi_2 + \frac{1}{4(\|\Delta\psi_0\|_{L_2}^2 + \overline{E}_1)}, \Phi_2 + \frac{C_{\mathrm{PF}}}{4(\|\Delta\psi_0\|_{L_2}^2 + \overline{E}_1)} \left\|\Delta\psi_0\right\|_{L_2}, \\ &\qquad\qquad \Phi_2 + \frac{1}{4}\, \frac{1}{(\|\Delta\psi_0\|_{L_2}^2 + \overline{E}_1)} (1 + \overline{E}_1) \bigg\}, \end{aligned} \tag{3.26}$$

which can be cast into the form

$$
\begin{aligned}
f(t) &\le \Phi_3 \left(\delta + \int_0^t \omega(\tau) f(\tau) \mathrm{d}\tau \right), \\
f(t) &= E_0(\phi(t), \psi(t)) + E_1(\phi(t), \psi(t)) + \int_0^t E_2(\phi(\tau), \psi(\tau)) \mathrm{d}\tau, \\
\delta &= E_1(\psi_0, \psi_0) + \left\| \nabla \Delta \psi_0 \right\|_{L_2}, \quad \omega(t) = 1 + \sqrt{E_2(\phi(t), \phi(t))};
\end{aligned}
$$

consequently, by Gronwall's inequality, we finally have

$$
\begin{aligned}
& f(t) \le \Phi\, \delta, \\
& \overline{\omega} = T + \sqrt{T}\, \sqrt{\overline{E}_2}, \quad \int_0^t \omega(\tau) \mathrm{d}\tau \le \overline{\omega}, \\
& \Phi = \Phi\Big(T, \|\Delta \psi_0\|_{L_2}, \left\|\nabla \Delta \psi_0\right\|_{L_2}, \overline{E}_1, \overline{E}_2\Big) = \Phi_3\, \mathrm{e}^{\overline{\omega}\, \Phi_3}.
\end{aligned} \tag{3.27}
$$

Summary. For convenience, we summarize the previous considerations; we recall that the constants C_0, C_1, C_2, C_3, C_4 and the quantities $\Phi_0, \Phi_1, \Phi_2, \Phi_3, \Phi$ are defined in (3.17b), (3.18a), (3.23) as well as (3.20), (3.22), (3.25)–(3.27). Under the smallness conditions

$$
\begin{aligned}
& C_0\, \overline{E}_1 \le \frac{1}{12}, \quad \frac{C_{\mathrm{PF}}^2 C_{L_4 \hookrightarrow H^1}^2 \beta_5(\sigma)}{\underline{\beta}_1} \sqrt{\overline{E}_0} \le \frac{1}{2}, \\
& \frac{C_2}{\underline{\beta}_1} \Big(\left\|\Delta \psi_0\right\|_{L_2}^2 + C_3\, T^2\, \overline{E}_1 \Big) \le \frac{1}{4}, \quad C_4 \left(\frac{1}{2} \left\|\nabla \Delta \psi_0\right\|_{L_2} + \sqrt{\overline{E}_1} \right) \le \frac{1}{2},
\end{aligned} \tag{3.28a}
$$

see (3.17b), (3.19), (3.21), (3.24), the energy estimate

$$
\begin{aligned}
& E_0(\phi(t), \psi(t)) + E_1(\phi(t), \psi(t)) + \int_0^t E_2(\phi(\tau), \psi(\tau)) \mathrm{d}\tau \\
& \quad \le \Phi(T, \|\Delta \psi_0\|_{L_2}, \left\|\nabla \Delta \psi_0\right\|_{L_2}, \overline{E}_1, \overline{E}_2) \Big(E_1(\psi_0, \psi_0) + \left\|\nabla \Delta \psi_0\right\|_{L_2} \Big)
\end{aligned} \tag{3.28b}
$$

holds for $a \in (0, \overline{a}]$; we note that the quantities $E_1(\psi_0, \psi_0)$ and $\|\nabla \Delta \psi_0\|_{L_2}$ only depend on the initial data and can be chosen sufficiently small.

3.4. *Existence result*

The proof of the following existence result uses Schauder's fixed point theorem and hence does not include uniqueness; as described in Remark 3.1 below, uniqueness can be established under stronger conditions on the initial data.

Proposition 3.1. *Consider the nonlinear damped wave equation* (3.1) *for* $a \in (0, \overline{a}]$, *and impose the homogeneous Dirichlet boundary conditions* (3.3) *as well as* *the initial conditions* (3.4). *Suppose that the prescribed initial data satisfy the regularity and compatibility conditions*

$$\psi_0, \psi_1 \in H^3(\Omega) \cap H_0^1(\Omega), \quad \Delta\psi_0, \Delta\psi_1, \psi_2 \in H_0^1(\Omega);$$

assume in addition that for $\|\Delta\psi_0\|_{L_2}, \|\nabla\Delta\psi_0\|_{L_2}$ *and upper bounds* $\overline{e}_0, \overline{e}_1 > 0$ *on the initial energies*

$$\left\|\psi_2\right\|_{L_2}^2 + \beta_2^{(a)}(\sigma_0)\left\|\Delta\psi_1\right\|_{L_2}^2 + \left\|\nabla\psi_1\right\|_{L_2}^2 \le \overline{e}_0,$$
$$\left\|\nabla\psi_2\right\|_{L_2}^2 + \beta_2^{(a)}(\sigma_0)\left\|\nabla\Delta\psi_1\right\|_{L_2}^2 + \left\|\Delta\psi_1\right\|_{L_2}^2 \le \overline{e}_1,$$

the quantity

$$M(\overline{e}_0, \overline{e}_1) = \frac{C_{PF}^2 C_{L_4 \hookrightarrow H^1}^2 \beta_5(\sigma)}{\underline{\beta}_1} \sqrt{\overline{e}_0} + C_0\,\overline{e}_1 + \frac{C_2}{\underline{\beta}_1}\left(\left\|\Delta\psi_0\right\|_{L_2}^2 + C_3\,T^2\,\overline{e}_1\right) + C_4\left(\frac{1}{2}\left\|\nabla\Delta\psi_0\right\|_{L_2} + \sqrt{\overline{e}_1}\right) \tag{3.29}$$

is sufficiently small, see (3.15), (3.17b), (3.18a), *and* (3.23) *for the definition of the arising constants. Then, there exists a weak solution*

$$\psi \in X = H^2([0,T], H_\diamond^2(\Omega)) \cap W_\infty^2([0,T], H_0^1(\Omega)) \cap W_\infty^1([0,T], H_\diamond^3(\Omega)),$$
$$H_\diamond^2(\Omega) = \{\chi \in H^2(\Omega) : \chi \in H_0^1(\Omega)\},\ H_\diamond^3(\Omega) = \{\chi \in H^3(\Omega) : \chi, \Delta\chi \in H_0^1(\Omega)\},$$

to the associated equation

$$\begin{aligned}\partial_{tt}\psi(t) - \psi_2 - \beta_1^{(a)}\,\Delta(\partial_t\psi(t) - \psi_1) + \beta_2^{(a)}(\sigma_0)\,\Delta^2(\psi(t) - \psi_0) - \beta_3\,\Delta(\psi(t) - \psi_0)\\ + \beta_4^{(a)}(\sigma_0)\int_0^t \Delta^2\psi(\tau)\mathrm{d}\tau + \beta_5(\sigma)(\partial_{tt}\psi(t)\,\partial_t\psi(t) - \psi_2\,\psi_1)\\ + 2\,\beta_6(\sigma)(\nabla\partial_t\psi(t)\cdot\nabla\psi(t) - \nabla\psi_1\cdot\nabla\psi_0) = 0,\end{aligned}$$

obtained by integration with respect to time. This solution satisfies a priori energy estimates of the form

$$\mathcal{E}_0(\psi(t)) = \left\|\partial_{tt}\psi(t)\right\|_{L_2}^2 + \beta_2^{(a)}(\sigma_0)\left\|\Delta\partial_t\psi(t)\right\|_{L_2}^2 + \left\|\nabla\partial_t\psi(t)\right\|_{L_2}^2,$$
$$\mathcal{E}_1(\psi(t)) = \left\|\nabla\partial_{tt}\psi(t)\right\|_{L_2}^2 + \beta_2^{(a)}(\sigma_0)\left\|\nabla\Delta\partial_t\psi(t)\right\|_{L_2}^2 + \left\|\Delta\partial_t\psi(t)\right\|_{L_2}^2,$$
$$\sup_{t\in[0,T]} \mathcal{E}_0(\psi(t)) \le \overline{E}_0, \quad \sup_{t\in[0,T]} \mathcal{E}_1(\psi(t)) \le \overline{E}_1, \quad \int_0^T \left\|\Delta\partial_{tt}\psi(t)\right\|_{L_2}^2 \mathrm{d}t \le \overline{E}_2, \tag{3.30}$$

which hold uniformly for $a \in (0, \overline{a}]$. *In particular, the quantity* $M(\overline{E}_0, \overline{E}_1)$ *remains sufficiently small to ensure uniform boundedness and hence non-degeneracy of the* *first time derivative*

$$0 < \underline{\alpha} = \frac{1}{2} \le \left\| 1 + \beta_5(\sigma)\, \partial_t \psi \right\|_{L_\infty([0,T], L_\infty(\Omega))} \le \overline{\alpha} = \frac{3}{2},$$

$$0 < \frac{1}{\overline{\alpha}} = \frac{2}{3} \le \left\| (1 + \beta_5(\sigma) \partial_t \psi)^{-1} \right\|_{L_\infty([0,T], L_\infty(\Omega))} \le \frac{1}{\underline{\alpha}} = 2.$$

Proof. As indicated before, our proof relies on a fixed-point argument. For suitably chosen positive constants $\overline{E}_0, \overline{E}_1, \overline{E}_2 > 0$ and suitably chosen initial data

$$\psi_0 \in H^3_\diamond(\Omega), \quad \psi_1 \in H^3_\diamond(\Omega), \quad \psi_2 \in H^1_0(\Omega),$$

such that $M(\overline{E}_0, \overline{E}_1)$ is sufficiently small, we introduce the nonempty closed subset

$$\mathcal{M} = \Bigg\{ \phi \in X : \phi(0) = \psi_0, \partial_t \phi(0) = \psi_1, \partial_{tt} \phi(0) = \psi_2,$$
$$\sup_{t \in [0,T]} \mathcal{E}_0(\phi(t)) \le \overline{E}_0, \sup_{t \in [0,T]} \mathcal{E}_1(\phi(t)) \le \overline{E}_1, \int_0^T \left\| \Delta \partial_{tt} \phi(t) \right\|^2_{L_2} \mathrm{d}t \le \overline{E}_2 \Bigg\}.$$

The nonlinear operator is defined by

$$\mathcal{T} : \mathcal{M} \to \mathcal{M} : \phi \mapsto \psi,$$

where ψ is the solution to

$$\begin{aligned}
&(1 + \beta_5(\sigma) \partial_t \phi) \partial_{ttt} \psi - \beta_1^{(a)}\, \Delta \partial_{tt} \psi + \beta_2^{(a)}(\sigma_0)\, \Delta^2 \partial_t \psi - \beta_3\, \Delta \partial_t \psi \\
&\quad + \beta_4^{(a)}(\sigma_0)\, \Delta^2 \psi + \beta_5(\sigma)\, \partial_{tt} \psi\, \partial_{tt} \phi + 2\, \beta_6(\sigma)\, \nabla \partial_{tt} \psi \cdot \nabla \phi \\
&\quad + 2\, \beta_6(\sigma)\, \nabla \partial_t \psi \cdot \nabla \partial_t \phi = 0;
\end{aligned} \tag{3.31}$$

that is, in (3.6b), we replace α and r by $\alpha^{(\phi)}$ and $r^{(\phi)}$, see also (3.8).

(i) *Well-definedness.* As common, existence of a solution to (3.31) is shown by Galerkin approximation in space and weak limits based on the *a priori* energy estimate (3.28) deduced before; relation (3.28) also implies uniqueness and ensures that $\mathcal{T}$ is a self-mapping on $\mathcal{M}$.

(ii) *Continuity.* The set $\mathcal{M}$ is a weak* compact and convex subset of the Banach space X; thus, for ensuring existence of a fixed point of $\mathcal{T}$ from the general version of Schauder's fixed point theorem in locally convex topological spaces, we have to prove weak* continuity of $\mathcal{T}$, see Ref. 10. For any sequence $(\phi^{(k)})_{k \in \mathbb{N}_{\ge 0}}$ in $\mathcal{M}$ converging weakly* to some $\phi_* \in \mathcal{M}$, the sequence of corresponding images defined by

$$\psi^{(k)} = \mathcal{T}(\phi^{(k)}) \in \mathcal{M}, \quad k \in \mathbb{N}_{\ge 0},$$

is bounded in X; hence, there exists a subsequence that converges to a function $\psi_* \in \mathcal{M}$ in the following sense:

$$\begin{aligned}
&\psi^{(k)} \overset{*}{\rightharpoonup} \psi_* \quad \text{in } X \quad \text{as } k \to \infty, \\
&\psi^{(k)} \to \psi_* \quad \text{in } \widetilde{X} = H^1([0,T], W^1_4(\Omega)) \quad \text{as } k \to \infty,
\end{aligned} \tag{3.32}$$

with compact embedding $X \hookrightarrow \widetilde{X}$. We apply a subsequence–subsequence argument for proving weak* convergence of $\psi^{(k)}$ to $\mathcal{T}(\phi_*)$. For this purpose, we consider an arbitrary weakly* convergent subsequence of $(\psi^{(k)})_{k\in\mathbb{N}_{\geq 0}}$ and prove that its limit ψ_* coincides with $\mathcal{T}(\phi_*)$. Due to boundedness in X, there is a sub-subsequence (not relabeled in the following) which converges in the sense of (3.32); the same type of convergence can be assumed for the corresponding subsequence of preimages (also not relabeled) $(\phi^{(k)})_{k\in\mathbb{N}_{\geq 0}}$ to ϕ_*. It remains to verify the solution property $\psi_* = \mathcal{T}(\phi_*)$.

(iii) *Verification of solution property.* We employ convenient abbreviations for the linear and the nonlinear terms

$$\begin{aligned}
\mathcal{L}^{(a)}\chi(t) &= \partial_{tt}\chi(t) - \beta_1^{(a)}\,\Delta\partial_t\chi(t) + \beta_2^{(a)}(\sigma_0)\,\Delta^2\chi(t) - \beta_3\,\Delta\chi(t)\\
&\quad + \beta_4^{(a)}(\sigma_0)\int_0^t \Delta^2\chi(\tau)\mathrm{d}\tau,\\
\mathcal{L}_0^{(a)} &= -\psi_2 + \beta_1^{(a)}\,\Delta\psi_1 - \beta_2^{(a)}(\sigma_0)\,\Delta^2\psi_0 + \beta_3\,\Delta\psi_0,\\
\mathcal{N}(\phi(t),\chi(t)) &= \beta_5(\sigma)\,\partial_{tt}\chi(t)\,\partial_t\phi(t) + 2\,\beta_6(\sigma)\,\nabla\partial_t\chi(t)\cdot\nabla\phi(t),\\
\mathcal{N}_0 &= -\beta_5(\sigma)\,\psi_2\,\psi_1 - 2\,\beta_6(\sigma)\,\nabla\psi_1\cdot\nabla\psi_0;
\end{aligned} \tag{3.33}$$

the relation

$$\mathcal{L}^{(a)}\psi^{(k)} + \mathcal{L}_0^{(a)} + \mathcal{N}(\phi^{(k)},\psi^{(k)}) + \mathcal{N}_0 = 0$$

thus corresponds to the given reformulation of the defining equation, obtained by integration with respect to time. In order to verify that ψ_* is a solution to

$$\mathcal{L}^{(a)}\psi_* + \mathcal{L}_0^{(a)} + \mathcal{N}\big(\phi_*,\psi_*\big) + \mathcal{N}_0 = 0,$$

we consider the difference

$$\begin{aligned}
&\mathcal{L}^{(a)}(\psi^{(k)} - \psi_*) + \mathcal{N}(\phi^{(k)},\psi^{(k)}) - \mathcal{N}(\phi_*,\psi_*)\\
&\quad = \mathcal{L}^{(a)}(\psi^{(k)} - \psi_*) + \mathcal{N}(\phi^{(k)} - \phi_*,\psi^{(k)}) + \mathcal{N}(\phi_*,\psi^{(k)} - \psi_*).
\end{aligned}$$

Due to the fact that $\phi^{(k)} \overset{*}{\rightharpoonup} \phi_*$ in X as $k\to\infty$, the linear contribution tends to zero in $L_\infty([0,T],H^{-1}(\Omega))$. The first terms in the nonlinearity satisfy

$$\begin{aligned}
&\beta_5(\sigma)\,\big\|\partial_{tt}\psi^{(k)}\big\|_{L_\infty([0,T],L_4(\Omega))}\,\big\|\partial_t(\phi^{(k)} - \phi_*)\big\|_{L_2([0,T],L_4(\Omega))}\\
&\quad + 2\,\beta_6(\sigma)\,\big\|\nabla\partial_t\psi^{(k)}\big\|_{L_\infty([0,T],L_4(\Omega))}\,\big\|\nabla(\phi^{(k)} - \phi_*)\big\|_{L_2([0,T],L_4(\Omega))}\\
&\quad + 2\,\beta_6(\sigma)\,\big\|\nabla\partial_t(\psi^{(k)} - \psi_*)\big\|_{L_2([0,T],L_4(\Omega))}\,\big\|\nabla\phi_*\big\|_{L_\infty([0,T],L_4(\Omega))}\\
&\leq C_{L_4\hookrightarrow H^1}\big((\beta_5(\sigma) + 2\,\beta_6(\sigma))\big\|\psi^{(k)}\big\|_X\,\big\|\phi^{(k)} - \phi_*\big\|_{\widetilde{X}}\\
&\quad + 2\,\beta_6(\sigma)\,\big\|\psi^{(k)} - \psi_*\big\|_{\widetilde{X}}\,\big\|\phi_*\big\|_X\big)
\end{aligned}$$

and therefore tend to zero by the strong convergence of $\phi^{(k)}$ and $\psi^{(k)}$ in $\widetilde{X}$; for any $v \in L_2([0,T], L_2(\Omega))$, due to the fact that

$$\partial_{tt}(\psi^{(k)} - \psi_*) \rightharpoonup 0 \quad \text{in } L_2([0,T], L_2(\Omega)) \quad \text{as } k \to \infty,$$
$$\partial_t \phi_* \, v \in L_2([0,T], L_2(\Omega)),$$

we further have

$$\beta_5(\sigma) \int_0^T \big(\partial_{tt}(\psi^{(k)}(t) - \psi_*(t)) \,|\, \partial_t \phi_*(t)\, v(t)\big)_{L_2} \,\mathrm{d}t \to 0 \quad \text{as } k \to \infty,$$

which concludes the proof. □

Remark 3.1. (i) By Morrey's inequality, for any element in the Sobolev space $W_p^1([0,T])$ with $p \in [1,\infty]$, there exists a unique representative that is Hölder continuous with exponent $\gamma = 1 - \frac{1}{p}$; in this sense, the continuous embeddings $H^1([0,T]) \hookrightarrow C^{0,\gamma}([0,T])$ with $\gamma \in [0, \frac{1}{2}]$ and $W_\infty^1 \hookrightarrow C^{0,1}([0,T])$ hold. As a consequence, the regularity

$$\psi \in H^2([0,T], H_\diamond^2(\Omega)) \cap W_\infty^2([0,T], H_0^1(\Omega)) \cap W_\infty^1([0,T], H_\diamond^3(\Omega)),$$

ensured by Proposition 3.1, implies

$$\psi \in C^{1,\gamma}([0,T], H_\diamond^2(\Omega)) \cap C^{1,1}([0,T], H_0^1(\Omega)) \cap C^{0,1}([0,T], H_\diamond^3(\Omega)).$$

Differentiability with respect to time follows by Rademacher's theorem, which states that any Lipschitz-continuous function is almost everywhere differentiable; more precisely, choosing the unique continuously differentiable representative, we have

$$\psi \in C^{1,\gamma}([0,T], H_\diamond^2(\Omega)) \cap C^2([0,T], H_0^1(\Omega)) \cap C^1([0,T], H_\diamond^3(\Omega)).$$

This also explains in which sense the initial conditions are satisfied.

(ii) Our result compares with Ref. 13, where under the stronger regularity requirements $\psi_0 \in H^4(\Omega)$, $\psi_1 \in H^3(\Omega)$, $\psi_2 \in H^2(\Omega)$ and additional compatibility conditions on the initial data existence and uniqueness of a solution

$$\begin{aligned}\psi &\in H^3((0,\infty), L_2(\Omega)) \cap W_\infty^2((0,\infty), H^1(\Omega)) \cap H^2((0,\infty), H^2(\Omega)) \\ &\quad \cap W_\infty^1((0,\infty), H^3(\Omega)) \cap H^1((0,\infty), H^4(\Omega)) \cap L_\infty((0,\infty), H^4(\Omega))\end{aligned}$$

to the general model is proven.

4. Limiting Systems

The transition from the Brunnhuber–Jordan–Kuznetsov equation to the Kuznetsov and Westervelt equations permits a significant reduction of the temporal order of differentiation from three to two, which is for instance of relevance with regard to numerical simulations. In this section, we rigorously justify this limiting process under a suitable compatibility condition on the initial data.

Situation. We consider the unifying representation (3.1) including (BJK), (BCK), (BJW), and (BCW), respectively; for the sake of clearness, we indicate the dependence of the solution on the decisive parameter $a > 0$. We suppose that the assumptions of Proposition 3.1 are satisfied; note that the prescribed initial data are independent of $a > 0$ and that the fundamental smallness requirement on $M(\overline{e}_0, \overline{e}_1)$ or $M(\overline{E}_0, \overline{E}_1)$, respectively, can be fulfilled uniformly for $a \in (0, \overline{a}]$. The main result of this work, given below, ensures convergence in a weak sense towards the solution of the Kuznetsov and Westervelt equation, respectively. In contrast to Proposition 3.1, the canonical solution space is now

$$X_0 = H^2([0,T], H^2_\diamond(\Omega)) \cap W^2_\infty([0,T], H^1_0(\Omega)),$$

that is, we employ the regularity properties

$$\int_0^T \left\|\Delta \partial_{tt} \psi^{(a)}(t)\right\|^2_{L_2} + \operatorname*{ess\,sup}_{t\in[0,T]} \left\|\nabla \partial_{tt} \psi^{(a)}(t)\right\|_{L_2} < \infty;$$

due to the fact that $\beta_2^{(a)}(\sigma_0) \to 0$ as $a \to 0_+$ and hence the terms

$$\beta_2^{(a)}(\sigma_0) \left\|\Delta \partial_t \psi(t)\right\|^2_{L_2}, \quad \beta_2^{(a)}(\sigma_0) \left\|\nabla \Delta \partial_t \psi(t)\right\|^2_{L_2}$$

arising in the energy estimates (3.30) vanish, the higher regularity of the solution space X cannot be achieved.

Theorem 4.1. *In the situation of Proposition* 3.1, *assume in addition that the prescribed initial data satisfy the consistency condition*

$$\psi_2 - \beta_1^{(0)} \Delta \psi_1 - \beta_3 \Delta \psi_0 + \beta_5(\sigma) \psi_2 \psi_1 + 2 \beta_6(\sigma) \nabla \psi_1 \cdot \nabla \psi_0 = 0. \tag{4.1}$$

For any $a \in (0, \overline{a}]$, *let* $\psi^{(a)} : [0,T] \to L_2(\Omega)$ *denote the solution to the nonlinear damped wave equation*

$$\begin{aligned}&\partial_{ttt} \psi^{(a)}(t) - \beta_1^{(a)} \Delta \partial_{tt} \psi^{(a)}(t) + \beta_2^{(a)}(\sigma_0) \Delta^2 \partial_t \psi^{(a)}(t) - \beta_3 \Delta \partial_t \psi^{(a)}(t)\\ &\quad + \beta_4^{(a)}(\sigma_0) \Delta^2 \psi^{(a)}(t) + \partial_{tt} \left(\frac{1}{2} \beta_5(\sigma) (\partial_t \psi^{(a)}(t))^2 + \beta_6(\sigma) \left|\nabla \psi^{(a)}(t)\right|^2\right) = 0\end{aligned}$$

under homogeneous Dirichlet boundary conditions and the initial conditions

$$\psi^{(a)}(0) = \psi_0, \quad \partial_t \psi^{(a)}(0) = \psi_1, \quad \partial_{tt} \psi^{(a)}(0) = \psi_2,$$

or of the following reformulation obtained by integration and application of (4.1)

$$\begin{aligned}&\partial_{tt} \psi^{(a)}(t) - \beta_1^{(0)} \Delta \partial_t \psi^{(a)}(t) - (\beta_1^{(a)} - \beta_1^{(0)})(\Delta \partial_t \psi^{(a)}(t) - \Delta \psi_1)\\ &\quad + \beta_2^{(a)}(\sigma_0)(\Delta^2 \psi^{(a)}(t) - \Delta^2 \psi_0) - \beta_3 \Delta \psi^{(a)}(t) + \beta_4^{(a)}(\sigma_0) \int_0^t \Delta^2 \psi^{(a)}(\tau) \mathrm{d}\tau\\ &\quad + \beta_5(\sigma) \partial_{tt} \psi^{(a)}(t) \partial_t \psi^{(a)}(t) + 2 \beta_6(\sigma) \nabla \partial_t \psi^{(a)}(t) \cdot \nabla \psi^{(a)}(t) = 0,\end{aligned}$$

respectively, see (3.1) *and* (3.5)*. Then, as* $a \to 0_+$*, the family* $(\psi^{(a)})_{a\in(0,\overline{a}]}$ *converges to the solution* $\psi^{(0)} : [0,T] \to L_2(\Omega)$ *of the limiting system*

$$\partial_{tt}\psi^{(0)}(t) - \beta_1^{(0)}\,\Delta\partial_t\psi^{(0)}(t) - \beta_3\,\Delta\psi^{(0)}(t) \\ + \beta_5(\sigma)\,\partial_{tt}\psi^{(0)}(t)\,\partial_t\psi^{(0)}(t) + 2\,\beta_6(\sigma)\,\nabla\partial_t\psi^{(0)}(t)\cdot\nabla\psi^{(0)}(t) = 0, \quad (4.2)$$

see (3.2)*; more precisely, for the solution to the associated weak formulation, obtained by testing with* $v \in L_1([0,T], H_0^1(\Omega))$ *and performing integration-by-parts, convergence is ensured in the following sense*:

$$\psi^{(a)} \overset{*}{\rightharpoonup} \psi^{(0)} \quad \textit{in } X_0 \quad \textit{as } a \to 0_+.$$

Proof.

(i) *Convergence.* In the present situation, as a consequence of Proposition 3.1, a sequence of positive numbers $(a_k)_{k\in\mathbb{N}}$ with limit zero exists such that the associated sequence $(\psi^{(a_k)})_{k\in\mathbb{N}}$ converges to a function $\psi^{(0)} \in X_0$ in the following sense:

$$\psi^{(a_k)} \overset{*}{\rightharpoonup} \psi^{(0)} \quad \text{in } X_0 \quad \text{as } k \to \infty,$$
$$\psi^{(a_k)} \to \psi^{(0)} \quad \text{in } \widetilde{X} = H^1([0,T], W_4^1(\Omega)) \quad \text{as } k \to \infty.$$

(ii) *Verification of solution property.* In order to verify that $\psi^{(0)}$ is a solution to (4.2), we make use of the fact that any function $\psi^{(a_k)}$ satisfies

$$\mathcal{L}^{(a_k)}\psi^{(a_k)} + \mathcal{L}_0^{(a_k)} + \mathcal{N}(\psi^{(a_k)}, \psi^{(a_k)}) + \mathcal{N}_0 = 0,$$

see (3.33), and prove that the difference

$$\mathcal{L}^{(a_k)}\psi^{(a_k)} - \mathcal{L}^{(0)}\psi^{(0)} + \mathcal{N}(\psi^{(a_k)}, \psi^{(a_k)}) - \mathcal{N}(\psi^{(0)}, \psi^{(0)}) \\ = (\mathcal{L}^{(a_k)} - \mathcal{L}^{(0)})\psi^{(a_k)} + \mathcal{L}^{(0)}(\psi^{(a_k)} - \psi^{(0)}) \\ + \mathcal{N}(\psi^{(a_k)} - \psi^{(0)}, \psi^{(a_k)}) + \mathcal{N}(\psi^{(0)}, \psi^{(a_k)} - \psi^{(0)})$$

tends to zero in a weak sense. On the one hand, testing the reformulation of the general model with $v \in L_1([0,T], H_0^1(\Omega))$ and employing integration-by-parts, yields

$$\int_0^T \left((\mathcal{L}^{(a_k)} - \mathcal{L}^{(0)})\psi^{(a_k)}(t) \,\middle|\, v(t) \right)_{L_2} \mathrm{d}t \\ = \int_0^T \Big((\beta_1^{(a_k)} - \beta_1^{(0)})(\nabla\partial_t\psi^{(a_k)}(t) \,|\, \nabla v(t))_{L_2} - \beta_2^{(a_k)}(\sigma_0) \\ \times (\nabla\Delta\psi^{(a_k)}(t) \,|\, \nabla v(t))_{L_2} \Big)\mathrm{d}t \\ - \frac{\beta_2^{(a_k)}(\sigma_0)}{\beta_0^{(a_k)}(\sigma_0)} \int_0^T \int_0^t (\nabla\Delta\psi^{(a_k)}(\tau) \,|\, \nabla v(t))_{L_2}\,\mathrm{d}\tau\,\mathrm{d}t,$$

which tends to zero, since

$$\left\|\nabla\partial_t\psi^{(a_k)}\right\|_{L_\infty([0,T],L_2(\Omega))}, \quad \sqrt{\beta_2^{(a_k)}(\sigma_0)}\left\|\nabla\Delta\psi^{(a_k)}\right\|_{L_\infty([0,T],L_2(\Omega))},$$

are uniformly bounded for $a_k \in (0,\overline{a}]$. On the other hand, it is seen that

$$\begin{aligned}
&\int_0^T \left(\mathcal{L}^{(0)}(\psi^{(a_k)}(t)-\psi^{(0)}(t))\,|\,v(t)\right)_{L_2}\mathrm{d}t\\
&\quad=\int_0^T \Big(\left(\partial_{tt}(\psi^{(a_k)}(t)-\psi^{(0)}(t))\,|\,v(t)\right)_{L_2}\\
&\qquad-\beta_1^{(0)}\left(\Delta\partial_t(\psi^{(a_k)}(t)-\psi^{(0)}(t))\,|\,v(t)\right)_{L_2}\\
&\qquad-\beta_3\left(\Delta(\psi^{(a_k)}(t)-\psi^{(0)}(t))\,|\,v(t)\right)_{L_2}\Big)\mathrm{d}t
\end{aligned}$$

tends to zero by the weak convergence in X_0. For the nonlinear part, the same argument as given in the proof of Proposition 3.1 applies. We finally note that convergence of the family $(\psi^{(a)})_{a\in(0,\overline{a}]}$ follows from a subsequence–subsequence argument and uniqueness of the solutions to the Kuznetsov and Westervelt equations. Altogether, we thus obtain

$$\begin{aligned}
&\int_0^T \Big((\partial_{tt}\psi^{(a)}(t)\,|\,v(t))_{L_2}+\beta_1^{(0)}\,(\nabla\partial_t\psi^{(a)}(t)\,|\,\nabla v(t))_{L_2}\\
&\quad+\beta_3\,(\nabla\psi^{(a)}(t)\,|\,\nabla v(t))_{L_2}+(\beta_1^{(a)}-\beta_1^{(0)})(\nabla\partial_t\psi^{(a)}(t)-\nabla\psi_1\,|\,\nabla v(t))_{L_2}\\
&\quad-\beta_2^{(a)}(\sigma_0)\,(\nabla\Delta\psi^{(a)}(t)-\nabla\Delta\psi_0\,|\,\nabla v(t))_{L_2}\\
&\quad-\beta_4^{(a)}(\sigma_0)\int_0^T\!\!\int_0^t(\nabla\Delta\psi^{(a)}(\tau)\,|\,\nabla v(t))_{L_2}\,\mathrm{d}\tau\\
&\quad+\beta_5(\sigma)\,(\partial_{tt}\psi^{(a)}(t)\,\partial_t\psi^{(a)}(t)\,|\,v(t))_{L_2}\\
&\quad+2\,\beta_6(\sigma)\,(\nabla\partial_t\psi^{(a)}(t)\cdot\nabla\psi^{(a)}(t)\,|\,v(t))_{L_2}\Big)\mathrm{d}t\\
&\quad\overset{a\to 0+}{\longrightarrow}\int_0^T\Big((\partial_{tt}\psi^{(0)}(t)\,|\,v(t))_{L_2}+\beta_1^{(0)}\,(\nabla\partial_t\psi^{(0)}(t)\,|\,\nabla v(t))_{L_2}\\
&\quad+\beta_3\,(\nabla\psi^{(0)}(t)\,|\,\nabla v(t))_{L_2}+\beta_5(\sigma)\,(\partial_{tt}\psi^{(0)}(t)\,\partial_t\psi^{(0)}(t)\,|\,v(t))_{L_2}\\
&\quad+2\,\beta_6(\sigma)\,(\nabla\partial_t\psi^{(0)}(t)\cdot\nabla\psi^{(0)}(t)\,|\,v(t))_{L_2}\Big)\mathrm{d}t,
\end{aligned}$$

which concludes the proof. □

Remark 4.1. Under stronger regularity and compatibility requirements on the initial data, the solution space considered in Ref. 16 for the Kuznetsov equation is

$$X_0 \cap W^3_\infty([0,T], L_2(\Omega)) \cap H^3([0,T], H^1_0(\Omega)) \cap W^2_\infty([0,T], H^1_0(\Omega)) \\ \cap W^1_\infty([0,T], H^2(\Omega)),$$

see Theorem 1.1 in Ref. 16 with $u = \partial_t \psi$; in this situation, also uniqueness of a solution in X_0 is proven in Ref. 16. Similar statements hold for the Westervelt equation, see Ref. 15. For more general results on the Westervelt and Kuznetsov equations, considered as equations in L_p-spaces, we refer to Refs. 22 and 23.

Appendix A. Detailed Derivation of Most General Model

In the following, we deduce the Blackstock–Crighton–Brunnhuber–Jordan–Kuznetsov equation (2.4) from the conservation laws for mass, momentum, and energy as well as a heuristic equation of state relating mass density, acoustic pressure, and temperature, see (2.1) and (2.2). For notational simplicity, we include detailed calculations for the one-dimensional case; the extension to higher space dimensions is then straightforward. In order to indicate that only terms which are linear or quadratic with respect to the fluctuating quantities are taken into account, we introduce a (small) positive real number $\varepsilon > 0$ and set

$$\varrho = \varrho_0 + \varepsilon\, \varrho_\sim, \quad v = \varepsilon\, \partial_x \psi, \quad p = p_0 + \varepsilon\, p_\sim, \quad T = T_0 + \varepsilon\, T_\sim; \tag{A.1}$$

here, we anticipate that inserting the Helmholtz composition (2.3) into the fundamental relations permits a decoupling into irrotational and rotational parts. Moreover, to identify terms that are related to dissipative effects, we replace μ_B, μ, c_V, c_p as well as $\hat{A}$ and a by

$$\delta\mu_B, \quad \delta\mu, \quad \delta c_V, \quad \delta c_p, \quad \gamma\hat{A}, \quad \lambda\, a,$$

where $\delta, \gamma, \lambda > 0$ denote (small) positive real numbers that will be adjusted later on.

Fundamental relations. For convenience, we restate the fundamental equations (2.1) and (2.2) employing (A.1). In a single space dimension, the relation reflecting conservation of mass (2.1a) reads

$$\varepsilon\, \partial_t \varrho_\sim + \varepsilon\, \varrho_0\, \partial_{xx}\psi + \varepsilon^2\, \partial_x \varrho_\sim\, \partial_x \psi + \varepsilon^2\, \varrho_\sim\, \partial_{xx}\psi = 0. \tag{A.2a}$$

Omitting higher-order contributions, i.e. terms of the form $o(\varepsilon^2)$, the relation describing conservation of momentum (2.1b) simplifies as follows:

$$\varepsilon\, \varrho_0\, \partial_{xt}\psi + \varepsilon\, \partial_x p_\sim - \varepsilon\, \delta \left(\mu_B + \frac{4}{3}\,\mu\right) \partial_{xxx}\psi \\ + \varepsilon^2\, \partial_t \varrho_\sim\, \partial_x \psi + \varepsilon^2\, \varrho_\sim\, \partial_{xt}\psi + 2\,\varepsilon^2\, \varrho_0\, \partial_x\psi\, \partial_{xx}\psi = 0.$$

Substracting the $\varepsilon\,\partial_x\psi$ multiple of (A.2a), leads to

$$\partial_x\left(\varepsilon\,\varrho_0\,\partial_t\psi+\varepsilon\,p_\sim-\varepsilon\,\delta\left(\mu_B+\frac{4}{3}\,\mu\right)\partial_{xx}\psi+\varepsilon^2\,\frac{\varrho_0}{2}(\partial_x\psi)^2\right)+\varepsilon^2\,\varrho_\sim\,\partial_{xt}\psi=0. \tag{A.2b}$$

Neglecting contributions of the form $o(\varepsilon^2)$, we obtain the following relation reflecting the conservation of energy (2.1c) in a single space dimension:

$$\varepsilon\,\delta\,\frac{c_p-c_V}{\alpha_V}\,\varrho_0\,\partial_{xx}\psi-\varepsilon\,\lambda\,a\,\partial_{xx}T_\sim+\varepsilon\,\delta\,c_V\,\varrho_0\,\partial_t T_\sim=0. \tag{A.2c}$$

Omitting higher-order contributions, the equation of state (2.2) reduces to

$$\varepsilon\,p_\sim=\varepsilon\,\frac{A}{\varrho_0}\,\varrho_\sim+\varepsilon^2\,\frac{B}{2\varrho_0^2}\,\varrho_\sim^2+\varepsilon\,\gamma\,\frac{\hat{A}}{T_0}\,T_\sim. \tag{A.2d}$$

Linear wave equation. Reconsidering Eqs. (A.2a)–(A.2d) and incorporating only first-order contributions, i.e. terms of the form $\mathcal{O}(\varepsilon)$, yields

$$\partial_t\varrho_\sim+\varrho_0\,\partial_{xx}\psi=0,\quad \partial_x(\varrho_0\,\partial_t\psi+p_\sim)=0,\quad p_\sim=\frac{A}{\varrho_0}\,\varrho_\sim.$$

Integration with respect to the space variable shows that a solution of the system

$$\partial_t\varrho_\sim=-\varrho_0\,\partial_{xx}\psi,\quad p_\sim=\frac{A}{\varrho_0}\,\varrho_\sim=-\varrho_0\,\partial_t\psi,$$

is also a solution of the original system. The relation for the acoustic pressure implies

$$\varrho_\sim=-\frac{\varrho_0^2}{A}\,\partial_t\psi;$$

together with the identity $A=c_0^2\,\varrho_0$, this leads to a linear wave equation for the acoustic velocity potential

$$\partial_{tt}\psi-c_0^2\,\partial_{xx}\psi=0.$$

Nonlinear damped wave equation. The above considerations explain the ansatz

$$\varrho_\sim=-\frac{\varrho_0^2}{A}\,\partial_t\psi+\varepsilon\,\varrho_0\,F$$

with space-time-dependent real-valued function F determined by (A.2a). Inserting this representation into (A.2a)–(A.2d), neglecting higher-order contributions, employing the identity

$$\partial_{xt}\psi\,\partial_t\psi=\frac{1}{2}\,\partial_x(\partial_t\psi)^2,$$

and integrating (A.2b) with respect to space, we arrive at

$$\varepsilon^2\,\partial_t F=\varepsilon\,\frac{\varrho_0}{A}\,\partial_{tt}\psi-\varepsilon\,\partial_{xx}\psi+\varepsilon^2\,\frac{\varrho_0}{A}\left(\partial_{xt}\psi\,\partial_x\psi+\partial_{xx}\psi\,\partial_t\psi\right), \tag{A.3a}$$

$$\varepsilon\,\varrho_0\,\partial_t\psi + \varepsilon\,p_\sim - \varepsilon\,\delta\left(\mu_B + \frac{4}{3}\,\mu\right)\partial_{xx}\psi + \varepsilon^2\,\frac{\varrho_0}{2}(\partial_x\psi)^2 - \varepsilon^2\,\frac{\varrho_0^2}{2A}(\partial_t\psi)^2 = 0, \tag{A.3b}$$

$$\varepsilon\,\delta\,\frac{c_p - c_V}{\alpha_V}\,\varrho_0\,\partial_{xx}\psi - \varepsilon\,\lambda\,a\,\partial_{xx}T_\sim + \varepsilon\,\delta\,c_V\,\varrho_0\,\partial_t T_\sim = 0, \tag{A.3c}$$

$$\varepsilon\,\varrho_0\,\partial_t\psi + \varepsilon\,p_\sim = \varepsilon^2\,A\,F + \varepsilon^2\,\frac{B}{A}\,\frac{\varrho_0^2}{2A}(\partial_t\psi)^2 + \varepsilon\,\gamma\,\frac{\hat{A}}{T_0}\,T_\sim. \tag{A.3d}$$

On the one hand, we insert (A.3d) into (A.3b), differentiate the resulting equation with respect to time, and insert (A.3a) to obtain

$$\begin{aligned}
&\varepsilon\,\varrho_0\,\partial_{tt}\psi - \varepsilon\,A\,\partial_{xx}\psi - \varepsilon\,\delta\left(\mu_B + \frac{4}{3}\,\mu\right)\partial_{xxt}\psi + \varepsilon\,\gamma\,\frac{\hat{A}}{T_0}\,\partial_t T_\sim\\
&\quad + \varepsilon^2\left(\frac{B}{A} - 1\right)\frac{\varrho_0^2}{A}\,\partial_{tt}\psi\,\partial_t\psi + 2\,\varepsilon^2\,\varrho_0\,\partial_{xt}\psi\,\partial_x\psi + \varepsilon^2\,\varrho_0\,\partial_{xx}\psi\,\partial_t\psi = 0;
\end{aligned}$$

replacing the second-order contribution $\varepsilon^2\,\varrho_0\,\partial_{xx}\psi\,\partial_t\psi$ with $\varepsilon^2\,\frac{\varrho_0^2}{A}\,\partial_{tt}\psi\,\partial_t\psi + o(\varepsilon^2)$, see (A.3a), further yields

$$\begin{aligned}
\varepsilon\,\gamma\,\partial_t T_\sim &= -\varepsilon\,\frac{\varrho_0 T_0}{\hat{A}}\,\partial_{tt}\psi + \varepsilon\,\frac{A T_0}{\hat{A}}\,\partial_{xx}\psi + \varepsilon\,\delta\left(\mu_B + \frac{4}{3}\,\mu\right)\frac{T_0}{\hat{A}}\,\partial_{xxt}\psi\\
&\quad - \varepsilon^2\,\frac{B}{A}\,\frac{\varrho_0 T_0}{\hat{A}}\,\frac{\varrho_0}{2A}\,\partial_t(\partial_t\psi)^2 - \varepsilon^2\,\frac{\varrho_0 T_0}{\hat{A}}\,\partial_t(\partial_x\psi)^2.
\end{aligned} \tag{A.4}$$

On the other hand, differentiating (A.3c) with respect to time, we have

$$\varepsilon\,\delta\,\frac{c_p - c_V}{\alpha_V}\,\varrho_0\,\partial_{xxt}\psi - a\,\partial_{xx}(\varepsilon\,\lambda\,\partial_t T_\sim) + c_V\,\varrho_0\,\partial_t(\varepsilon\,\delta\,\partial_t T_\sim) = 0;$$

with the help of (A.4), this yields

$$\begin{aligned}
&\varepsilon\,\partial_{ttt}\psi - \left(\frac{\varepsilon\lambda}{\delta}\,\frac{a}{c_V\,\varrho_0} + \varepsilon\,\delta\,\frac{1}{\varrho_0}\left(\mu_B + \frac{4}{3}\,\mu\right)\right)\partial_{xxtt}\psi\\
&\quad + \varepsilon\,\lambda\,\frac{a}{c_V\,\varrho_0}\,\frac{1}{\varrho_0}\left(\mu_B + \frac{4}{3}\,\mu\right)\partial_{xxxxt}\psi\\
&\quad - \left(\varepsilon\,\frac{A}{\varrho_0} + \varepsilon\,\gamma\,\frac{c_p - c_V}{\alpha_V}\,\frac{\hat{A}}{c_V\,\varrho_0 T_0}\right)\partial_{xxt}\psi + \frac{\varepsilon\lambda}{\delta}\,\frac{a}{c_V\,\varrho_0}\,\frac{A}{\varrho_0}\,\partial_{xxxx}\psi\\
&\quad + \varepsilon^2\,\frac{B}{A}\,\frac{\varrho_0}{2A}\,\partial_{tt}(\partial_t\psi)^2 + \varepsilon^2\,\partial_{tt}(\partial_x\psi)^2 - \frac{\varepsilon^2\lambda}{\delta}\,\frac{B}{A}\,\frac{a}{2Ac_V}\,\partial_{xxt}(\partial_t\psi)^2\\
&\quad - \frac{\varepsilon^2\lambda}{\delta}\,\frac{a}{c_V\,\varrho_0}\,\partial_{xxt}(\partial_x\psi)^2 = 0.
\end{aligned}$$

With the special scaling

$$\delta = \sqrt{\varepsilon}, \quad \gamma = \sqrt{\varepsilon}\,\varepsilon, \quad \lambda = \varepsilon,$$

we arrive at the relation

$$\begin{aligned}
&\varepsilon\,\partial_{ttt}\psi - \varepsilon\,\sqrt{\varepsilon}\left(\frac{a}{c_V\,\varrho_0} + \frac{1}{\varrho_0}\left(\mu_B + \frac{4}{3}\,\mu\right)\right)\partial_{xxtt}\psi \\
&\quad + \varepsilon^2\,\frac{a}{c_V\,\varrho_0}\,\frac{1}{\varrho_0}\left(\mu_B + \frac{4}{3}\,\mu\right)\partial_{xxxxt}\psi \\
&\quad - \left(\varepsilon\,\frac{A}{\varrho_0} + \varepsilon^2\,\sqrt{\varepsilon}\,\frac{c_p - c_V}{\alpha_V}\,\frac{\hat{A}}{c_V\,\varrho_0 T_0}\right)\partial_{xxt}\psi + \varepsilon\sqrt{\varepsilon}\,\frac{a}{c_V\,\varrho_0}\,\frac{A}{\varrho_0}\,\partial_{xxxx}\psi \\
&\quad + \varepsilon^2\,\frac{B}{A}\,\frac{\varrho_0}{2A}\,\partial_{tt}(\partial_t\psi)^2 + \varepsilon^2\,\partial_{tt}(\partial_x\psi)^2 - \varepsilon^2\sqrt{\varepsilon}\,\frac{B}{A}\,\frac{a}{2Ac_V}\,\partial_{xxt}(\partial_t\psi)^2 \\
&\quad - \varepsilon^2\sqrt{\varepsilon}\,\frac{a}{c_V\,\varrho_0}\,\partial_{xxt}(\partial_x\psi)^2 = 0;
\end{aligned}$$

neglecting the higher-order terms

$$\varepsilon^2\,\sqrt{\varepsilon}\,\frac{c_p - c_V}{\alpha_V}\,\frac{\hat{A}}{c_V\,\varrho_0 T_0}\,\partial_{xxt}\psi, \quad \varepsilon^2\sqrt{\varepsilon}\,\frac{B}{A}\,\frac{a}{2Ac_V}\,\partial_{xxt}(\partial_t\psi)^2, \quad \varepsilon^2\sqrt{\varepsilon}\,\frac{a}{c_V\,\varrho_0}\,\partial_{xxt}(\partial_x\psi)^2,$$

omitting then $\varepsilon > 0$ and employing the relations

$$\frac{1}{\varrho_0}\left(\mu_B + \frac{4}{3}\,\mu\right) = \nu\Lambda, \quad A = c_0^2\,\varrho_0, \quad \frac{a}{c_V\,\varrho_0} = a\left(1 + \frac{B}{A}\right),$$

see Table 1, finally leads to the nonlinear damped wave equation

$$\begin{aligned}
&\partial_{ttt}\psi - \left(a\left(1 + \frac{B}{A}\right) + \nu\Lambda\right)\partial_{xxtt}\psi + a\left(1 + \frac{B}{A}\right)\nu\Lambda\,\partial_{xxxxt}\psi - c_0^2\,\partial_{xxt}\psi \\
&\quad + a\left(1 + \frac{B}{A}\right)c_0^2\,\partial_{xxxx}\psi + \partial_{tt}\left(\frac{1}{2c_0^2}\,\frac{B}{A}(\partial_t\psi)^2 + (\partial_x\psi)^2\right) = 0,
\end{aligned}$$

see also (2.4); it is remarkable that the differential operator defining the linear contributions factorises as follows:

$$\begin{aligned}
&\left(\partial_t - a\left(1 + \frac{B}{A}\right)\partial_{xx}\right)(\partial_{tt} - \nu\Lambda\,\partial_{xxt} - c_0^2\,\partial_{xx})\psi \\
&\quad + \partial_{tt}\left(\frac{1}{2c_0^2}\,\frac{B}{A}(\partial_t\psi)^2 + (\partial_x\psi)^2\right) = 0.
\end{aligned}$$

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