THEORETICAL STUDY AND NUMERICAL SIMULATION OF PATTERN FORMATION IN THE DETERMINISTIC AND STOCHASTIC GRAY–SCOTT EQUATIONS

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ABSTRACT. Mathematical models based on systems of reaction-diffusion equations provide fundamental tools for the description and investigation of various processes in biology, biochemistry, and chemistry; in specific situations, an appealing characteristic of the arising nonlinear partial differential equations is the formation of patterns, reminiscent of those found in nature. The deterministic Gray–Scott equations constitute an elementary two-component system that describes autocatalytic reaction processes; depending on the choice of the specific parameters, complex patterns of spirals, waves, stripes, or spots appear.

In the derivation of a macroscopic model such as the deterministic Gray–Scott equations from basic physical principles, certain aspects of microscopic dynamics, e.g. fluctuations of molecules, are disregarded; an expedient mathematical approach that accounts for significant microscopic effects relies on the incorporation of stochastic processes and the consideration of stochastic partial differential equations.

The present work is concerned with a theoretical and numerical study of the stochastic Gray–Scott equations driven by independent spatially time-homogeneous Wiener processes. Under suitable regularity assumptions on the prescribed initial states, existence and uniqueness of the solution processes is proven. Numerical simulations based on the application of a time-adaptive first-order operator splitting method and the fast Fourier transform illustrate the formation of patterns in the deterministic case and their variation under the influence of stochastic noise.

Keywords: Mathematical and theoretical biology, Mathematical biochemistry, Mathematical chemistry, Reaction-diffusion systems, Gray–Scott equations, Turing patterns, Stochastic partial differential equations, Wiener processes, Numerical approximation, Operator splitting methods, Fast Fourier transform.

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1. INTRODUCTION

This work is concerned with the theoretical study and numerical simulation of the stochastic Gray–Scott equations, which constitute a two-component system of reaction-diffusion equations driven by stochastic processes; in spite of its comparatively simple structure, the underlying system of deterministic nonlinear partial differential equations exhibits a fascinating variety of complex patterns for different choices of the decisive parameters.

Deterministic models and pattern formation Processes in biochemical and chemical kinetics have been a rich source for the observation of beautiful spatial-temporal patterns; the derivation and investigation of suitable mathematical models for such phenomena remains a challenging question.

A famous example of non-equilibrium thermodynamics is the Belousov–Zhabotinsky reaction, discovered by BORIS BELOUSOV in the beginning of the 1950s; he succeeded in stimulating a reaction of chemical substances that led to periodic changes of their concentrations, visible as oscillations in colour.

An elementary mathematical model for this kind of nonlinear chemical oscillators is the Brusselator, a system of reaction-diffusion equations proposed by PRIGOGINE, LEFEVER [28, Eq. (3.6)]; in a dimensionless

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formulation, the considered system of nonlinear partial differential equations has the structure

$$\begin{cases} \partial_t u(x,t) = r_u\, \Delta u(x,t) + h_u\big(u(x,t), v(x,t)\big)\,, \\ \partial_t v(x,t) = r_v\, \Delta v(x,t) + h_v\big(u(x,t), v(x,t)\big)\,, \end{cases} \tag{1.1a}$$

where the real-valued space-time-dependent functions $u, v : I \times [0,T] \subset \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ are associated with the concentrations of the chemical substances, Δ represents the Laplacian with respect to the spatial variables, the constants $r_u, r_v > 0$ denote the diffusion coefficients, and the nonlinear functions $h_u, h_v : \mathbb{R}^2 \to \mathbb{R}$ describe the reactions.

Alan Turing suggested that the main mechanisms of morphogenesis are captured by mathematical models for systems of chemical substances, which react together and diffuse through tissue. In a seminal work [46], he studies reaction-diffusion equations that have a similar form as (1.1a) on different geometries, amongst others on spheres and rings, and explains the development of patterns from almost uniform initial states by instabilities of homogeneous equilibria; we refer to such patterns as Turing patterns.

Deterministic Gray–Scott equations In the present work, we focus on a classical mathematical model for isothermal autocatalytic reaction processes that goes back to Gray, Scott [18, 19, 20, 21]; depending on the choice of the feed and removal rates of the reactants, Turing patterns of spirals, waves, stripes, or spots appear. The deterministic Gray–Scott equations are cast into the form (1.1a) with cubic reaction terms

$$\begin{gathered} h_u(u,v) = \alpha_u\,(1-u) - g(u,v)\,, \quad h_v(u,v) = -\,\alpha_v\, v + g(u,v)\,, \\ g(u,v) = u\, v^2\,, \end{gathered} \tag{1.1b}$$

involving certain constants $\alpha_u, \alpha_v > 0$.

Related systems of reaction-diffusion equations are also studied in other contexts. Kierstead, Slobodkin [25] describe the survival of phytoplankton populations in bodies of water. Segel, Jackson [40] consider predator-prey interaction models with diffusion; based on a linear stability analysis, they demonstrate that spatially uniform equilibria which are stable for homogeneously distributed populations become unstable through dispersal effects. Levin, Segel [29] study the dynamics of plankton populations. Klausmeier [26] discusses a model for semi–arid ecosystems on slope terrains. Murray [33, 34] describes coat patterns in animal tails; numerical simulations on surfaces with periodic and homogeneous Neumann boundary conditions, respectively, show patterns of stripes and spots that are similar to the markings observed on the tails of felines.

Stochastic models and pattern formation Reaction-diffusion systems like (1.1a) constitute prevalent macroscopic models for microscopic phenomena; however, as their derivation relies on fundamental balance laws and Fick's law of diffusion, significant aspects of microscopic dynamics such as fluctuations of molecules are disregarded. An appropriate mathematical approach to establish more realistic models is the incorporation of stochastic processes.

Biancalani et al. [9] introduce a microscopic model of the Brusselator that includes stochastic fluctuations. Compartment-based approaches use a division of the domain into certain compartments and a simulation of the number of molecules in each compartment; Cao, Erban [10] investigate the dependence of stochastic Turing patterns on the compartment size. In McKane et al. [32], it is shown how a stochastic amplification of a Turing instability gives rise to spatial-temporal patterns. Treatments of the stochastic Brusselator in different respects are found in [2, 39, 45].

Stochastic Gray–Scott equations In the present work, our main objective is to contribute to a rigorous analysis of the Gray–Scott equations (1.1) driven by independent spatially time-homogeneous Wiener processes; the core part of this manuscript is dedicated to the derivation of an existence and uniqueness result.

The theory of stochastic partial differential equations provides the basis of our investigations; for a comprehensive treatment of the fundamentals as well as an extensive bibliography, we refer to the monographs [12, 14, 30].

Outline. This manuscript has the following structure. In Section 2, we introduce the stochastic Gray–Scott equations as well as the needed hypotheses on the driving Wiener processes and the initial states; the restriction to bounded space domains of a special form permits to use Fourier series expansions. Moreover, we deduce our main result ensuring existence, uniqueness, and positivity of the solution processes. In Section 3, to complement our theoretical analysis, we present numerical simulations for the Gray–Scott equations in two space dimensions that illustrate the formation of patterns in the deterministic case and their variation under the influence of stochastic noise. The employed numerical approximation is based on a first-order operator splitting method and the fast Fourier transform; in order to enhance reliability of the computations, we adapt the time stepsizes accordingly to the sizes of the nonlinear terms for particular realisations. Auxiliary considerations are collected in the appendix.

2. Stochastic Gray–Scott equations

In this section, we state the mathematical formulation of the stochastic Gray–Scott equations, introduce the underlying spaces, review basic auxiliary results on spatially time-homogeneous Wiener processes as well as associated stochastic integrals, and specify the hypotheses under which a solution exists. We begin with the mathematical formulation of the stochastic Gray–Scott equations. In the system, u and v are concentrations of two reactants U and V, normalized as dimensionless units. The parameters f and k represent the feed rate and removal rate of the reactants. We recall that the parameters $r_u, r_v > 0$ correspond to the diffusion coefficients. These parameters have a significant effect at the form of the observed patterns. The equation is given as follows

$$
\left\{
\begin{aligned}
du(x,t) &= \big(r_u \Delta u(x,t) - u(x,t)v^2(x,t) + f(1-u(x,t))\big)\, dt \\
&\qquad + \sigma_u u(x,t) \circ dW_1(x,t), \quad x \in I, t > 0, \\
dv(x,t) &= \big(r_v \Delta v(x,t) + u(x,t)v^2(x,t) - (f+k)v(x,t)\big)\, dt \\
&\qquad + \sigma_v v(x,t) \circ dW_2(x,t), \quad x \in I, t > 0,
\end{aligned}
\right.
\tag{2.1}
$$

where $I = [0,1]^d$ be a bounded domain, $d = 1, 2$, $A = \Delta$ be the Laplace operator with periodic, or Dirichlet boundary conditions. The initial conditions are given by u_0 and v_0. Since the white noise is an approximation of a continuously fluctuating noise with finite memory being much shorter than the dynamical timescales, the representation of the stochastic integral as a Stratonovich stochastic integral is appropriate.

Starting with some initial condition, the Gray Scott system will generate some patterns depending on the coefficients. For convenience, we suppose that the constants that determine the strength of the multiplicative stochastic noise are positive, i.e. $\sigma_u, \sigma_v \geq 0$; evidently, the deterministic Gray–Scott equations (1.1) are retained from (2.1) for the special case $(\sigma_u, \sigma_v) = (0, 0)$.

In this work, we focus on situations where the Gray–Scott equations (2.1) are driven by independent spatially time-homogeneous Wiener processes; as relevant concrete examples, we study the Gray Scott system driven by fractional Gaussian field.

Let

$$
\mathfrak{A} = \Big(\Omega, \mathcal{A}, \big(\mathcal{A}(t)\big)_{t\in[0,T]}, \mathbb{P}\Big)
$$

be a complete probability space with associated filtration satisfying the standard assumptions; for our purposes, it suffices to consider a finite time interval. Let $\{\beta_k : k \in \mathbb{Z}^d\}$ be a family of one–dimensional standard Brownian motions defined over $\mathfrak{A}$. Here, we consider our equation on the d dimensional torus. In the case of a single dimension, a complete orthonormal system of the underlying Lebesgue space $L^2(I) := L^2(I, \mathbb{R})$ is given by sine and cosine functions

$$
\psi_m(x) = \begin{cases} \sqrt{2}\, \sin\big(2\pi m x\big) & \text{if } m \geq 1\,, \\ \sqrt{2} & \text{if } m = 0\,, \\ \sqrt{2}\, \cos\big(2\pi m x\big) & \text{if } m \leq -1\,, \end{cases}
\tag{2.2}
$$

The extension to higher space dimensions relies on tensor products, i.e., for a multiindex $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$ we have

$$
\psi_m(x) = \prod_{j=1}^{d} \psi_{m_j}(x_j)\,, \qquad x \in I.
\tag{2.3}
$$

The corresponding eigenvalues are given by

$$\lambda_m = -\,4\pi^2 \sum_{j=1}^{d} m_j^2\,, \quad m=(m_1,\dots,m_d)\in\mathbb{Z}^d\,. \tag{2.4}$$

The spatially time–homogenous Wiener process can be expressed in terms of the orthogonal system, i.e.,

$$W(x,t) = \sum_{k\in\mathbb{Z}^d} \delta_k \psi_k(x)\beta_k(t).$$

where the family $\{\beta_k : k\in\mathbb{Z}^d\}$ is a family of independent and identically distributed standard Brownian motions. For simplicity, we assume in our work that $\delta_k=(1-\lambda_k)^{-\gamma}$, $k\in\mathbb{Z}^d$.

Going back to our equation (2.1), we impose the following hypothesis.

Hypothesis 2.1. *The Wiener processes W_1 and W_2 are spatially time–homogenous Wiener processes such that*

$$W_j(x,t) = \sum_{k\in\mathbb{Z}^d} (1-\lambda_k)^{-\gamma_j}\psi_k(x)\beta_k(t), \quad j=1,2,$$

with $\gamma_j > \frac{d}{2}$.

In our case, Hypothesis 2.1 means that the sum defined by

$$S(\gamma_j) := \sum_{k\in\mathbb{Z}^d} (1-\lambda_k)^{-\gamma_j} \tag{2.5}$$

is bounded. For simplicity we assume that $\gamma_1=\gamma_2=\gamma$. Since the solutions u and v of the Gray Scott system have to be non-negative, the initial conditions u_0 and v_0 have to be non-negative. Besides, we have to impose some regularity assumptions on the initial condition to get existence and uniqueness of the solution.

Hypothesis 2.2. *Let $u_0, v_0\in L^2(I)$ such that*

(1) $u_0\ge 0$ and $v_0\ge 0$;
(2) u_0 and v_0 belong to $L^6(I)$, in particular we have $\mathbb{E}|u_0|^6_{L^6}<\infty$ and $\mathbb{E}|v_0|^6_{L^6}<\infty$.
(3) u_0 and v_0 belong to $H_4^{\frac{d}{2}}(I)$, in particular $\mathbb{E}|u_0|^4_{H_4^{\frac{d}{2}}}<\infty$ and $\mathbb{E}|v_0|^4_{H_4^{\frac{d}{2}}}<\infty$.

In system (2.1) we interpreted the stochastic integral as a Stratonovich integral. White noise is an idealisation; real fluctuating forcing has a finite amplitude and a finite timescale. If the white noise is approximated by a continuously fluctuating noise with finite memory (much shorter than dynamical timescales), i.e., by noise with a finite correlation time τ, and then the limit is taken for $\tau\to 0$, the Wong-Zakai Theorem in [48] gives as the appropriate representation of the white noise the Stratonovich integral. In this sense, the Startonovich integral models the natural one, the drawback is that the Stratonovich integral is not a martingale, and, therefore, the Itô isometry and Burkholder–Davis–Gundy inequality cannot be applied to the Stratonovich integral. Although here in the article we analyse a more general system, where the integral is interpreted as an Itô integral. To show that the system (2.1) has a unique solution, we first transform the system (2.1) into a system, where the integral can be interpreted in the Itô sense by adding a correction term, and, then, we show that the correction term behaves nicely. One can find a survey of some facts about the Stratonovich integral in Chapter 4.5.2 in [15]. In this way, it can be shown that the solution to (2.1) and the solution to

$$\begin{cases} du(x,t) &=& \big(r_u\Delta u(x,t) - u(x,t)v^2(x,t) + f - (f+\delta_u)u(x,t)\big)\,dt \\ && \quad + \sigma_u u(x,t)dW_1(x,t), \quad x\in I, t>0, \\ dv(x,t) &=& \big(r_v\Delta v(x,t) + u(x,t)v^2(x,t) - (f+k-\delta_v)v(x,t)\big)\,dt \\ && \quad + \sigma_v v(x,t)dW_2(x,t), \quad x\in I, t>0, \end{cases}$$

with

$$\delta_u = \sum_{m\in\mathbb{Z}^d} (1-\lambda_m)^{-2\gamma_1}, \quad \delta_v = \sum_{m\in\mathbb{Z}^d} (1-\lambda_m)^{-2\gamma_2},$$

are equivalent. For simplicity we will combine the coefficient and consider the following system

$$(2.6) \quad \begin{cases} du(x,t) &= \big(r_u\Delta u(x,t) - u(x,t)v^2(x,t) + \rho + \alpha_u u(x,t))\big)\, dt \\ & \qquad + \sigma_u u(x,t)dW_1(x,t), \quad x\in I, t>0, \\ dv(x,t) &= \big(r_v\Delta v(x,t) + u(x,t)v^2(x,t) + \alpha_v v(x,t)\big)\, dt \\ & \qquad + \sigma_v v(x,t)dW_2(x,t), \quad x\in I, t>0 \end{cases}$$

where ρ, α_u and α_v are real–valued number, not necessarily positive and the stochastic integral is interpreted in the sense of Itô. For this system we can show the following Theorem.

Theorem 2.1. *Let us assume that u_0, v_0 are satisfying the Hypothesis 2.2 and the Wiener processes W_1 and W_2 the Hypothesis 2.1. Then there exists a couple of progressively measurable processes (u,v) solving the system of equations (2.6) and for all $\delta < 1$ $\mathbb{P}(u \in C(0,T; H_2^\delta(I))) = 1$. In addition, we have*

(1) for $p = 2, 4,$ or 6, and for all $T > 0$, there exists a constant $C > 0$ such that

$$\mathbb{E} \sup_{0\le t\le T} |u(t)|^p_{L^p} \le C \quad \text{and} \quad \mathbb{E} \sup_{0\le t\le T} |v(t)|^p_{L^p} \le C.$$

(2) for $p = 4$, there exists a constant $C > 0$ such that for all $T > 0$,

$$\mathbb{E} \sup_{0\le t\le T} |u(t)|^2_{H^1_p} \le C \quad \text{and} \quad \mathbb{E} \sup_{0\le t\le T} |v(t)|^2_{H^1_p} \le C.$$

From Theorem 2.1 and the assumption on the Wiener processes we can prove the existence of a unique solution to the original equation.

Corollary 2.2. *If Hypothesis 2.1 and Hypothesis 2.2 are satisfied, then there exists a couple of progressively measurable processes (u,v) solving the system of equations (2.1). In addition, we have*

(1) for $p = 2, 4,$ or 6, and for all $T > 0$, there exists a constant $C > 0$ such that

$$\mathbb{E} \sup_{0\le t\le T} |u(t)|^p_{L^p} \le C \quad \text{and} \quad \mathbb{E} \sup_{0\le t\le T} |v(t)|^p_{L^p} \le C.$$

(2) for $p = 4$, there exists a constant $C > 0$ such that for all $T > 0$,

$$\mathbb{E} \sup_{0\le t\le T} |u(t)|^2_{H^1_p} \le C \quad \text{and} \quad \mathbb{E} \sup_{0\le t\le T} |v(t)|^2_{H^1_p} \le C.$$

Proof of Corollary 2.2: In particular, assuming, for the time being, that the correction term is finite, we get as a new system

$$(2.7) \quad \begin{cases} du(x,t) &= \big(r_u\Delta u(x,t) - u(x,t)v^2(x,t) + f(1-u(x,t))\big)\, dt \\ & \qquad + \sigma_u u(x,t)dW_1(x,t) + \sigma_u S(\gamma_1)u(x,t)\, dt, \quad x\in I, t>0, \\ dv(x,t) &= \big(r_v\Delta v(x,t) + u(x,t)v^2(x,t) - (f+k)v(x,t)\big)\, dt \\ & \qquad + \sigma_v v(x,t)dW_2(x,t) + \sigma_v S(\gamma_2)v(x,t)\, dt, \quad x\in I, t>0. \end{cases}$$

Replacing f by $f - \sigma_u S(\gamma_1)$ and $(f+k)$ by $(f+k) - \sigma_v S(\gamma_2)$ an application of Theorem 2.1 gives that there is a solution (u,v) to (2.7) both processes being $\mathbb{P}$–a.s. continuous in $H_2^\delta(I)$ and satisfying (1) and (2). Now, if the process arising by the correction term given by (A.23), i.e.,

$$\xi_j(t) = \frac{1}{2}\int_0^t \sum_{k\in\mathbb{Z}} \big(u(s)\big|\psi_k\big)_{\mathcal{H}} \sum_{l\in\mathbb{Z}} (1-\lambda_l)^{-2\gamma_j}\, ds$$

is continuous in $H_2^\delta(I)$ and satisfies the properties (1) and (2), then we are done. However, this follows by the properties of u and v. $\square$

Proof. The proof of Theorem 2.1 consists of several steps. First, we show that the system with a truncated nonlinearity can be uniquely solved. In a second step, we show that the solution is a.s. non–negative. In the third step, we give an uniform estimate of $u+v$ in $H^1_4(I)$. From Sobolev embeddings we get uniform bounds with the L^∞-norm. Finally, by these uniform bounds we can globalize the solution in the last step.

Step (i) Fix $m \in \mathbb{N}$. Since we would like to relax the condition on the initial conditions, we first approximate the nonlinear term uv^2 as follows. Let us define

$$g_m(x) := \begin{cases} x & \text{if} \quad 0 < x \le m, \\ \in (m, (m+1)) & \text{if} \quad m < x < m+1, \\ (m+1) & \text{if} \quad m+1 \le x. \end{cases}$$

Between the interval $(m, m+1)$ we interpolate the function by a polynomial function, such that g_m is twice continuously differentiable. In particular,

$$g'_m(x)\mid_{x=x_0} = 1, \quad \text{for} \quad x_0 = m, \quad g'_m(x)\mid_{x=x_0} = 0, \quad \text{for} \quad x_0 = m+1,$$

and

$$g''{}_m(x)\mid_{x=x_0} = 0, \quad \text{for} \quad x_0 = m, m+1.$$

Let us define the mapping F_m by

$$\begin{aligned} F_m : L^2(I) \times L^4(I) &\longrightarrow L^1(I), \\ (u,v) &\longmapsto F_m(u,v); \end{aligned}$$

by

$$F_m(u,v)(x) := g_m(u(x)) g_m^2(v(x)), \quad x \in I.$$

The mapping F_m is Lipschitz with Lipschitz constant $2(m+1)^2$. By Theorem 6.24 [14, p. 178] the following system

$$\begin{cases} du_m(x,t) = [r_u \Delta u_m(x,t) - F_m(u_m(x,t), v_m(x,t)) + \rho + \alpha_u u_m(x,t)]\, dt + \sigma_u\, u_m(x,t) dW_1(x,t), \\ u_m(0,x) = u_0(x), \quad x \in I, \end{cases} \tag{2.8}$$

and

$$\begin{cases} dv_m(x,t) = [r_v \Delta v_m(x,t) + F_m(u_m(x,t), v_m(x,t)) + \alpha_v v_m(x,t)]\, dt + \sigma_v v_m(x,t) dW_2(x,t), \\ v_m(0,x) = v_0(x), \quad x \in I, \end{cases} \tag{2.9}$$

has a unique pair of solution $\{u_m, v_m\}$, each component belonging to $C([0,T]; L^2(I)) \cap L^2([0,T]; H^1_2(I))$.

Step (ii) As the next step, we show that each component of the pair of the solution $\{u_m, v_m\}$ are non–negative. To show this, we can follow e.g. Theorem 2.3 in [42], or [3, Theorem 2.6.2, p. 42]. Here, we summarize only the idea. In fact it remains to approximate the operator Δ by, e.g., its Yosida approximation to be able to apply the Itô formula. Let

$$g_\delta(r) = \frac{r^2}{\delta + r}, \quad r \in (-\delta, \infty),$$

and

$$G_\delta(r) := g_\delta((r^-)^2), \quad r \in \mathbb{R}.$$

Then, G_δ belongs to C^2 and $G_\delta(r) = G'_\delta(r) = G''_\delta(r) = 0$ for all $r \in [0,\infty)$, $|G'_\delta(r)| \le 2r^-$, and $0 \le G''_\delta(r) \le 8$. Now, define $\phi_\delta : L^2(I) \to \mathbb{R}$ by

$$\phi_\delta(w) = \int_I G_\delta(w(\xi))\, d\xi, \quad w \in L^2(I).$$

Observe, ϕ_δ is twice uniformly continuous on bounded subsets, and hence the Itô formula can be applied (see Theorem 4.32 [14, p. 107]). Applying the Itô formula to $\phi_\delta(u_m(t))$ where $u_m(t)$ solves (2.8), we get

$$\mathbb{E}\phi_\delta(u_m(t)) + r_u\mathbb{E}\int_0^t \langle \Delta u_m(s), D\phi_\delta(u_m(s))\rangle\, ds = \phi_\delta(u_0) - \mathbb{E}\int_0^t \langle u_m(s)v_m(s)^2, D\phi_\delta(u_m(s))\rangle\, ds$$
$$+\alpha_u\mathbb{E}\int_0^t \langle u_m(s), D\phi_\delta(u_m(s))\rangle\, ds + \frac{\sigma_u^2}{2}\mathbb{E}\int_0^t \operatorname{Tr}\left(D^2G_\delta(u_m(s))[M(u_m(s))Q^{\frac12}][M(u_m(s))Q^{\frac12}]^*\right) ds\,;$$

the defining relation for the covariance operator is found in the appendix, see (A.5). Note, that

$$\langle \Delta u_m(s), D\phi_\delta(u_m(s))\rangle = \int_I (\nabla u_m(x,s))^2\phi_\delta''(u_m(x,s))\, dx \ge 0.$$

Due to (**??**), we know

$$\mathbb{E}\int_0^t \operatorname{Tr}\left(D^2G_\delta(u_m(s))[M(u_m(s))Q^{\frac12}][M(u_m(s))Q^{\frac12}]^*\right) ds \le 8\mathbb{E}\int_0^t |u_m^-(s)|_{L^2}^2\, ds.$$

A similar arguments works for v_m.

$$\langle u_m(s)v_m(s)^2, D\phi_\delta(u_m(s))\rangle = \int_I (u_m(x,s)^-)^2 v_m^2(x,s)\, dx \le (m+1)^2\int_I |u_m(s)^-|_{L^2}^2,$$

and

$$\langle u_m(s), D\phi_\delta(u_m(s))\rangle \le |u_m^-(s)|_{L^2}^2.$$

Collecting all together and applying the Grownwall Lemma give $\mathbb{E}\phi_\delta(u_m(t)) = 0$. Taking the limit $\delta\to 0$ gives the assertion. Similarly, one can proof that v_m is $\mathbb{P}$–a.s. non–negative.

Step (iii) In this step we will show that there exists some bounds on $\mathbb{E}|u_m|_{L^p}^p$, which are uniform in $m\in\mathbb{N}$.

Claim 2.1. *For any even integer $2\le p<\infty$ and initial condition satisfying $\mathbb{E}|u_0|_{L^p}^p, \mathbb{E}|v_0|_{L^p}^p<\infty$, there exist constants $C_1, C_2, C_3>0$ such that*

$$\mathbb{E}\sup_{0\le s\le T}|u_m(s)|_{L^p}^p \quad\le\quad C(T)(C_0+\mathbb{E}|u_0|_{L^p}^p),\quad \forall\, m\in\mathbb{N}.$$

For any even integer $2\le p<\infty$, there exist constants $C_1, C_2, C_3>0$ such that

$$\mathbb{E}\int_0^T\int_I u_m^{p-2}(x,s)(\nabla u_m(x,s))^2\, dx\, ds \quad\le\quad C(T)(C_0+\mathbb{E}|u_0|_{L^p}^p),\quad \forall\, m\in\mathbb{N}.$$

Proof. Let us put first $p=2$. The calculations are straight forward using the variational approach. Let us remind that we have equation (2.8) and the definition of the multiplication operator M defined in (A.10)

$$du_m(t) = r_uAu_m(t)\, dt - F_m(u_m,v_m)(t) + \alpha_u u_m(t) + \sigma_u M(u_m(t))dW_1(t),$$

respectively,

$$du_m(t) = r_uAu_m(t)\, dt - F_m(u_m,v_m)(t) + \alpha_u u_m(t) + \sum_{k,l\in\mathbb{Z}}\langle u_m(t),\psi_l\rangle\psi_l h_k\beta_k(t),$$

with $h_k = (1-\lambda_k)^{-\frac{\beta}{2}}\psi_k$ and β_k are i.i.d. mutually independent standard Brownian motion. Now, since $\Phi(x) = |x|_{L^2}^2$, $D\Phi(x)[h] = \langle x,h\rangle$, $D^2\Phi(x)[h^1,h^2] = \langle h^1,h^2\rangle$, applying the Itô formula (see Theorem 4.17, [14, p. 105]) to $\Phi(x)=|x|_{L^2}^2$ and integration by parts give

$$
\begin{aligned}
d\Phi(u_m(t)) &= d\int_I u_m^2(x,t)\,dx = 2\int_I u_m(x,t)\Delta u_m(x,t)\,dx\,dt\\
&\quad -2\int_I u_m(x,t)g_m(u_m(x,t))g_m^2(v_m(x,t))\,dx\,dt + 2\int_I u_m(x,t)(\rho+\alpha_u u_m(x,t))\,dx\,dt\\
&\quad +2\sum_{k\in\mathbb{Z}}\langle u_m(t), M(u_m(t))h_k\rangle d\beta_k(t) + \mathrm{Tr}\left[D^2\Phi(u_m(t))[M(u_m(t))Q^{\frac12}][M(u_m(t))Q^{\frac12}]^*\right]\,dt\\
= &\quad -2\int_I(\nabla u_m(x,t))^2\,dx\,dt - 2\int_I u_m(x,t)g_m(u_m(x,t))g_m^2(v_m(x,t))\,dx\,dt\\
&\quad +2\int_I u_m(x,t)(\rho+\alpha_u u_m(x,t))\,dx\,dt\\
&\quad +2\sum_{k\in\mathbb{Z}}\langle u_m(t), M(u_m(t))h_k\rangle d\beta_k(t) + \mathrm{Tr}\left[D^2\Phi(u_m(t))[M(u_m(t))Q^{\frac12}[M(u_m(t))Q^{\frac12}]^*\right]\,dt.
\end{aligned}
$$

Taking the expectation, integrating, and taking into account that the stochastic integral vanishes, we get

$$
\begin{aligned}
&\frac12\mathbb{E}\int_0^t\int_I u_m^2(x,s)\,dx\,ds + 2\int_0^t\int_I(\nabla u_m(x,s))^2\,dx\,ds\\
&\leq\ \mathbb{E}|u_0|_{L^2}^2 + 2\int_0^t\int_I u_m(x,s)f(1-u_m(x,s))\,dx\,ds + \sigma_u\int_0^t\sum_{k\in\mathbb{Z}}|M(u_m(s))h_k|_{L^2}^2\,ds.
\end{aligned}
$$

By estimate (**??**) and Hypothesis 2.1 we have

$$
\int_0^t \mathrm{Tr}\left[D^2\Phi(u_m(t))[M(u_m(s))Q^{\frac12}[M(u_m(s))Q^{\frac12}]^*\right]\,ds \le S(\gamma_1)\int_0^t |u_m(s)|_{L^2}^2\,ds,
$$

and therefore, by the Young inequality, we get

$$
\begin{aligned}
&\frac12\mathbb{E}|u_m(t)|_{L^2}^2 + 2\int_0^t\mathbb{E}|u_m(s)|_{H_2^1}^2ds + 2\mathbb{E}\int_0^t\int_I u_m(x,s)g_m(u_m(x,s))g_m^2(v_m(x,s))\,dx\,ds\\
&\leq\ \mathbb{E}|u_0|_{L^2}^2 + C(\varepsilon)\,(2\rho)^2 + \int_0^t\mathbb{E}|u_m(s)|_{L^2}^2\,ds + (\alpha_u + CS(\gamma))\int_0^t\mathbb{E}|u_m(s)|_{L^2}^2\,ds.
\end{aligned}
$$

Grownwall's Lemma gives that there exists a constant $C=C(T)>0$ such that

$$
\frac12\mathbb{E}|u_m(t)|_{L^2}^2 + 2\int_0^t\mathbb{E}|u_m(s)|_{H_2^1}^2ds \le \mathbb{E}|u_0|_{L^2}^2 + C(T),\quad \forall t\in[0,T]. \tag{2.10}
$$

To estimate the supremum over the time, i.e. $\mathbb{E}\sup_{0\le t\le T}|u_m(t)|_{L^2}^2$, we have to apply the Burkholder–Davis–Gundy inequality to estimate the stochastic integral

$$
\sum_{k\in\mathbb{Z}}\langle u_m(t), M(u_m(t))(1-\lambda_k)^{-\gamma}h_k\rangle d\beta_k(t)\,.
$$

Thus, inequality (**??**) gives

$$
\mathbb{E}\sup_{0\le s\le t}\Big|\int_0^s\sum_{k\in\mathbb{Z}}\langle u_m(t),u_m(t)h_k\rangle d\beta_k(t)\Big|_{L^2}\ \le\ C\,\mathbb{E}\left(\int_0^t|u_m(s)|_{L^2}^4\,ds\right)^{\frac12} \le C\,\mathbb{E}\sup_{0\le s\le t}|u_m(s)|_{L^2}^2 t^{\frac12}.
$$

Again, we have by (**??**)

$$
\int_0^t\mathrm{Tr}\left(D^2\Phi(u_m(s))[M(u_m(s))Q^{\frac12}[M(u_m(s))Q^{\frac12}]^*\right)\,ds \le S(\gamma_1)\int_0^t|u_m(s)|_{L^2}^2\,ds,
$$

Fix $T^* > 0$. Integrating up to time T^*, taking expectation, rearranging, using the Hölder and Young inequality, and taking into account the positivity of $u_m(x,t)$, lead to

$$\begin{aligned}(2.11)\quad & \mathbb{E}\sup_{0\le t\le T^*}\int_I u_m^2(x,t)\,dx + 2r_u\int_0^{T^*}\mathbb{E}\int_I(\nabla u_m(x,t))^2\,dx\,dt\\ &\quad + 2\int_0^{T^*}\mathbb{E}\int_I u_m(x,t)g_m(u_m(x,t))g_m^2(u_m(x,t))\,dx\,dt\\ &\le\ \mathbb{E}|u_0|_{L^2}^2 + 2\rho\int_0^{T^*}\mathbb{E}\int_I u_m(x,t)\,dx\,dt\\ &\quad + 2\alpha_u\int_0^{T^*}\mathbb{E}\int_I u_m^2(x,t)\,dx\,dt + 2C_1^Q\sigma_u\mathbb{E}\int_0^{T^*}|u_m(t)|_{L^2}^2\,dt + C\,\mathbb{E}\sup_{0\le s\le T^*}|u_m(s)|_{L^2}^2 {T^*}^{\frac12}.\end{aligned}$$

Rearranging we get

$$\begin{aligned}& \mathbb{E}\sup_{0\le t\le T^*}\int_I u_m^2(x,t)\,dx + 2r_u\int_0^{T^*}\mathbb{E}\int_I(\nabla u_m(x,t))^2\,dx\,dt\\ &\quad + 2\int_0^{T^*}\mathbb{E}\int_I u_m(x,t)g_m(u_m(x,t))g_m^2(u_m(x,t))\,dx\,dt\\ &\le\ \mathbb{E}|u_0|_{L^2}^2 + C\,(2\rho)^2 + \int_0^{T^*}\mathbb{E}|u_m(t)|_{L^2}^2\,dt\\ &\quad + 2\alpha_u\int_0^{T^*}\mathbb{E}|u_m(t)|_{L^2}^2\,dt + 2C_1^Q\mathbb{E}\int_0^{T^*}|u_m(t)|_{L^2}^2\,dt + C\,\mathbb{E}\sup_{0\le s\le T^*}|u_m(s)|_{L^2}^2 {T^*}^{\frac12}.\end{aligned}$$

In case $\sqrt{T^*}C \le \frac12$, we get by subtracting $\mathbb{E}\sup_{0\le s\le T^*}|u_m(s)|_{L^2}^2$ on both sides

$$\begin{aligned}(2.12)\quad & \frac12\mathbb{E}\sup_{0\le t\le T^*}\int_I u_m^2(x,t)\,dx + 2r_u\int_0^{T^*}\mathbb{E}\int_I(\nabla u_m(x,t))^2\,dx\,dt\\ &\quad + 2\int_0^{T^*}\mathbb{E}\int_I u_m(x,t)g_m(u_m(x,t))g_m^2(u_m(x,t))\,dx\,dt\\ &\le\ \mathbb{E}|u_0|_{L^2}^2 + C_1 + C_2\int_0^{T^*}\mathbb{E}|u_m(t)|_{L^2}^2\,dt.\end{aligned}$$

Taking into account (2.10) we get

$$\begin{aligned}(2.13)\quad & \frac12\mathbb{E}\sup_{0\le t\le T^*}\int_I u_m^2(x,t)\,dx + 2r_u\int_0^{T^*}\mathbb{E}\int_I(\nabla u_m(x,t))^2\,dx\,dt\\ &\quad + 2\int_0^{T^*}\mathbb{E}\int_I u_m(x,t)g_m(u_m(x,t))g_m^2(u_m(x,t))\,dx\,dt + 2f\int_0^{T^*}\mathbb{E}|u_m(t)|_{L^2}^2\,dt\\ &\le\ \mathbb{E}|u_0|_{L^2}^2 + C_1\mathbb{E}|u_0|_{L^2}^2 + C_2 + C(T^*).\end{aligned}$$

Given T, we can decompose $[0,T]$ as $\cup_{0\le k\le N-1}[kT^*,(k+1)T^*]$, and apply inequality (2.13) to each interval $[kT^*,(k+1)T^*]$, $k=0,\dots,N-1$. In this way, we extend the estimate to the whole interval $[0,T]$ to prove that the family $\{u_m : m\in\mathbb{N}\}$ can be bounded uniformly for all $m\in\mathbb{N}$ in the supremums norm over time. In particular, we proved the assertion (1) of Theorem 2.1 for the family $\{u_m : m\in\mathbb{N}\}$.

Let $p=4$ and $\Phi(u)=\int_I u^p(x)\,dx$. Then $D\Phi(u)[h] = p\int_I u^3(x)\,h(x)\,dx$ and $D^2\Phi(u)[h^1,h^2] = 12\int_I u^2(x)\,h^1(x)h^2(x)\,dx$. Recalling that u_m is non–negative, we obtain by the Itô formula applied to

$\Phi(x)=|x|^p_{L^p}$

$$\Phi(u(T))-\Phi(u_0)=\int_I u_m^4(T,x)\,dx-\int_I u_m^4(0,x)\,dx=\int_0^T\int_I\Big[r_u 4u_m^3(x,t)\nabla^2 u_m(x,t)$$
$$-4u_m^3(x,t)g_m(u_m(x,t))g_m^2(v_m(x,t))\;+4(\rho-\alpha_u u_m(x,t))u_m^3(x,t)\Big]\,dx\,dt$$
$$+\int_0^T 4\,\sigma_u u_m^4(x,t)dW_1(x,t)+\int_0^T \operatorname{Tr}\left(D^2\Phi(u_m(t))[M(u_m(t))Q^{\frac12}[M(u_m(t))Q^{\frac12}]^*\right)\,dt.$$

Continuing gives

$$\begin{aligned}\Phi(u(t))-\Phi(u_0)\quad+\quad & r_u 12\int_0^t\int_I u_m^2(x,s)(\nabla u_m(x,s))^2\,dx\,ds\\ &+p\int_0^t\int_I u_m^3(x,t)g_m(u_m(x,t))g_m^2(v_m(x,t))\,dx\,ds\\ =\quad & \Phi(u(T))-\Phi(u_0)+4\int_0^t\int_I(\rho-\alpha_u u_m(x,t))u_m^3(x,s)\,dx\,ds+p\sigma_u\int_0^t\int_I u_m^4(x,s)dW_1(x,s)\\ &+\int_0^t \operatorname{Tr}\left(D^2\Phi(u_m(s))[M(u_m(s))Q^{\frac12}[M(u_m(s))Q^{\frac12}]^*\right)\,ds.\end{aligned}$$

Taking expectation and using integration by parts give

$$\begin{aligned}\mathbb{E}|u_m(t)|^4_{L_4}+r_u 12\int_0^t\mathbb{E}\int_I u_m^2(x,s)(\nabla u_m(x,s))^2\,dx\,ds&\\ +4\mathbb{E}\int_0^t\int_I F_m(u_m,v_m)u_m^3(x,s)\,dx\,ds\le\mathbb{E}|u_0|^4_{L_4}&\\ +C_1\mathbb{E}\int_0^t\int_I u_m^3(x,s)\,dx\,ds+C_3\mathbb{E}\int_0^t|u_m(s)|^4_{L^4}\,ds.&\end{aligned}$$

We get by some rearrangements and Gronwall's Lemma

$$\mathbb{E}|u_m(t)|^4_{L_4}+r_u 12\int_0^t\mathbb{E}\int_I u_m^2(x,s)(\nabla u_m(x,s))^2\,dx\,ds\le\mathbb{E}|u_0|^4_{L^4}+C(T). \tag{2.14}$$

To estimate the supremum, we apply again A.18 and get

$$\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_I u_m^4(x,s)dW_1(x,s)\right|\le S(\gamma_1)\mathbb{E}\left(\int_0^T|u_m^4(s)|^2_{L^2}\,ds\right)^{\frac12}.$$

Applying the Hölder inequality, Sobolev embedding, and then the Young inequality gives for $\varepsilon,\tilde\varepsilon>0$

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_I u_m^4(x,s)dW_1(x,s)\right|&\le S(\gamma_1)\,\mathbb{E}\left(\int_0^T|u_m^2(s)|^2_{L^\infty}\,|u_m^2(s)|^2_{L^2}ds\right)^{\frac12}\\ \le\quad & S(\gamma_1)\,\mathbb{E}\left(\int_0^T|u_m^2(s)|^2_{H^1_2}\,|u_m(s)|^2_{L^4}ds\right)^{\frac12}\\ \le\quad & S(\gamma_1)\,\mathbb{E}\left(\int_0^T\left(\int_I u_m^2(x,s)(\nabla u_m(x,s))^2\,dx\right)\,ds\,\sup_{0\le s\le T}|u_m(s)|^2_{L^4}\right)^{\frac12}\\ \le\quad & \varepsilon S(\gamma_1)\,\mathbb{E}\int_0^T\left(\int_I u_m^2(x,s)(\nabla u_m(x,s))^2\,dx\right)ds+C(\varepsilon)\mathbb{E}\sup_{0\le s\le T}|u_m(s)|^2_{L^4}\\ \le\quad & \varepsilon S(\gamma_1)\,\mathbb{E}\int_0^T\int_I u_m^2(x,s)(\nabla u_m(x,s))^2\,dx\,ds+\tilde\varepsilon\mathbb{E}\sup_{0\le s\le T}|u_m(s)|^4_{L^4}+C(\varepsilon,\tilde\varepsilon).\end{aligned}$$

Again, the trace is given by

$$\frac{1}{2}\mathrm{Tr}\left(D^2\Phi(u_m(s))[M(u_m(s))Q^{\frac{1}{2}}][M(u(s))Q^{\frac{1}{2}}]^*\right) = S(\gamma)\,|u_m(s)|_{L^4}^4.$$

Therefore, taking ε and $\tilde{\varepsilon}$ sufficiently small

$$\begin{aligned}\frac{1}{2}\mathbb{E}\sup_{0\le t\le T}|u_m(t)|_{L_4}^4 &+ r_u\mathbb{E}\int_0^T\int_I u_m^2(x,s)[\nabla u_m(x,s)]^2\,dx\,ds\\ &+4\mathbb{E}\int_0^T\int_I F_m(u_m,v_m)(s)u_m^3(x,s)\,dx\,ds \le \mathbb{E}|u_0|_{L_4}^4\\ &+4\mathbb{E}\int_0^T\int_I(\rho+\alpha_u u_m(x,s))u_m^3(x,s)\,dx\,ds + C\,\mathbb{E}\int_0^T\int_I u_m^4(x,s)\,dx\,ds + C(\varepsilon,\tilde{\varepsilon}).\end{aligned}$$

Due to (2.14) the terms in the RHS are bounded and there exists a constant $C = C(T) > 0$ such that

$$\mathbb{E}\sup_{0\le t\le T}|u_m(t)|_{L_4}^4 \le C(T)\,\mathbb{E}|u_0|_{L^4}^4.$$

□

Step (iv) Let us define $w_m = u_m + v_m$ and $w_0 = u_0 + v_0$. Here, we will prove the following claim:

Claim 2.2. *Under the Hypothesis 2.2-(ii), the following estimates are valid:*

(1) There exists a constant $C = C(T) > 0$ such that

$$\mathbb{E}\sup_{0\le t\le T}|w_m(t)|_{L^2}^2,\ \int_0^T\mathbb{E}|\nabla u_m(s)|_{L^2}^2\,ds,\ \mathbb{E}\int_0^T|\nabla v_m(s)|_{L^2}^2\,ds \le C,\quad m\in\mathbb{N}.$$

(2) for any even integer $p\ge 2$, there exists a constant $C = C(T,p) > 0$ such that

$$\mathbb{E}\sup_{0\le t\le T}|w_m(t)|_{L^p}^p \le C,\quad m\in\mathbb{N},$$

and

$$\begin{aligned}\mathbb{E}\int_0^t\int_I|u_m^{k-1}(x,s)v_m^{p-1-k}(x,s)(\nabla u_m(x,s))^2|\,dx\,ds \le C,\quad k=1\dots,p-1,\quad m\in\mathbb{N},\\ \mathbb{E}\int_0^t\int_I|u_m^{k}(x,s)v_m^{p-2-k}(x,s)(\nabla v_m(x,s))^2|\,dx\,ds \le C,\quad k=0,\dots,p-2,\quad m\in\mathbb{N}.\end{aligned}$$

(3) In addition, there exists a constant $C = C(T) > 0$ such that

$$\mathbb{E}\int_0^t\int_I\left|u_m^{k-1}(x,s)v_m^{p-1-k}(x,s)\nabla u_m(x,s)\nabla v_m(x,s)\right|\,dx\,ds \le C,\quad k=0,\dots,p-2,\quad m\in\mathbb{N}.$$

Proof of Claim 2.2: To show (1) and (2) first note that w_m solves

$$(2.15)\quad \begin{cases} dw_m(x,t) &= (r_u\Delta u_m(x,t) + r_v\Delta v_m(x,t) + \alpha_u w_m(x,t) - (\alpha_u-\alpha_v)v_m(x,t) + \rho)\,dt\\ &\quad +\sigma_u u_m(x,t)\,dW_1(x,t) + \sigma_v v_m(x,t)\,dW_2(x,t),\\ w_m(0,x) &= u_0(x) + v_0(x),\end{cases}$$

We denote the inner product in $L^2(I)$ by $\langle\cdot,\cdot\rangle$. Now, an application of the Itô formula with $k=-(\alpha_u-\alpha_v)$ gives

$$\begin{aligned}
|w_m(t)|^2_{L^2} &+ \int_0^t \left(r_u|\nabla u_m(s)|^2_{L^2}+r_v|\nabla v_m(s)|^2_{L^2}\right)\,ds\\
&+\int_0^t \alpha_u\langle w_m(s),w_m(s)\rangle\,ds\\
\leq\ & |w_0|^2_{L^2}+\int_0^t (r_u+r_v)\langle\nabla u_m(s),\nabla v_m(s)\rangle\,ds+\int_0^t\langle w_m(s),\alpha_u\rangle\,ds\\
&+k\int_0^t\langle w_m(s),v_m(s)\rangle\,ds+\int_0^t\langle w_m(s),\sigma_u u_m(s)dW_1(s)\rangle+\int_0^t\langle w_m(s),\sigma_u v_m(s)dW_2(s)\rangle\\
&+2\sigma_u\sum_{k\in\mathbb{Z}}(1-\lambda_k)^{-\gamma_1}\langle w_m(t),M(u_m(t))h_k\rangle d\beta^1_k(t)+2\sigma_v\sum_{k\in\mathbb{Z}}(1-\lambda_k)^{\gamma_2}\langle w_m(t),M(v_m(t))h_k\rangle d\beta^2_k(t)\\
&+\sigma_u\int_0^t \operatorname{Tr}\left(D^2\Phi(w_m(s))[M(u_m(s))Q_2^{\frac12}[M(u_m(s))Q_2^{\frac12}]^*\right)\,ds+\sigma_v\int_0^t\operatorname{Tr}\left(D^2\Phi(w_m(s))[M(v_m(s))[M(v_m(s))]\right)\,ds.
\end{aligned}$$

Since $v_m(s)\geq 0$ and $u_m(s)\geq 0$ $\mathbb{P}\times$ Leb–a.e., it follows that $\mathbb{P}$–a.e. $\langle w_m(s),v_m(s)\rangle\geq 0$. The Young inequality and taking expectation give

$$\begin{aligned}
\mathbb{E}\,|w_m(t)|^2_{L^2}&+\mathbb{E}\int_0^t\left(r_u|\nabla u_m(s)|^2_{L^2}+\frac{r_v}{4}|\nabla v_m(s)|^2_{L^2}\right)\,ds\\
&+\frac{r}{2}\int_0^t\mathbb{E}|w_m(s)|^2_{L^2}\,ds\leq\frac{r_v}{2(r_u+r_v)}\,\mathbb{E}\int_0^t|\nabla u_m(s)|^2_{L^2}\,ds+k\int_0^t\mathbb{E}\langle w_m(s),v_m(s)\rangle\,ds\\
&+C\int_0^t\mathbb{E}|w_m(s)|^2_{L^2}\,ds+C\,\alpha_u\,t+\mathbb{E}|w_0|^2_{L^2}.
\end{aligned}$$

In addition,

$$\left|\int_0^t\mathbb{E}\langle w_m(s),v_m(s)\rangle\,ds\right|\leq\int_0^t\mathbb{E}|w_m(s)|^2_{L^2}\,ds+\int_0^t\mathbb{E}|v_m(s)|^2_{L^2}\,ds.$$

Applying Claim 2.1 and Grownwall's Lemma give

$$\mathbb{E}\,|w_m(t)|^2_{L^2}+\mathbb{E}\int_0^t\left(r_u|\nabla u_m(s)|^2_{L^2}+\frac{r_v}{4}|\nabla v_m(s)|^2_{L^2}\right)\,ds\quad\leq\quad C_1(T)\,\mathbb{E}|w_0|^2_{L^2}+C_2(T).$$

Note, that we took into account that $|u_m|_{L^2},|v_m|_{L^2}\leq|w_m|_{L^2}$.

Again, to estimate the supremum, we have to apply the Burkholder–Davis–Gundy inequality (A.18) inequality and get

$$\begin{aligned}
\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t\int_I w_m(x,s)u_m(x,s)dW_1(x,s)\right|\quad&\leq\quad C_1\mathbb{E}\left(\int_0^T|w_m(s)u_m(s)|^2_{L^2}\,ds\right)^{\frac12}\\
\leq C_1\mathbb{E}\left(\int_0^T|w_m(s)|^2_{L^2}\,|u_m(s)|^2_{L^\infty}\,ds\right)^{\frac12}\quad&\leq\quad C_1\mathbb{E}\int_0^T|w_m(s)|^2_{L^2}\,ds+\mathbb{E}\sup_{0\leq s\leq T}|u_m(s)|^2_{H^1_2}.
\end{aligned}$$

To estimate the supremum in the second stochastic integral, we apply the Burkholder–Davis–Gundy and the Young inequality, but taking into account that the term containing v_m have to be cancelled with the LHS, we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t\int_I w_m(x,s)\sigma_v v_m(x,s)dW_1(x,s)\right|_{L^2}\\
&\quad\leq\quad\frac{r_v}{4}\mathbb{E}\sup_{0\leq s\leq T}|v_m(s)|^2_{H^1_2}+C(T)C_2\mathbb{E}\int_0^T|w_m(s)|^2_{L^2}\,ds.
\end{aligned}$$

The Young inequality and taking expectation give

$$\mathbb{E}\sup_{0\le t\le T}|w_m(t)|_{L^2}^2+\mathbb{E}\int_0^T\left(r_u|\nabla u_m(s)|_{L^2}^2+\frac{r_v}{4}|\nabla v_m(s)|_{L^2}^2\right)ds+k\int_0^T\mathbb{E}\langle w_m(s),v_m(s)\rangle\,ds$$
$$+\frac{r}{2}\int_0^T\mathbb{E}|w_m(s)|_{L^2}^2\,ds\le\frac{r_v}{2(r_u+r_v)}\;\mathbb{E}\int_0^T|\nabla u_m(s)|_{L^2}^2\,ds$$
$$+C_1\int_0^T\mathbb{E}|w_m(s)|_{L^2}^2\,ds+C_2\,T+\mathbb{E}|w_0|_{L^2}^2.$$

Applying Claim 2.1 and the Grownwall Lemma give

$$\mathbb{E}\sup_{0\le t\le T}|w_m(t)|_{L^2}^2+\mathbb{E}\int_0^T\left(r_u|\nabla u_m(s)|_{L^2}^2+\frac{r_v}{4}|\nabla v_m(s)|_{L^2}^2\right)ds\quad\le\quad C_1(T)\,\mathbb{E}|w_0|_{L^2}^2+C_2(T).$$

It remains to show Claim 2.2-(2) and (3). For simplicity, we omit in the following the dependence on x and t. To show (ii) observe first, that we have for any $u,v\in H_2^2(I)$ by integration by parts

$$\begin{aligned}&\int_I(u+v)^{p-1}\left(\Delta r_u u+\Delta r_v v\right)dx\\&\quad=\sum_{k=0}^{p-1}\binom{p-1}{k}\int_I u^k v^{p-1-k}\left(\Delta r_u u+\Delta r_v v\right)dx\\&\quad=-\sum_{k=0}^{p-1}\binom{p-1}{k}\int_I\nabla(u^kv^{p-1-k})\left(\nabla r_u u+\nabla r_v v\right)dx.\end{aligned}$$

We rewrite the inner part of the sum as follows

$$\begin{aligned}&\int_I\nabla(u^kv^{p-1-k})\left(\nabla r_u u+\nabla r_v v\right)dx\\&\quad=\int_I\left(ku^{k-1}v^{p-1-k}\nabla u+(p-1-k)u^kv^{p-2-k}\nabla v\right)\left(r_u\nabla u+r_v\nabla v\right)dx\\&=\int_I\left(r_uku^{k-1}v^{p-1-k}(\nabla u)^2+r_v(p-1-k)u^kv^{p-2-k}(\nabla v)^2\right)dx\\&+\int_I\left(r_vku^{k-1}v^{p-1-k}\nabla u\nabla v+r_u(p-1-k)u^kv^{p-2-k}\nabla v\nabla u\right)dx.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_I(u+v)^{p-1}\left(\Delta r_u u+\Delta r_v v\right)dx\\&\qquad+\sum_{k=0}^{p-1}\binom{p-1}{k}\int_I\left(r_uku^{k-1}v^{p-1-k}(\nabla u)^2+r_v(p-1-k)u^kv^{p-2-k}(\nabla v)^2\right)dx\\&=\quad-\sum_{k=0}^{p-1}\binom{p-1}{k}\int_I\left(r_vku^{k-1}v^{p-1-k}\nabla u\nabla v+r_u(p-1-k)u^kv^{p-2-k}\nabla v\nabla u\right)dx.\end{aligned}$$

Renumbering gives

$$
\begin{aligned}
&\int_I (u+v)^{p-1}\left(\Delta r_u u + \Delta r_v v\right)\, dx \\
&\qquad + \sum_{k=0}^{p-1} \binom{p-1}{k} \int_I \left(r_u k u^{k-1} v^{p-1-k} (\nabla u)^2 + r_v (p-1-k) u^k v^{p-2-k} (\nabla v)^2\right)\, dx \\
&= \quad -\sum_{k=0}^{p-1} \binom{p-1}{k} \int_I r_u (p-1-k) u^k v^{p-2-k} \nabla v \nabla u \, dx \\
&\qquad - \sum_{k=0}^{p-2} \binom{p-1}{k+1} \int_I \left(r_v (k+1) u^k v^{p-2-k} \nabla u \nabla v\right)\, dx.
\end{aligned}
$$

We now estimate the RHS. We get for any $\varepsilon > 0$

$$
\begin{aligned}
\int_I \left| r_u u^k v^{p-2-k} \nabla v \nabla u \right| \, dx &\le r_u \left(\int_I \left| u^k v^{p-2-k} (\nabla u)^2 \right| \, dx \right)^{\frac{1}{2}} \left(\int_I \left| u^k v^{p-2-k} (\nabla v)^2 \right| \, dx \right)^{\frac{1}{2}} \\
&\le r_u C_\varepsilon \int_I \left| u^k v^{p-2-k} (\nabla u)^2 \right| \, dx + \varepsilon r_u \int_I \left| u^k v^{p-2-k} (\nabla v)^2 \right| \, dx.
\end{aligned}
$$

Taking into account that

$$
\binom{p-1}{k+1} = \frac{p-1-k}{k+1} \binom{p-1}{k},
$$

and choosing $\varepsilon = \frac{r_v}{4 r_u}$ we get

$$
\begin{aligned}
&\int_I (u+v)^{p-1}\left(\Delta r_u u + \Delta r_v v\right)\, dx \\
&\qquad + \sum_{k=0}^{p-1} \binom{p-1}{k} \int_I \left(r_u k u^{k-1} v^{p-1-k} (\nabla u)^2 + r_v (p-1-k) u^k v^{p-2-k} (\nabla v)^2\right)\, dx \\
&\le \quad \sum_{k=0}^{p-1} \binom{p-1}{k} \Big\{ r_v C_\gamma k \int_I \left| u^{k-1} v^{p-1-k} (\nabla u)^2 \right| \, dx + r_u C_\varepsilon (p-1-k) \int_I \left| u^k v^{p-2-k} (\nabla u)^2 \right| \, dx \Big\} \\
&\qquad + \sum_{k=1}^{p-1} \binom{p-1}{k} \frac{r_v}{4} (p-1-k) \int_I \left| u^{k-1} v^{p-1-k} (\nabla v)^2 \right| \, dx \\
&\qquad + \sum_{k=0}^{p-1} \binom{p-1}{k} \frac{r_v}{4} (p-1-k) \int_I \left| u^k v^{p-2-k} (\nabla v)^2 \right| \, dx.
\end{aligned}
$$

Subtracting the last two terms in the inequality, the terms containing ∇u are remaining on the RHS. In particular, we have

$$
\begin{aligned}
&\int_I (u+v)^{p-1}\left(\Delta r_u u + \Delta r_v v\right)\, dx \\
&\qquad + \sum_{k=0}^{p-1} \binom{p-1}{k} \int_I \Big(r_u k u^{k-1} v^{p-1-k} (\nabla u)^2 + \frac{r_v}{2} (p-1-k) u^k v^{p-2-k} (\nabla v)^2\Big)\, dx \\
&\le \quad \sum_{k=0}^{p-1} \binom{p-1}{k} \Big\{ r_v C_\gamma k \int_I \left| u^{k-1} v^{p-1-k} (\nabla u)^2 \right| \, dx + r_u C_\varepsilon (p-1-k) \int_I \left| u^k v^{p-2-k} (\nabla u)^2 \right| \, dx \Big\}.
\end{aligned}
$$

Here, applying the Hölder inequality and Young inequality with $q = \frac{p-2}{p-1-k}$ and $q' = \frac{p-2}{k-1}$, we get for $\gamma = \frac{1}{q}$

$$\int_I \left|u^{k-1} v^{p-1-k} (\nabla u)^2\right| dx \quad \leq \quad \int_I \left| v^{p-1-k} (\nabla u)^{2\gamma} u^{k-1} (\nabla u)^{2(1-\gamma)} \right| dx$$
$$\leq \varepsilon \int_I \left|v^{p-2} (\nabla u)^2\right| dx \quad + \quad C(\varepsilon) \int_I \left| u^{(p-2)\frac{k-1}{k}} (\nabla u)^2 \right| dx.$$

Again, the first term can be cancelled (by taking $\varepsilon > 0$ sufficiently small) with the term

$$r_v v^{p-2} (\nabla v)^2$$

appearing in the sum (for $k = 0$)

$$\sum_{k=0}^{p-1} \left(r_u k u^{k-1} v^{p-1-k} (\nabla u)^2 + r_v (p-1-k) u^k v^{p-2-k} (\nabla v)^2 \right).$$

To handle the second term on the RHS we observe

$$\int_I \left| u^{(p-2)\frac{k-1}{k}} (\nabla u)^2 \right| dx \leq \int_I \left| u^{p-2} (\nabla u)^2 \right| dx + \int_I \left| (\nabla u)^2 \right| dx.$$

The term on the right hand side can be estimated by Claim 2.1. Going back to problem the original problem and applying the Itô formula to $\Phi(u) = \int_I u^p(x)\, dx$, we obtain

$$\begin{aligned}
& |w_m(t)|_{L^p}^p + \sum_{k=0}^{p-1} \binom{p+1}{k} \int_0^t \int_I \Big(r_u k\, u_m^{k-1}(x,s)\, v_m^{p-1-k}(x,s)\, (\nabla u_m(x,s))^2 \\
& + r_v (p-1-k)\, u_m^k(x,s)\, v_m^{p-2-k}(x,s)\, (\nabla v_m(x,s))^2 \Big)\, dx\, ds \\
\leq \quad & |w_0|_{L^p}^p + \sum_{k=0}^{p-1} \binom{p-1}{k} \Big\{ r_v C_\gamma k \int_0^t \int_I \left| u_m^{k-1}(s) v_m^{p-1-k}(s) (\nabla u_m(s))^2 \right| dx \\
& + r_u C_\varepsilon (p-1-k) \int_0^t \int_I \left| u_m^k(s) v_m^{p-2-k}(s) (\nabla u_m(s))^2 \right| dx \Big\} \\
& + \int_I \left| u_m^{p-2}(s) (\nabla u_m(s))^2 \right| dx + \int_I \left| (\nabla u_m(s))^2 \right| dx \\
& + kp \int_0^t \int_I w_m^{p-1}(x,s) v_m(x,s) dx\, ds + \alpha_u p \int_0^t |w_m^p(s)|_{L^p}\, ds \\
& + p\sigma_u \int_0^t \int_I w_m^{p-1}(x,s) u_m(x,s)\, dW_1(x,s) + \int_0^t \mathrm{Tr}\left(D^2\Phi(w_m(t)) M(u_m(s)) Q^{\frac{1}{2}} [M(u_m(s)) Q^{\frac{1}{2}}]^* \right) ds \\
& + p\sigma_v \int_0^t \int_I w_m^{p-1}(x,s) u_m(x,s)\, dW_2(x,s) + \int_0^t \mathrm{Tr}\left(D^2\Phi(w_m(t)) M(v_m(s)) Q^{\frac{1}{2}} [M(v_m(s)) Q^{\frac{1}{2}}]^* \right) ds.
\end{aligned}$$

Estimating the trace by (**??**)

$$\mathrm{Tr}\left(D^2\Phi(w_m(t)) [M(u_m(s)) Q^{\frac{1}{2}} [M(u_m(s)) Q^{\frac{1}{2}}]^* \right), \mathrm{Tr}\left(D^2\Phi(w_m(t)) M(v_m(s)) Q^{\frac{1}{2}} [M(v_m(s)) Q^{\frac{1}{2}}]^* \right) \leq S(\gamma) |w_m(s)|_{L^p}^p$$

Taking expectation gives, Gronwall's Lemma, and Claim 2.2 we verify that there exists a $C > 0$ such that

$$\sup_{0 \leq t \leq T} \mathbb{E} |w_m(t)|_{L^p}^p \leq C.$$

□

Step (v) In the next step, in order to control the L^∞–norm, we will give an estimate of the $H_p^\gamma(I)$ norm for $\gamma > \frac{d}{p}$. In particular, we will proof the following Claim.

Claim 2.3. *There exists a constant $C > 0$ such that*
(1) for $u_0 \in H^1_4(I)$ there exists a constant $C > 0$ such that

$$\mathbb{E} \sup_{0\le t\le T} \int_I |\nabla u_m(x,t)|^4 \, dx \le C(T) \left(1 + \mathbb{E}|\nabla u_0|^4_{L^4}\right), \quad m \in \mathbb{N}.$$

(2) for $v_0 \in H^1_4(I)$ there exists a constant $C = C(T) > 0$ such that

$$\mathbb{E} \sup_{0\le t\le T} \int_I |\nabla v_m(x,t)|^4 \, dx \le C(T) \left(1 + \mathbb{E}|\nabla v_0|^4_{L^4}\right), \quad m \in \mathbb{N}.$$

Proof. Since to show the uniform bounds for u_m and v_m are quite similar, we will only tackle the proof of the uniform bound for v_m. Before showing the assertion, we have to show that there exists a constant $C > 0$ such that

$$\mathbb{E} \int_I |\nabla v_m(x,t)|^2 \, dx \le C \left(1 + \mathbb{E}|\nabla v_0|^2_{L^2}\right), \quad m \in \mathbb{N}. \tag{2.16}$$

Here, first, note that by the Itô formula we have $p = 2$

$$\begin{aligned} |\nabla v_m(t)|^2_{L^2} + 2\int_0^t \int_I (\Delta v_m(x,s))^2 \, dx \, ds \le |\nabla v_0|^2_{L^2} \qquad (2.17) \\ + \int_0^t \int_I \nabla v_m(x,s) \, \nabla \left(u_m(x,s) \, v_m^2(x,s)\right) \, dx \, ds - 2(f+k) \int_0^t \int_I (\nabla v_m(x,s))^2 \, dx \, ds \\ + \int_0^t \int_I \sigma_v(\nabla v_m(x,s)) \nabla v_m(x,s) dW_2(x,s) \\ + \sigma_u \int_0^t \mathrm{Tr}\left(D\Phi(\nabla v_m(s))[M(v_m(s))Q^{\frac12}][M(v_m(s))Q^{\frac12}]^*\right) \, ds. \end{aligned}$$

The Cauchy–Schwarz and Young inequality give

$$\begin{aligned} \int_I \Delta v_m(x,s) u_m(x,s) v_m^2(x,s) \, dx &\le \varepsilon \int_I (\Delta v_m(x,s))^2 dx + C(\varepsilon) \int_I u_m^2(x,s) v_m^4(x,s) \, dx \\ &\le \varepsilon \int_I (\Delta v_m(x,s))^2 dx + C(\varepsilon) \int_I w_m^6(x,s) \, dx. \end{aligned}$$

Due to Claim 2.2-(ii), the second term is bounded uniformly in $m \in \mathbb{N}$, the first term can be cancelled. Next, we have by the Burkholder–Davis–Gundy inequality

$$\mathbb{E} \sup_{0\le t\le T} \left| \int_0^t \int_I \sigma_v(\nabla v_m(x,s)) \nabla v_m(x,s) dW_2(x,s) \right| \le \int_0^t |(\nabla v_m(s))^2|^2_{L^2} \, ds.$$

The Hölder inequality, Sobolev embedding, and the Young inequality for convolution give

$$\ldots \quad \le \quad \varepsilon \mathbb{E} \sup_{0\le s\le t} |\nabla v_m(s)|^2_{L^2} + C(\varepsilon) \mathbb{E} \left(\int_0^t |\Delta v_m(s)|^2_{L^2} \, ds \right).$$

If $\varepsilon > 0$ is chosen sufficiently small, the first term can be cancelled with the left hand side. Finally, we use Hypothesis 2.1 and inequality (**??**) to get

$$\mathbb{E} \int_0^t \mathrm{Tr}\left(D\Phi(\nabla v_m(s))[M(v_m(s))Q^{\frac12}][M(v_m(s))Q^{\frac12}]^*\right) \, ds \le \mathbb{E} \int_0^t |\nabla v_m(s)|^2_{L^2} \, ds$$

In this way we have shown (2.16).

Let $p=4$ and $\Phi(x)=|x|^p_{L^p}$. Note, that by the Itô formula we have

$$
\begin{aligned}
&|\nabla v_m(t)|^p_{L^p} + p(p-1)\int_0^t \int_I (\nabla v_m)^{p-2}(x,s)(\Delta v_m(x,s))^2\,dx\,ds \\
\le\;& |\nabla v(0)|^p_{L^p} + \int_0^t \int_I (\nabla v(x,s))^{p-2}\Delta v_m(x,s)\,u_m(x,s)\,v_m^2(x,s)\,dx\,ds \\
&- p(f+k)\int_0^t \int_I (\nabla v_m(x,s))^{p-1}\nabla v_m(x,s)\,dx\,ds \\
&+ p\int_0^t \int_I \sigma_v (\nabla v_m(x,s))^{p-1}\nabla v_m(x,s) dW_2(x,s) \\
&+ \sigma_v \int_0^t \mathrm{Tr}[\Phi(v_m)[M(v_m(s))Q^{\frac12}][M(v_m(s))Q^{\frac12}]^*\,ds.
\end{aligned}
\tag{2.18}
$$

The Cauchy–Schwarz inequality gives

$$
\begin{aligned}
&\int_I (\nabla v(x,s))^{p-2}\Delta v_m(x,s)u_m(x,s)v_m(x,s)^2\,dx \\
\le\;& \left(\int_I ((\nabla v(x,s))^{p-2}\Delta v_m(x,s))^2 dx\right)^{\frac12}\left(\int_I ((\nabla v(x,s))^{p-2}u_m^2(x,s)v_m^4(x,s)\,dx\right)^{\frac12}.
\end{aligned}
$$

The Young inequality gives

$$
\begin{aligned}
&\int_I (\nabla v(x,s))^{p-2}\Delta v_m(x,s)u_m(x,s)v_m(x,s)^2\,dx \\
\le\;& \varepsilon\int_I [(\nabla v(x,s)]^{p-2}\Delta v_m(x,s))^2 dx + C(\varepsilon)\int_I ((\nabla v(x,s))^{p-2}u_m^2(x,s)v_m^4(x,s)\,dx.
\end{aligned}
$$

In addition, the Burkholder–Davis–Gundy inequality and Hypothesis 2.1 give

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t \int_I \sigma_v(\nabla v_m(x,s))^{p-1}\nabla v_m(x,s)dW_2(x,s)\right|^2 \\
\le\;& \sigma_v S(\gamma_2)\mathbb{E}\left(\int_0^t \left|(\nabla v_m(s))^{p-1}\nabla v_m(s)\right|^2_{L^2}ds\right)^{\frac12}.
\end{aligned}
$$

The Hölder inequality, Sobolev embedding, and the Young inequality give

$$
\ldots \;\le\; C(\varepsilon)\sigma_v^2 S^2(\gamma_2)\mathbb{E}\left(\int_0^t |\nabla v_m(s)|^p_{L^p}\,ds\right) + \varepsilon\sigma_v^2 S^2(\gamma_2)\mathbb{E}\left(\sup_{0\le s\le t}|\nabla v_m(s)|^p_{L^p}\right).
$$

Taking ε small enough, the second term can be cancelled with the left hand side of equation (2.18). Finally, estimate (**??**) gives

$$
\mathbb{E}\int_0^t \mathrm{Tr}[\Phi(v_m)[M(v_m(s))Q^{\frac12}][M(v_m(s))Q^{\frac12}]^*\,ds \;\le\; S(\gamma)\,C\mathbb{E}\int_0^t |\nabla v_m(s)|^p_{L^p}\,ds.
$$

Going back to equation (2.18), we obtain

$$\begin{aligned}
&C_1\mathbb{E}\sup_{0\le s\le t}|\nabla v_m(s)|^p_{L^p}+C_2p(p-1)\mathbb{E}\int_0^t\int_I\left|(\nabla v_m)^{p-2}(s)(\Delta v_m(x,s))^2\right|\,dx\,ds\\
\le\;& \mathbb{E}\,|\nabla v_0|^p_{L^p}+C(\varepsilon)\mathbb{E}\int_0^t\int_I((\nabla v_m(x,s))^{p-2}u_m^2(x,s)v_m^4(x,s)\,dx\,ds\\
&-p(f+k)\mathbb{E}\int_0^t\int_I(\nabla v_m(x,s))^{p-1}\nabla v_m(x,s)\,dx\,ds\\
&+\sigma_vC(\varepsilon)\mathbb{E}\int_0^t\int_I\left|(\nabla v_m(x,s))^{p-2}u_m^2(x,s)v_m^4(x,s)\right|^2\,dxds\\
&+(C(\varepsilon)+S(\gamma))\mathbb{E}\int_0^t\int_I|\nabla v_m(x,s)|^p\,dx\,ds.
\end{aligned}$$

Observe that the terms

$$\int_0^t\int_I((\nabla v(x,s))^{p-2}u_m^2(x,s)v_m^4(x,s)\,dx\,ds$$

and

$$\mathbb{E}\int_0^t\int_I\left|(\nabla v_m(x,s))^{p-2}v_m^2(x,s)\right|^2dx\,ds$$

can be estimated by Claim 2.2-(iii). Gronwall's Lemma gives the assertion.

□

Step (vi) In the next step we will define the stopping time depending on the $C_b(I)$–norm of the solutions process. However, in order that these stopping times are well defined we have to verify that the solutions processes (u_m, v_m) are $\mathbb{P}$–a.s. continuous in $C_b(I)$. This is done by showing that (u_m, v_m) are $\mathbb{P}$–a.s. continuous in $H^\delta_4(I)$, where $\delta < 1$. Since $d < 3$, the continuity in $C_b(I)$ follows by embedding Theorems.

Claim 2.4. *For any $\delta < 1$, there exists a function $C : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ such that $C(h) \to 0$ as $h \to 0$ and*

$$\mathbb{E}\sup_{t\le s\le(t+h)\wedge T}|u_m(s)-u_m(t)|^4_{L^4}\le C(h)\left(1+\mathbb{E}|\nabla u_m(t)|^4_{L^4}\right),\; m\in\mathbb{N},\; t\in[0,T].$$

For any $\delta < 1$, there exists a function $C : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ such that $C(h) \to 0$ as $h \to 0$ and

$$\mathbb{E}\sup_{t\le s\le(t+h)\wedge T}|v_m(s)-v_m(t)|^4_{L^4}\le C(h)\left(1+\mathbb{E}|v_m(t)|^4_{L^4}\right),\; m\in\mathbb{N},\; t\in[0,T].$$

Let us assume by the time being that Claim 2.4 is true. Fist, we have by interpolation of $H^\delta_4(I)$ for every $s \in [t,(t+h)\wedge T$

$$\begin{aligned}
|u_m(s)-u_m(t)|_{H^\delta_4}&\le |u_m(s)-u_m(t)|^{1-\delta}_{L^4}|\nabla u_m(s)-\nabla u_m(t)|^\delta_{L^4}\\
&\le |u_m(s)-u_m(t)|^{1-\delta}_{L^4}\left(|\nabla u_m(s)|_{L^4}+|\nabla u_m(t)|^\delta_{L^4}\right).
\end{aligned}$$

In addition, let q and q' be integers such that $2 \le q, q' < \infty$ and $\frac{1}{q}+\frac{1}{q'}=1$. We take the supremum for every $s \in [t,(t+h)\wedge T$, the expectation, and we use the Hölder inequality on the RHS to get

$$\begin{aligned}
\mathbb{E}\sup_{t\le s\le(t+h)\wedge T}|u_m(s)-u_m(t)|^\gamma_{H^\delta_4}&\le\left(\mathbb{E}\sup_{t\le s\le(t+h)\wedge T}|u_m(s)-u_m(t)|^{\gamma(1-\delta)q}_{L^4}\right)^{\frac{1}{q}}\\
&\times\left(\mathbb{E}\sup_{t\le s\le(t+h)\wedge T}\left(|\nabla u_m(s)|^\delta_{L^4}+|\nabla u_m(t)|^\delta_{L^4}\right)^{\gamma q'}\right)^{\frac{1}{q'}}.
\end{aligned}$$

In the last line we used the identity $(a+b)^n \le C(n)(a^n+b^n)$, $n\in\mathbb{N}$. Now we fix γ, q, and q', such that $q(1-\delta)\gamma\le 4$ and $\gamma q'\delta \le 4$. Under these conditions the RHS can be estimated by Claim 2.4 and Claim 2.3. In particular, if $\gamma=q=q'=2$ we have

$$\mathbb{E}\sup_{t\le s\le (t+h)\wedge T} |u_m(s)-u_m(t)|^2_{H^\delta_4} \le C(h)\left(1+\mathbb{E}|\nabla u_m(t)|^4_{H^1_4}\right),\ m\in\mathbb{N},\ t\in[0,T],$$

and

$$\mathbb{E}\sup_{t\le s\le (t+h)\wedge T} |v_m(s)-v_m(t)|^2_{H^\delta_4} \le C(h)\left(1+\mathbb{E}|\nabla v_m(t)|^4_{H^1_4}\right),\ m\in\mathbb{N},\ t\in[0,T]$$

Proof. The proof is similar to the proof of the Claim 2.4. Without restriction to the general case we consider the time interval $[0,h\wedge T]$.

Again an application of the Itô formula to $\Phi(x)=|x|^4_{L^4}$ gives

$$\begin{aligned}
&|v_m(t)-v_0|^p_{L^p} + p(p-2)\int_0^t\int_I (v_m(s)-v_0)^{p-2}(x,s)(\Delta v_m(x,s)-\Delta v(0,x))^2\,dx\,ds\\
\le\ & p(p-2)\int_0^t\int_I (v_m(s)-v_0)^{p-2}(x,s)[\Delta v(0,x)]^2\,dx\,ds\\
&+p\int_0^t\int_I (v_m(x,s)-v_0(0,x))^{p-1}[u_m(x,s)\,v_m^2(x,s)]\,dx\,ds\\
&-p(f+k)\int_0^t\int_I (v_m(x,s)-v_0(0,x))^{p-1}v_m(x,s)\,dx\,ds\\
&+p\int_0^t\int_I \sigma_v (v_m(x,s)-v_0(0,x))^{p-1}v_m(x,s)dW_2(x,s)\\
&+\sigma_v\int_0^t \operatorname{Tr}\left(D^2\Phi(v_m(s)-v_0(s))[M(v_m(s))Q^{\frac12}][M(v_m(s))Q^{\frac12}]^*\right)\,ds.
\end{aligned}$$

The Young inequality gives for $p=4$

$$\left|\int_0^t\int_I (v_m(t)-v_0)^{p-2}(x,s)[\Delta v_0(0,x)]^2\,dx\,ds\right| \le \int_0^t |v_m(s)-v_0|^4_{L^4}\,ds + t\,|\Delta v_0|^4_{L^4}$$

Next,

$$\begin{aligned}
&\int_I (v_m(x,s)-v_0(0,x))^{p-1}[u_m(x,s)\,v_m^2(x,s)]\,dx\\
&\quad\le\ C_1|v_m(s)-v_0]|^p_{L^p} + C_2\left(1+|\nabla u_m(s)|^2_{L^2}\right) + C_3\left(1+|v_m(s)|^4_{L^4}\right),
\end{aligned}$$

and

$$\left|\int_I (v_m(x,s)-v(0,x))^{p-1}v_m(x,s)\,dx\right| \quad\le\quad |v_m(s)-v_0|^p_{L^p}+|v_m(s)|^p_{L^p}.$$

By Claim 2.3 it follows that there exists a $t>0$ such that $\mathbb{E}\int_0^t|\nabla u_m(s)|^p_{L^p}\,ds\le C\mathbb{E}|\nabla u_0|^2_{L^2}\,t$. Besides, by Claim 2.2 $\mathbb{E}\int_0^t|v_m(s)|^4_{L^4}\,ds\le C|v_0+u_0|^4_{L^4}\,t$. In addition, the Burkholder–Davis–Gundy inequality and Hypothesis 2.1 give

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\int_I \sigma_v(v_m(x,s)-v_0(0,x))^{p-1}v_m(x,s)dW_2(x,s)\right|^2\\
&\quad\le\ \sigma_v S(\gamma)\,\mathbb{E}\left(\int_0^t \left|(v_m(s)-v_0)^{p-1}v_m(s)\right|^2_{L^2}\,ds\right)^{\frac12}.
\end{aligned}$$

By similar calculation as in the step before we get

$$\ldots \leq \sigma_v S(\gamma)\,\mathbb{E}\left(\int_0^t |v_m(s)-v_0|_{L^4}^2|v_m-v_0|_{L^4}^2|\nabla v_m(s)|_{L^2}^2\,ds\right)^{\frac12}$$

The Young inequality gives

$$\begin{aligned}\ldots &\leq \frac{\sqrt{t}}{4}\,\mathbb{E}\sup_{0\le s\le t}|v_m(s)-v_0|_{L^4}^2 + C\sqrt{t}\left(\mathbb{E}\int_0^t |\nabla v_m(s)|_{L^2}^2\,ds\right)^{\frac12}\\ &\leq \frac{\sqrt{t}}{4}\mathbb{E}\left(\sup_{0\le s\le T}|v_m(s)-v_0|_{L^4}^4\right)+C\,t\,(1+\sup_{0\le s\le t}\mathbb{E}\,|\nabla v_m(s)|_{L^2}^2\,.\end{aligned}$$

Observe, the first term can be cancelled with the left hand side of equation (2.18). Due to Claim 2.3, the second term is controlled by $C\,t$. Finally, we obtain by estimate (**??**)

$$\begin{aligned}&\int_0^t \mathbb{E}\mathrm{Tr}\left(D^2\Phi(v_m(s)-v_0(s))[M(v_m(s))Q^{\frac12}][M(v_m(s))Q^{\frac12}]^*\right)ds\\ &\quad\leq C\int_0^t \mathbb{E}|v_m(x,s)-v_0(x)|^{p-2}v_m^2(x,s)\,dx\,ds.\end{aligned}$$

Applying the Hölder inequality gives

$$\begin{aligned}&\int_0^t \mathbb{E}\mathrm{Tr}\left(D^2\Phi(v_m(s)-v_0(s))[M(v_m(s))Q^{\frac12}][M(v_m(s))Q^{\frac12}]^*\right)ds\\ &\quad\leq C\int_0^t \mathbb{E}|v_m(s)-v_0|_{L^4}^4\,ds+\int_0^t \mathbb{E}|v_m(s)|_{L^4}^4\,ds.\end{aligned}$$

Collecting all together and analysing term by term, the assertion is shown. □

Step (vii) Let $\tau_m^u := \{t\in(0,T] : |u_m(t)|_{C_b}\ge m\}$, $\tau_m^v := \{t\in(0,T] : |v_m(t)|_{C_b}\ge m\}$ and $\tau_m := \min(\tau_m^u,\tau_m^v)$. In this step we will show that for $m\to\infty$ we have $\mathbb{P}(\tau_m<T)\to 0$. Observe, that for $\delta<1$ the trajectories $[0,T]\ni t\mapsto (u(t),v(t))\in H_4^\delta(I)\times H_4^\delta(I)$ are continuous. Besides, due to the fact that $H_4^\delta(I)\hookrightarrow C_b(I)$ for $\frac d4<\delta\le$, the trajectories $[0,T]\ni t\mapsto (u(t),v(t))\in C_b(I)\times C_b(I)$ are continuous and the stopping times are well defined. In addition, the estimate on u_m and v_m in Claim 2.3 were independent of m. Hence, for all $\delta\in\mathbb{R}$ with $\frac d4<\delta<1$ there exists a constant $C>0$ such that

$$\mathbb{E}\sup_{0\le t\le T}|u_m(t)|_{H_4^\delta}^4,\mathbb{E}\sup_{0\le t\le T}|v_m(t)|_{H_4^\delta}^4\le C,\quad m\in\mathbb{N}.$$

Due to the embedding $H_p^\delta(I)\hookrightarrow C_b(I)$, there exists a constant $C>0$ such that

$$\mathbb{E}\sup_{0\le t\le T}|u_m(t)|_{C_b}^4,\mathbb{E}\sup_{0\le t\le T}|v_m(t)|_{C_b}^4\le C,\quad m\in\mathbb{N}.$$

Let us define the stopping time

$$\tau_m^u := \inf_{t\ge0}\{|u_m(t)|_{H_4^\delta}\ge m\}\quad\text{and}\quad \tau_m^v := \inf_{t\ge0}\{|v_m(t)|_{H_4^\delta}\ge m\}.$$

By the definition of g_m it follows that for $s\le\tau_m:=\min(\tau_m^u,\tau_m^v)$ we get

$$F_m(u(s),v(s))=F_{m+1}(u(s),v(s))=F(u(s),v(s))$$

where $F: L^2(I)\times L^\infty(I)\to L^2(I)$ is the Nemityski operator defined by

$$F(u,v)(x):=f(u(x),v(x))=u(x)v^2(x).$$

Hence, on $[0,\tau_m)$ the processes (u_m,v_m) and (u_{m+1},v_{m+1}) are identical. In addition, we have $\tau_m\le\tau_{m+1}$ for all $m\in\mathbb{N}$. Fix $m\in\mathbb{N}$ and put

$$\Omega_m := \{\omega\in\Omega : |u_m(s)|_{C_b}\le m \text{ and } |v_m(s)|_{C_b}\le m\}\,.$$

It is straightforward that there exists a progressively measurable process (u_m, v_m) over $\mathfrak{A} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that (u_m, v_m) solves $\mathbb{P}$–a.s. the integral equation given by (2.7) up to time τ_m. In particular, we have for the conditioned probability

$$\mathbb{P}\left(\{\text{a solution } u \text{ to (2.7) exists}\} \mid \Omega_m\right) = 1.$$

Hence, for any $m \in \mathbb{N}$ we can glue together the solution to one process (u, v) with

$$u(t) = u_m(t) \text{ and } v(t) = v_m(t) \text{ when } t \in [\tau_{m-1} \wedge T, \tau_m \wedge T).$$

Then, it is straightforward to verify, that

$$\begin{aligned} &\mathbb{P}\left(\{\text{there exists solution to (2.7)}\}\right) \\ &\quad = \lim_{m\to\infty} \mathbb{P}\left(\{\text{a solution } u \text{ to (2.7) exists}\} \mid \Omega_m\right) \mathbb{P}\left(\Omega_m\right). \end{aligned}$$

Since $\mathbb{P}\left(\{\text{a solution } u \text{ to (2.7) exists}\} \mid \Omega_m\right) = 1$, it remains to show that $\lim_{m\to\infty} \mathbb{P}(\Omega_m) = 1$. Then, as $\Omega_m \supset \Omega_{m+1}$, it follows automatically that

$$\mathbb{P}\left(\{\text{there exists solution to (2.7)}\}\right) = 1.$$

However, since there exists a constant $C(T) > 0$ such that

$$\mathbb{E} \sup_{0\leq t\leq T} |u_m(t)|^4_{C_b}, \mathbb{E} \sup_{0\leq t\leq T} |u_m(t)|^4_{C_b} \leq C(T), \quad m \in \mathbb{N},$$

and, hence,

$$\mathbb{P}\left(\Omega \setminus \Omega_m\right) \leq \frac{C(T)}{m^4} \to 0.$$

The solution process is well defined on $\mathcal{A} = \lim_{m\to\infty} \mathcal{A}_m$, where $\mathbb{P}(\mathcal{A}) = 1$. In addition, by Claim 2.2 and the positivity of the solution follows item (i), (ii) and (iii).

□

3. Numerical simulations

In this section, we illustrate the formation of patterns in the deterministic Gray–Scott equations and their variation under stochastic perturbations; for this purpose, we employ operator splitting methods which have been applied successfully to stochastic systems, see for instance [4, 5, 6, 7, 8, 11, 24, 22].

Operator splitting methods for deterministic evolution equations. The approach of operator splitting may lead to favourable discretisations for various classes of deterministic evolution equations, see [23] and references given therein. We mention the works [43, 44], which illustrate the use of operator splitting methods in the context of nonlinear Schrödinger equations and confirm that time-adaptivity enhances reliability and efficiency of the numerical simulations; in [44], it is also demonstrated that Fourier spectral space approximations, although constrained to uniform meshes, are superiour to locally adaptive finite element space discretisations, due to the retained spectral convergence rate and the applicability of fast Fourier transform techniques. We expect similar conclusions to hold for diffusion-reaction equations with pattern formation, requiring as well high resolution in space and time, and hence favour the Fourier spectral method over the finite difference and finite element methods; we point out once again that the simple structure of the space domain and the imposed periodic boundary conditions permit solution representations by Fourier series expansion.

Lie–Trotter splitting method. As indicated by the nomenclature, operator splitting methods employ a decomposition of the defining operator into two parts

$$\begin{cases} U'(t) = F\big(U(t)\big) = F_1\big(U(t)\big) + F_2\big(U(t)\big), \quad t \in (0, T), \\ U(0) = U_0; \end{cases}$$

for each subinterval, defined by a suitably chosen time stepsize $h_n > 0$, the associated subproblems are solved separately

$$V_1'(t) = F_1\big(V_1(t)\big), \quad V_2'(t) = F_2\big(V_2(t)\big), \quad t \in (t_n, t_n + h_n),$$

potentially with specific numerical solvers. With regard to the low regularity of the solution to a stochastic evolution equation, we restrict ourselve to the presentation of the first-order Lie–Trotter splitting method; here, starting from a certain initial approximation $U_n \approx U(t_n)$, the solutions to the two subproblems are composed, which yields the following approximation to the exact solution value

$$\begin{cases} V_1'(t) = F_1\big(V_1(t)\big)\,, \quad t \in (t_n, t_n + h_n)\,, \\ V_1(t_n) = U_n\,, \end{cases}$$
$$\begin{cases} V_2'(t) = F_2\big(V_2(t)\big)\,, \quad t \in (t_n, t_n + h_n)\,, \\ V_2(t_n) = V_1(t_n + h_n)\,, \end{cases}$$
$$U_{n+1} = V_2(t_n + h_n) \approx U(t_n + h_n)\,.$$

Higher-order approximations are specified in [43, 44].

Deterministic Gray–Scott equations. The following compact formulation of the deterministic Gray–Scott equations as an evolution equation

$$A_u = r_u\,\Delta - \alpha_u\,, \quad A_v = r_v\,\Delta - \alpha_v\,, \quad g(u,v) = u\,v^2\,,$$
$$U(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \quad A_U = \begin{pmatrix} A_u & 0 \\ 0 & A_v \end{pmatrix}, \quad G\big(U(t)\big) = \begin{pmatrix} \alpha_u - g\big(u(t), v(t)\big) \\ g\big(u(t), v(t)\big) \end{pmatrix},$$
$$U'(t) = A_U\,U(t) + G\big(U(t)\big)\,, \quad t \in (0,T)\,, \quad U(0) = U_0\,,$$

makes the natural decomposition of the right-hand side into two parts evident. The solution of the linear subproblem

$$V_1'(t) = A_U\,V_1(t)\,, \quad t \in (t_n, t_n + h_n)\,,$$

comprising two decoupled diffusion equations, is formally given by

$$V_1(t_n + h_n) = \mathrm{e}^{h_n A_U}\,V_1(t_n)\,;$$

for each component, an explicit solution representation based on a Fourier series expansion is available. Our numerical approximation relies on a truncation of the infinite series and an application of the trapezoidal rule on an equidistant space grid; we use the fast Fourier transform and its inverse for an efficient implementation. For the numerical solution of the second subproblem

$$V_2'(t) = G\big(V_2(t)\big)\,, \quad t \in (t_n, t_n + h_n)\,,$$

comprising the nonlinear reaction terms, we employ a standard explicit solver. More precisely, we retain the equidistant space grid used for the discretisation of the linear subproblem; pointwise evaluation at each grid point yields a system of ordinary differential equations, which we resolve by an explicit Runge–Kutta method.

Stochastic Gray–Scott equations. Accordingly, we rewrite the stochastic Gray–Scott equations with stochastic integral interpreted as an Itô integral as

$$\Sigma\big(U(t)\big) = \begin{pmatrix} \sigma_u\,u(t) & 0 \\ 0 & \sigma_v\,v(t) \end{pmatrix}, \quad W_U(t) = \begin{pmatrix} W_u(t) \\ W_v(t) \end{pmatrix}, \quad t \in [0,T]\,,$$
$$\mathrm{d}U(t) = \Big(A_U\,U(t) + G\big(U(t)\big)\Big)\,\mathrm{d}t + \Sigma\big(U(t)\big)\,\mathrm{d}W_U(t)\,, \quad t \in (0,T)\,, \quad U(0) = U_0\,.$$

Here, we consider the linear subproblem

$$\mathrm{d}V_1(t) = A_U\,V_1(t)\,\mathrm{d}t + \Sigma\big(V_1(t)\big)\,\mathrm{d}W_U(t)\,, \quad t \in (t_n, t_n + h_n)\,;$$

in view of the formal representation for the mild solution

$$V_1(t_n + h_n) = \mathrm{e}^{h_n A_U}\,V_1(t_n) + \int_0^{h_n} \mathrm{e}^{(h_n - s)A_U}\,\Sigma\big(V_1(t_n + s)\big)\,\mathrm{d}W_U(t_n + s)\,,$$

we employ the following approximation of the Itô integral

$$V_1(t_n + h_n) \approx \mathrm{e}^{h_n A_U}\,\Big(V_1(t_n) + \Sigma\big(V_1(t_n)\big)\,\big(W_U(t_n + h_n) - W_U(t_n)\big)\Big)\,.$$

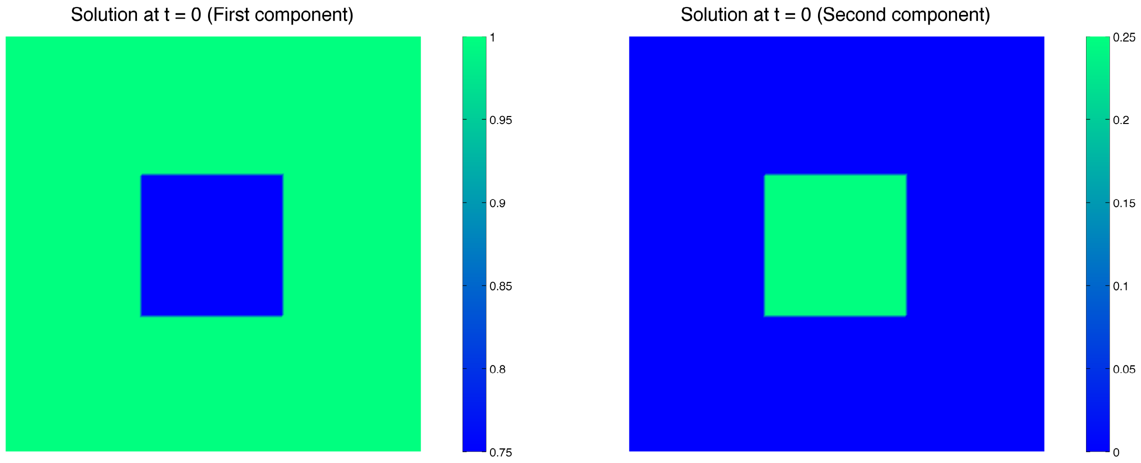

FIGURE 1. Deterministic and stochastic Gray–Scott equations. Prescribed initial states.

For the realisation of the increment $W_U(t_n + h_n) - W_U(t_n)$, we generate normally distributed numbers and apply the inverse Laplacian $(1 - \Delta)^{-\gamma}$; this as well as the action of the evolution operator $\mathrm{e}^{h_n A_U}$ is implemented by fast Fourier transforms.

Time stepsize control. In our implementation of operator splitting methods for the deterministic and stochastic Gray–Scott equations, we take into account that reaction equations may show blow-up and that limitations of the time stepsizes are needed to prevent failure of the time integration; that is, in order to enhance efficiency in the deterministic case and reliability in the stochastic case, respectively, we incorporate the possibility to increase or decrease the time stepsize accordingly. For the deterministic Gray–Scott equations, we apply adaptive splitting methods as described in [44]. For the stochastic Gray–Scott equations, our pathwise local error control in essence uses that the solution to the ordinary differential equation associated with the second component of the reaction term is well-defined on a finite time interval

$$\xi'(t) = \eta\, \xi^2(t)\,, \quad \xi(t) = \frac{\xi(t_n)}{1 - (t - t_n)\,\eta\, \xi(t_n)}\,, \quad t \in \big[t_n, t_n + \tfrac{1}{\eta\, \xi(t_n)}\big)\,, \quad \eta > 0\,, \quad \xi(t_n) > 0\,;$$

this explains the stepsize restriction $h_n \leq \alpha \, \frac{1}{\eta\, \xi(t_n)}$ with a certain safety factor $\alpha \in (0, 1)$, where the constant $\eta > 0$ corresponds to the maximum value of v on the space grid at time t_n.

Numerical results. In Figure 1, we display the initial states prescribed for the two-dimensional deterministic and stochastic Gray–Scott equations. For two exponents $\gamma > 0$, the effect of the inverse Laplacian $(1 - \Delta)^{-\gamma}$ on a set of normally distributed numbers is illustrated in Figure 2. Realisations of the numerical solution processes are shown in Figures 5-7 for different choices of the parameters $r_u, r_v, \alpha_u, \alpha_v, \sigma_u, \sigma_v, \gamma$; in comparison with the deterministic case, the loss of symmetry is most apparent, see Figures 3-4. In the captions of the figures, we provide the links to movies that visualise the creation of patterns and their variation under the influence of stochastic noise.

Itô versus Stratonovich formulation. We note that the presented numerical results correspond to the Gray–Scott equations with stochastic integral interpreted as Itô integral; in order to obtain the Stratonovich form, the arising parameters have to be adapted.

APPENDIX A. STOCHASTIC INTEGRAL AND AUXILIARY RESULTS

Meanwhile, we denote by $\mathcal{H}$ a separable infinite-dimensional Hilbert space and by $(h_m)_{m \in \mathcal{M}}$ a complete orthonormal system of $\mathcal{H}$. Provided that a stochastic process $(Y(t))_{t \in [0,T]}$ with values in the space of Hilbert–Schmidt operators from $\mathcal{H}$ to another Hilbert space $\widetilde{\mathcal{K}}$ is progressively measurable on the underlying

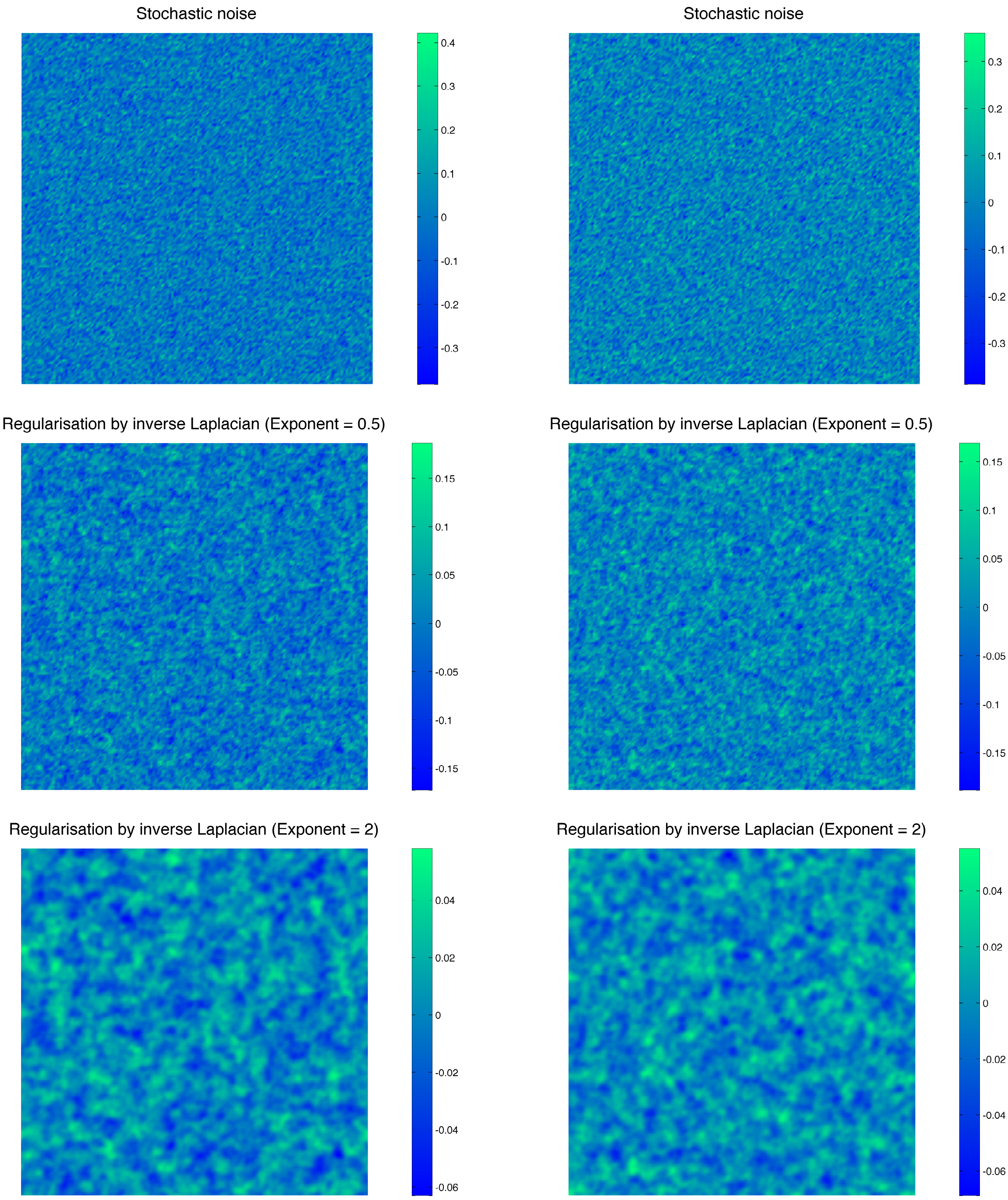

FIGURE 2. Two realisations of stochastic noise and regularisations by powers of inverse Laplacian $(1-\Delta)^{-\gamma}$, $\gamma \in \{0.5, 2\}$.

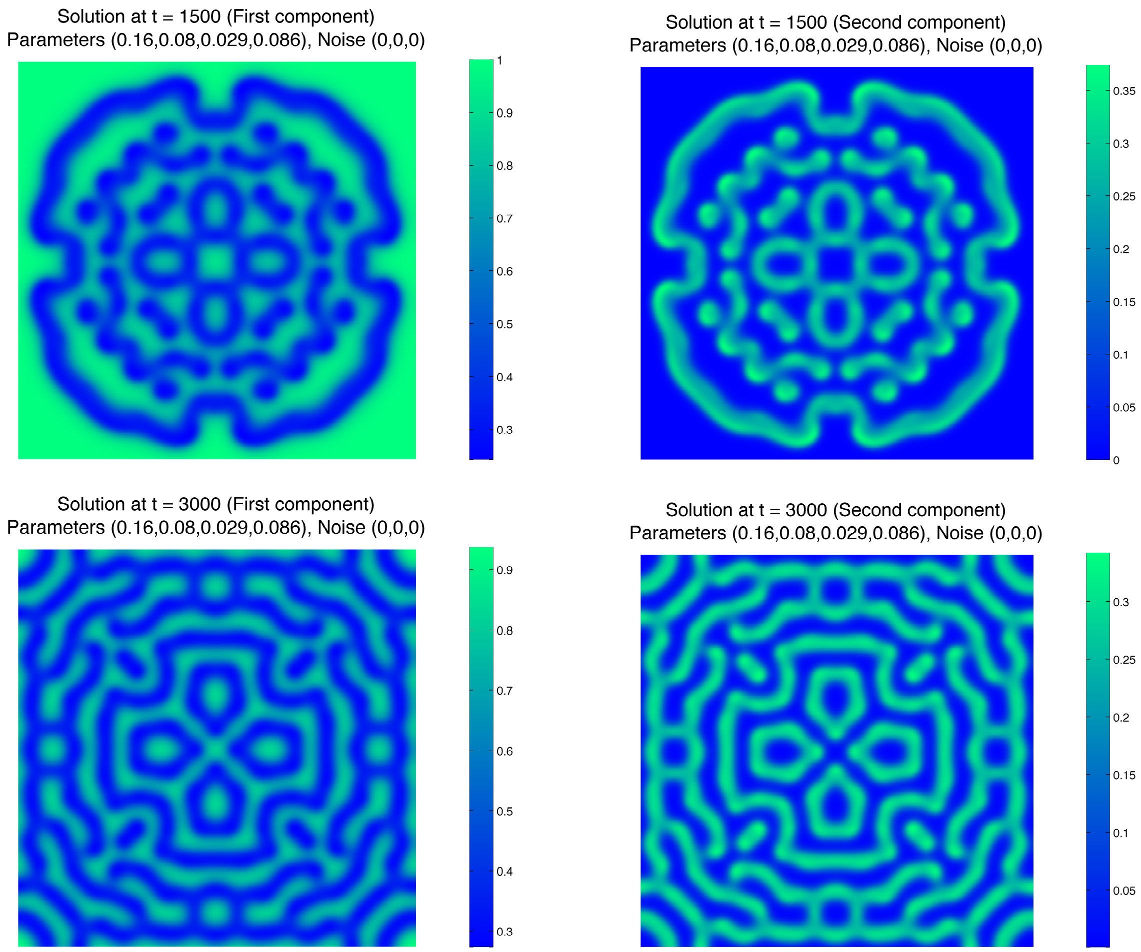

FIGURE 3. Deterministic Gray–Scott equations with first choice of parameters $(r_u, r_v, \alpha_u, \alpha_v) = (0.16, 0.08, 0.029, 0.086)$. Components of numerical solution at different times. Movie available at http://techmath.uibk.ac.at/mecht/MyHomepage/Research/GS/MovieMyCase1.mov

probability space and fulfills a certain integrability condition

$$Y : \Omega \times [0,T] \longrightarrow L_{\mathrm{HS}}(\mathcal{H}, \widetilde{\mathcal{K}}), \quad \mathbb{E}\,\big\|Y\big\|^2_{L^2([0,T],L_{\mathrm{HS}}(\mathcal{H},\widetilde{\mathcal{K}}))} < \infty, \tag{A.1a}$$

the stochastic integral, denoted by

$$J : \Omega \times [0,T] \longrightarrow \widetilde{\mathcal{K}} : (\omega, t) \longmapsto \int_0^t Y(\omega, s)\, \mathrm{d}W(\omega, s), \tag{A.1b}$$

is given as the limit of the infinite series

$$\sum_{m \in \mathcal{M}} \int_0^t Y(\omega, s)\, h_m\, \mathrm{d}\big(W(\omega, s)\big|h_m\big)_{\mathcal{H}}$$

in $L^2(\Omega, \widetilde{\mathcal{K}})$ and leads to a well-defined continuous square-integrable martingale in $\widetilde{\mathcal{K}}$. In addition, fundamental results such as the Itô isometry

$$\mathbb{E}\,\big\|J(T)\big\|^2_{\widetilde{\mathcal{K}}} = \mathbb{E}\,\big\|Y\big\|^2_{L^2([0,T],L_{\mathrm{HS}}(\mathcal{H},\widetilde{\mathcal{K}}))} \tag{A.1c}$$

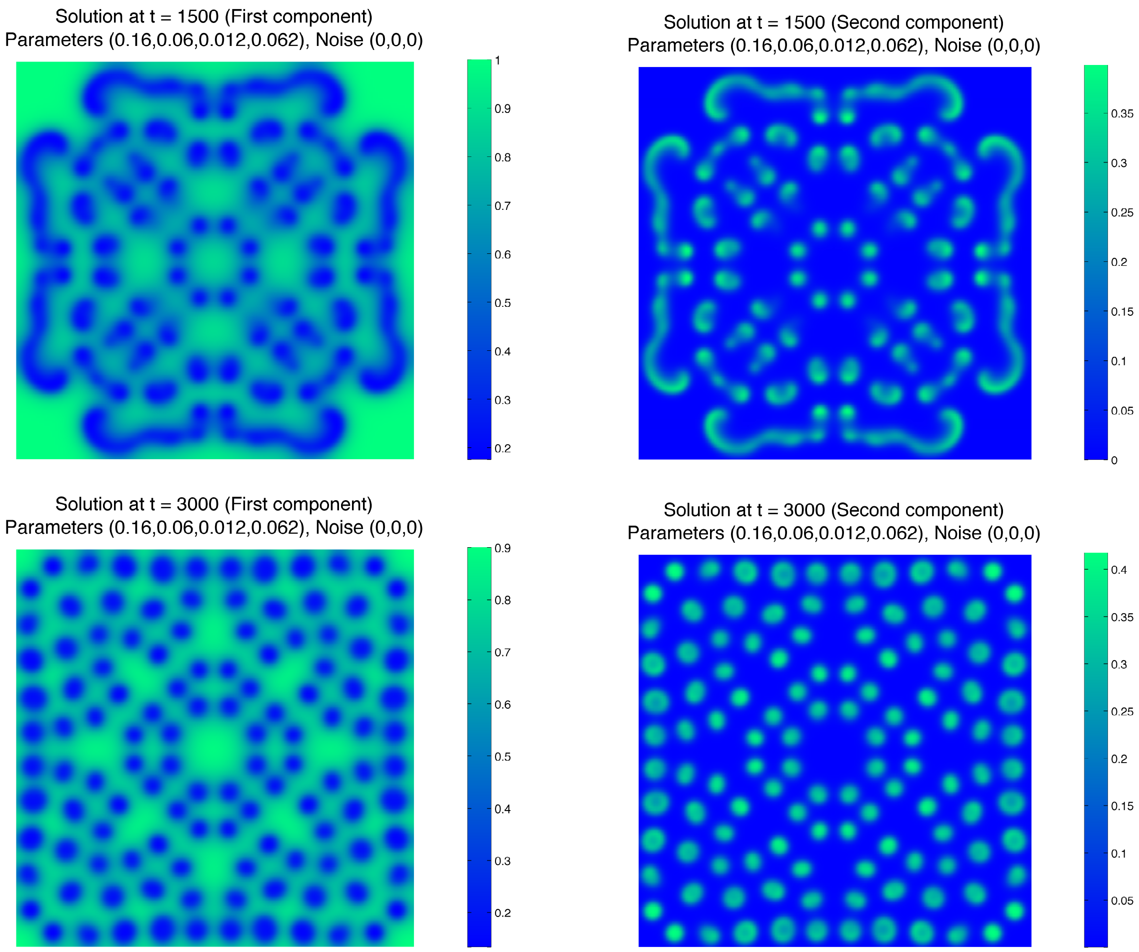

FIGURE 4. Deterministic Gray–Scott equations with second choice of parameters $(r_u, r_v, \alpha_u, \alpha_v) = (0.16, 0.06, 0.012, 0.062)$. Components of numerical solution at different times. Movie available at http://techmath.uibk.ac.at/mecht/MyHomepage/Research/GS/MovieMyCase2.mov

and the Burkholder–Davis–Gundy inequality

$$\mathbb{E} \sup_{t\in[0,T]} \big\|J(t)\big\|_{\widetilde{\mathcal{K}}}^p \leq C_p \, \mathbb{E} \, \big\|Y\big\|^p_{L^2([0,T],L_{\mathrm{HS}}(\mathcal{H},\widetilde{\mathcal{K}}))} \,, \quad p \in [1,\infty)\,, \tag{A.1d}$$

are valid.

In view of the investigation of the stochastic Gray–Scott equations with multiplicative noise, we restrict ourselves to the consideration of the bounded space domain $I = [0,1]^d$ and the Lebesgue space $\mathcal{H} = L^2(I,\mathbb{R})$. In the case of a single dimension, a complete orthonormal system of eigenfunctions associated with the Laplacian is given by sine and cosine functions

$$\psi_m(x) = \begin{cases} \sqrt{2}\,\sin\big(2\pi m x\big) & \text{if } m \geq 1\,, \\ \sqrt{2} & \text{if } m = 0\,, \\ \sqrt{2}\,\cos\big(2\pi m x\big) & \text{if } m \leq -1\,, \end{cases} \qquad x \in [0,1]\,, \quad m \in \mathbb{Z}\,; \tag{A.2}$$

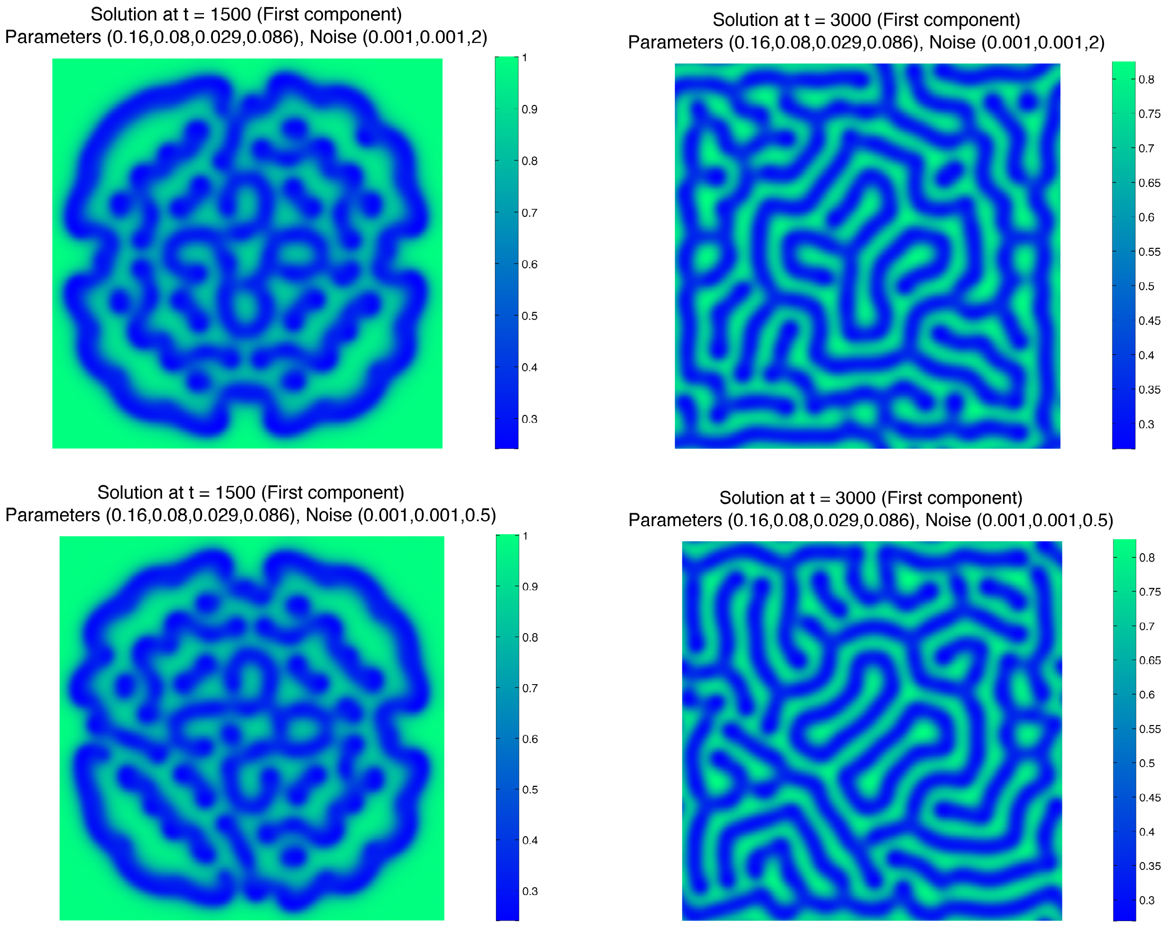

FIGURE 5. Stochastic Gray–Scott equations with parameters $(r_u, r_v, \alpha_u, \alpha_v) = (0.16, 0.08, 0.029, 0.086)$ and $(\sigma_u, \sigma_v, \gamma) = (0.001, 0.001, 2)$ (first row) or $(\sigma_u, \sigma_v, \gamma) = (0.001, 0.001, 0.5)$ (second row), respectively. First component of numerical solution at different times. Movies available at http://techmath.uibk.ac.at/mecht/MyHomepage/Research/GS/ MovieMyCase11.mov & MovieMyCase111.mov

below, in order to simplify calculations, we tacitly use representations based on the complex exponential. The extension to higher space dimensions relies on tensor products

$$\psi_m(x) = \prod_{j=1}^{d} \psi_{m_j}(x_j)\,, \quad x \in I = [0,1]^d\,, \quad m = (m_1, \dots, m_d) \in \mathcal{M} = \mathbb{Z}^d\,, \tag{A.3}$$

and the corresponding eigenvalues associated with the Laplacian are given by

$$\lambda_m = -\,4\pi^2 \sum_{j=1}^{d} m_j^2\,, \quad m = (m_1, \dots, m_d) \in \mathcal{M}\,. \tag{A.4}$$

The considered Wiener processes have the following representation

$$W(x,t) = \sum_{m \in \mathcal{M}} (1 - \lambda_m)^{-\gamma} \psi_m(x)\, \beta_m(t)$$

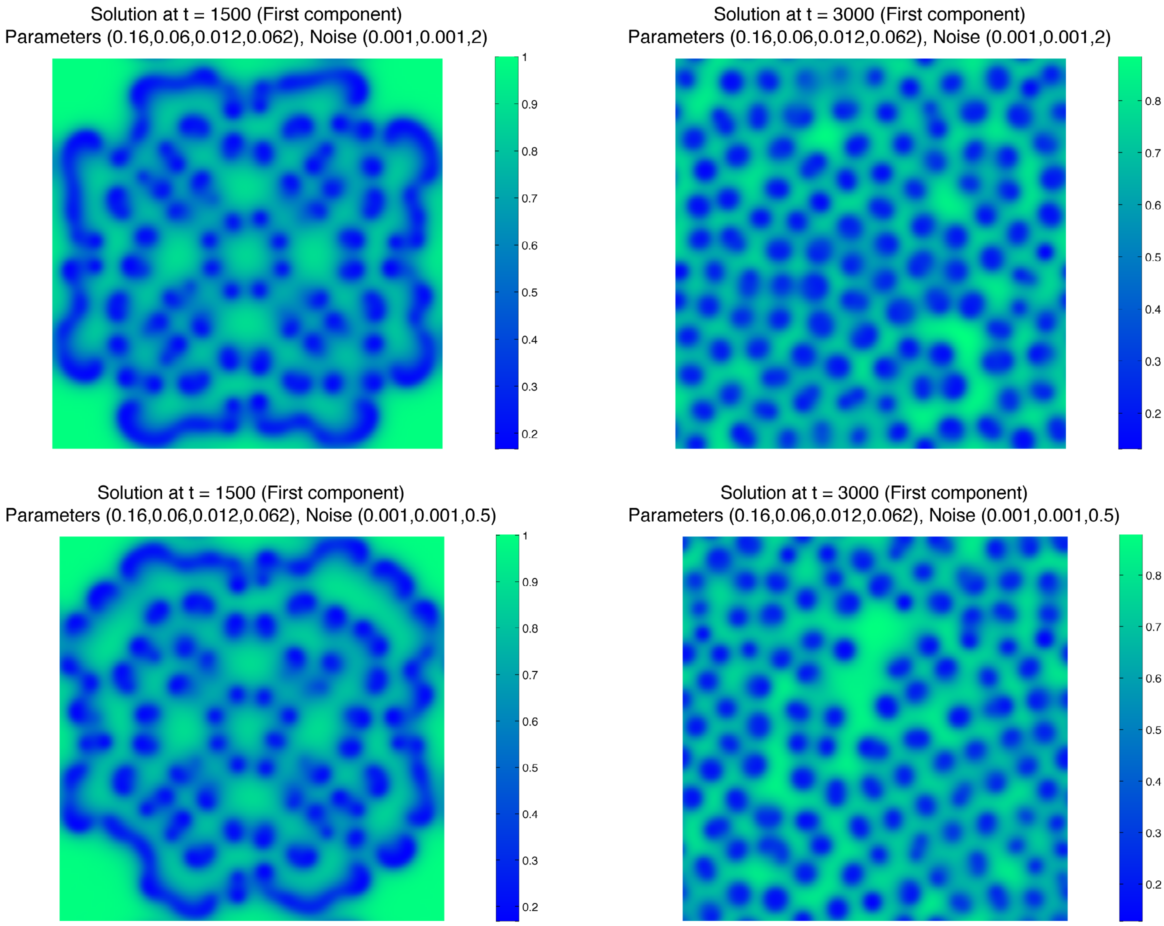

FIGURE 6. Stochastic Gray–Scott equations with parameters $(r_u, r_v, \alpha_u, \alpha_v) = (0.16, 0.06, 0.012, 0.062)$ and $(\sigma_u, \sigma_v, \gamma) = (0.001, 0.001, 2)$ (first row) or $(\sigma_u, \sigma_v, \gamma) = (0.001, 0.001, 0.5)$ (second row), respectively. First component of numerical solution at different times. Movies available at http://techmath.uibk.ac.at/mecht/MyHomepage/Research/GS/ MovieMyCase21.mov & MovieMyCase211.mov

with exponent $\gamma > \frac{d}{2}$, $(\beta_m)_{m\in\mathcal{M}}$ denoting a family of one–dimensional Brownian motions, and covariance operator Q defined by

$$\big(Q\,\psi_\ell\big|\psi_m\big)_{L^2} = (1-\lambda_m)^{-\gamma}\,\delta_{\ell m}\,, \quad \ell, m \in \mathcal{M}\,. \tag{A.5}$$

With regard to concrete representations, it is useful to relate the inner product and norm of fractional Sobolev spaces to fractional Laplace operators. That is, we set

$$\begin{gathered}\big(\phi_1\big|\phi_2\big)_{H_2^\kappa(I)} = \big((1-\Delta)^\kappa\,\phi_1\big|\phi_2\big)_{L^2}\,, \quad \big\|\phi\big\|_{H_2^\kappa} = \big\|(1-\Delta)^{\frac{\kappa}{2}}\,\phi\big\|_{L^2}\,, \\ \phi, \phi_1, \phi_2 \in H_2^\kappa(I,\mathbb{R})\,, \quad \kappa \in \mathbb{R}\,.\end{gathered} \tag{A.6}$$

For scaled Fourier functions, we employ the abbreviation

$$\psi_m^{(\kappa)} = (1-\lambda_m)^{-\frac{\kappa}{2}}\,\psi_m\,, \quad m \in \mathcal{M}\,. \tag{A.7}$$

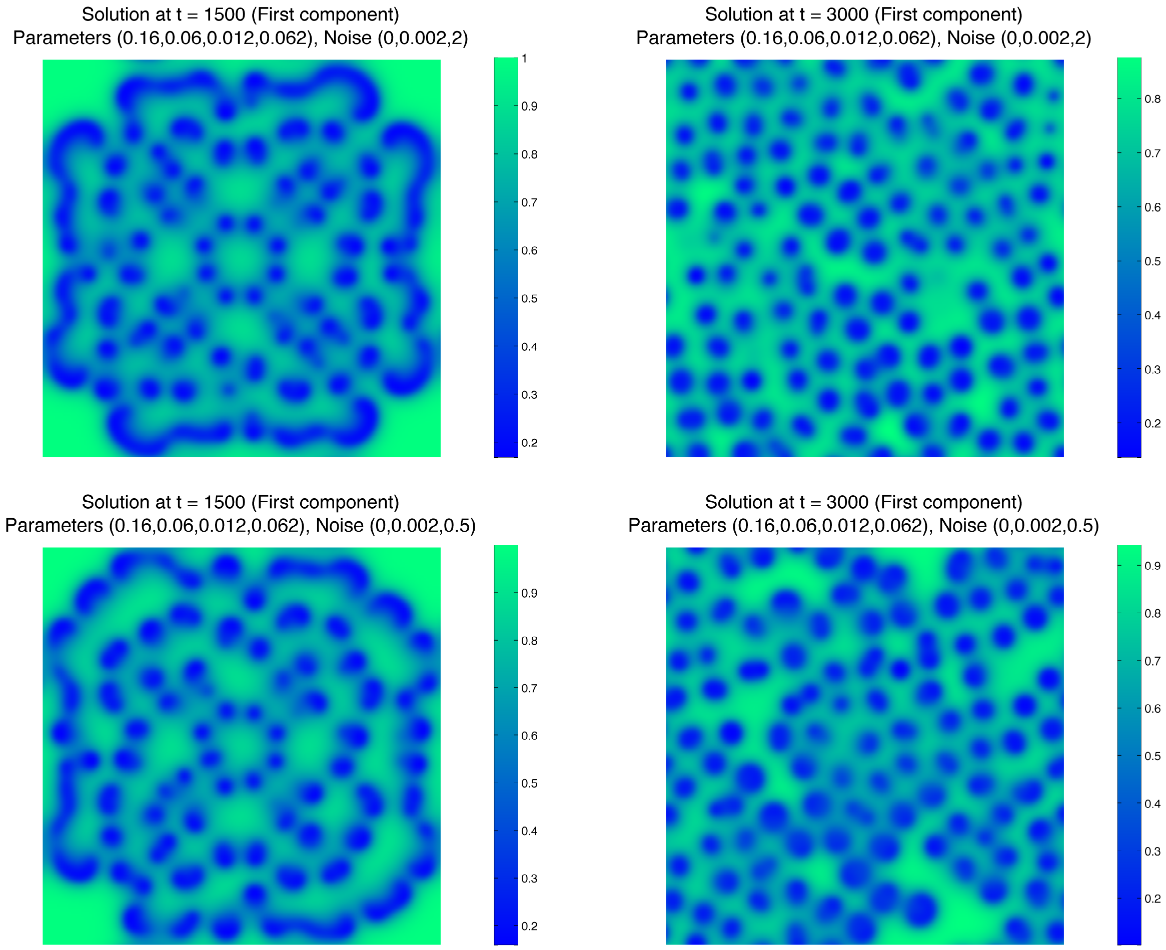

FIGURE 7. Stochastic Gray–Scott equations with parameters $(r_u, r_v, \alpha_u, \alpha_v) = (0.16, 0.06, 0.012, 0.062)$ and $(\sigma_u, \sigma_v, \gamma) = (0, 0.002, 2)$ (first row) or $(\sigma_u, \sigma_v, \gamma) = (0, 0.002, 0.5)$ (second row), respectively. First component of numerical solution at different times. Movies available at http://techmath.uibk.ac.at/mecht/MyHomepage/Research/GS/MovieMyCase22.mov & MovieMyCase221.mov

The eigenvalue relation

$$(1-\Delta)^{\frac{\kappa}{2}}\,\psi_m = (1-\lambda_m)^{\frac{\kappa}{2}}\,\psi_m\,, \quad m \in \mathcal{M}\,, \tag{A.8}$$

implies that $(\psi_m^{(\kappa)})_{m\in\mathcal{M}}$ forms a complete orthonormal system of the fractional Sobolev space $H_2^\kappa(I,\mathbb{R})$, that is

$$\big(\psi_\ell^{(\kappa)}\big|\psi_m^{(\kappa)}\big)_{H_2^\kappa} = \delta_{\ell m}\,, \quad \ell, m \in \mathcal{M}\,. \tag{A.9}$$

A.1. **The multiplication operator.** In our equation, the diffusion coefficient in front of the stochastic perturbation is given by the multiplication operator defined by a function ϕ, which is interpreted as a mapping from the Hilbert space $\mathcal{H}$ to the other Hilbert space $L^2(I)$. To be more precise,

$$M(\phi) : \mathcal{H} \longrightarrow \mathcal{K} : \chi \longmapsto \phi\,\chi\,, \tag{A.10}$$

which complies with [36, Eq. (1.4)]; within the article, however, we often wrote

$$\phi = M(\phi)$$

for short. Let $\gamma > \frac{d}{2}$ and $\mathcal{H} = H_2^\gamma(I)$. Besides, let us denote the orthonormal basis in $\mathcal{H}$ by $\{\psi_m^{(\gamma)} : m \in \mathbb{Z}\}$, which is given by

$$\psi_m^{(\gamma)} = (1-\lambda_m)^{-\frac{\gamma}{2}}\,\psi_m\,, \quad m \in \mathcal{M}\,, \tag{A.11}$$

Then arguments detailed below show that for the particular case

$$M(\phi) : \mathcal{H} \longrightarrow L^2(I) : \chi \longmapsto \phi\,\chi\,, \quad \phi \in L^2(I)\,, \quad \gamma > \tfrac{d}{2}\,, \tag{A.12}$$

the associated Hilbert–Schmidt norm is finite, since the estimate

$$\big\|M(\phi)\big\|_{L_{\mathrm{HS}}(\mathcal{H},L^2)} = \Big(\sum\nolimits_{m\in\mathcal{M}} \big\|\phi\,\psi_m^{(\gamma)}\big\|_{L^2}^2\Big)^{\frac{1}{2}} \leq C\,\sqrt{S(\gamma)}\,\big\|\phi\big\|_{L^2}\,, \quad \gamma > \tfrac{d}{2}\,, \tag{A.13}$$

holds. Indeed, the stated bound is obtained from a representation of the defining real-valued function with respect to the complex-valued Fourier functions

$$\phi = \sum_{\ell\in\mathcal{M}} \big(\phi\big|\psi_\ell\big)_{L^2}\,\psi_\ell\,.$$

Let $\gamma > \frac{d}{2}$ and $\mathcal{H} = H_2^\gamma(I)$. Besides, let us denote the orthonormal basis in $\mathcal{H}$ by $\{\psi_m^{(\gamma)} : m \in \mathbb{Z}\}$, which is given by

$$\psi_m^{(\gamma)} = (1-\lambda_m)^{-\frac{\gamma}{2}}\,\psi_m\,, \quad m \in \mathcal{M}\,, \tag{A.14}$$

The associated Hilbert–Schmidt norm is finite. To be more precise, since the estimate

$$\big\|M(\phi)\big\|_{L_{\mathrm{HS}}(\mathcal{H},L^2)} = \Big(\sum\nolimits_{m\in\mathcal{M}} \big\|\phi\,Q^{\frac{1}{2}}\psi_m\big\|_{L^2}^2\Big)^{\frac{1}{2}} \leq C\,\sqrt{S(\gamma)}\,\big\|\phi\big\|_{L^2}\,, \quad \gamma > \tfrac{d}{2}\,, \tag{A.15}$$

holds. What happens if the underlying Hilbert space is $H_2^\rho(I)$ instead of $L^2(I)$.

We can write ϕ in terms of the orthonormal basis, i.e.

$$\phi = \sum_{\ell\in\mathcal{M}} \big(\phi\big|\psi_\ell\big)_{L^2}\,\psi_\ell\,.$$

In case, the underlying space is $H_2^\delta(I)$, we get

$$\big\|M(\phi)\big\|_{L_{\mathrm{HS}}(\mathcal{H},L^2)} = \Big(\sum\nolimits_{m\in\mathcal{M}} \big\|\phi\,Q^{\frac{1}{2}}\psi_m\big\|_{L^2}^2\Big)^{\frac{1}{2}} \leq C\,\sqrt{S(\gamma)}\,\big\|\phi\big\|_{L^2}\,, \quad \gamma > \tfrac{d}{2}\,, \tag{A.16}$$

For the following considerations, it is convenient to represent sine and cosine functions by means of the complex exponential; the corresponding complete orthonormal set is meanwhile denoted by $(\chi_m)_{m\in\mathcal{M}}$ and satisfies the relation $\chi_\ell\,\chi_m = \chi_{\ell+m}$, which implies

$$\begin{aligned}\phi\,\chi_m^{(\gamma)} &= \sum_{\ell\in\mathcal{M}} \big(\phi\big|\chi_\ell\big)_{L^2}\,\chi_\ell\,\chi_m^{(\gamma)}\\ &= \sum_{\ell\in\mathcal{M}} (1-\lambda_m)^{-\frac{\gamma}{2}}\,\big(\phi\big|\chi_\ell\big)_{L^2}\,\chi_{\ell+m}\,, \quad m \in \mathcal{M}\,.\end{aligned}$$

Parseval's identity, summation, and an integrability criterium for infinite series confirms the given result

$$\begin{aligned}\big\|M(\phi)\big\|_{L_{\mathrm{HS}}(\mathcal{H},L^2)}^2 &= \sum_{m\in\mathcal{M}} \big\|\phi\,\chi_m^{(\gamma)}\big\|_{L^2}^2\\ &= \Big(\sum_{\ell\in\mathcal{M}} \big|\big(\phi\big|\chi_\ell\big)_{L^2}\big|^2\Big)\Big(\sum_{m\in\mathcal{M}} (1-\lambda_m)^{-\gamma}\Big)\\ &\leq C\,S(\gamma)\,\big\|\phi\big\|_{L^2}^2\,.\end{aligned}$$

By the very same calculation, one can show that we have

$$\|M(\phi)\|_{L_{\mathrm{HS}}(\mathcal{H},H^1_2(I))} \le C\,\sqrt{S(\gamma)}\,\|\phi\|_{H^1_2(I)}\,,\quad \gamma>\tfrac{d}{2}\,. \tag{A.17}$$

Let us denote for a Hilbert space H the space of progressively measurable processes

$$Y:\Omega\times[0,T]\to L_{HS}(\mathcal{H},H)$$

such that

$$\mathbb{E}|Y|_{L^2([0,T];L_{HS}(\mathcal{H},H)}<\infty,$$

by $M^2_{\mathcal{H}}(0,T;H)$. Having a process $Y\in M^2_{\mathcal{H}}(0,T;L^2(\mathbb{R}))$ we get

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t M(Y(s))\,dW(s)\right|^p_{L^2}\le C_p\,S(\gamma)\,\mathbb{E}\left(\int_0^T \|Y(t)\|^2_{L^2}\,dt\right)^{\frac{p}{2}},\quad p\in[1,\infty)\,, \tag{A.18}$$

Similarly, we have

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t M(Y(s))\,dW(s)\right|^p_{H^1_2}\le C_p\,S(\gamma)\,\mathbb{E}\left(\int_0^T \|Y(t)\|^2_{H^1_2}\,dt\right)^{\frac{p}{2}},\quad p\in[1,\infty)\,, \tag{A.19}$$

The Itô formula Within the proof we apply the Itô formula for the function $\Phi(x)=|x|^p_{L^p}$, $p\ge 2$ to a given process driven by a Wiener process. The diffusion operator will be the multiplication operator defined in (A.10). To be precise, let us put $\Phi(u)=\int_I u^p(x)\,dx$. Then $D\Phi(u)[h]=p\int_I u^{p-1}(x)\,h(x)\,dx$ and $D^2\Phi(u)[h^1,h^2]=p(p-1)\int_I u^{p-2}(x)\,h^1(x)h^2(x)\,dx$. The correction term in the Itô formula is now defined by

$$\mathrm{Tr}\left[D^2\phi(\xi(s))[M(u(s))Q^{\frac12}][M(u(s)Q^{\frac12}]^*\right]=\frac{p(p-1)}{2}\sum_{k\in\mathbb{Z}}\int_I \xi(x,s)^{p-2}(x)\,[M(u)h_k](x)[M(u)h_k](x)\,dx.$$

The definition of the multiplication operator gives

$$\mathrm{Tr}\left[D^2\phi(\xi(s))[M(u(s))Q^{\frac12}][M(u(s)Q^{\frac12}]^*\right]\quad\le\quad\frac{p(p-1)}{2}\sum_{k\in\mathbb{Z}}\int_I |\xi(x,s)|^{p-2}(x)u^2(x)h_k^2(x)\,dx$$

The Hölder inequality gives

$$\mathrm{Tr}\left[D^2\phi(\xi(s))[M(u(s))Q^{\frac12}][M(u(s)Q^{\frac12}]^*\right]\quad\le\quad\frac{p(p-1)}{2}S(\gamma)\int_I |\xi(x,s)|^{p-2}(x)u^2(x)\,dx.$$

In the case $\xi=u$ we get

$$\mathrm{Tr}\left[D^2\phi(\xi(s))[M(u(s))Q^{\frac12}][M(u(s)Q^{\frac12}]^*\right]\quad\le\quad\frac{p(p-1)}{2}S(\gamma)|u|^p_{L^p}.$$

In the case, $\Phi(u)=\int_I(\nabla u)^p(x)\,dx$. Then $D\Phi(u)[h]=p\int_I(\nabla u)^{p-1}(x)\,\nabla h(x)\,dx$ and $D^2\Phi(u)[h^1,h^2]=p(p-1)\int_I(\nabla u)^{p-2}(x)\,(\nabla h^1)(x)(\nabla h^2)(x)\,dx$, we obtain

$$\mathrm{Tr}\left[D^2\phi(\xi)[M(u(s))Q^{\frac12}][M(u(s)Q^{\frac12}]^*\right]\quad\le\quad\frac{p(p-1)}{2}\sum_{k\in\mathbb{Z}}\int_I|\nabla\xi(s)|^{p-2}(x)[\nabla(uh_k)]^2(x)\,dx$$

Again, the Hölder inequality gives

$$\begin{aligned}&\mathrm{Tr}\left[D^2\phi(\xi)[M(u(s))Q^{\frac12}][M(u(s)Q^{\frac12}]^*\right]\\ &\le\quad\frac{p(p-1)}{2}\left(S(\gamma)\int_I|\nabla\xi(s)|^{p-2}(x)[\nabla u]^2(x)+S(\gamma+1)\lambda_k,dx\right)\le\frac{p(p-1)}{2}S(\gamma)|u|^p_{L^p}.\end{aligned} \tag{A.20}$$

The Itô-Correction Term

Let us assume that the process X solves an infinite dimensional differential equation driven given as follows:

$$dX(t)=\Delta X(t)\,dt+\Sigma(X(t))\circ dW(t),\quad X(0)=X_0, \tag{A.21}$$

where Δ denotes the Laplacian operator with periodic boundary conditions. As before, $\{\psi_m : m \in \mathcal{M}\}$ denote the eigenfunctions of the Laplacian and $\{\lambda_m : m \in \mathcal{M}\}$ the corresponding eigenvalues. In this way, the solution process ξ of equation (A.21) can be described by the SPDE given in terms of the Itô–integral by adding a correction term. The correction term can be calculated explicitly (see [15, p. 65, Section 4.5.1]), i.e., the equivalent Itô equation of (A.21) is given by

$$d\xi(t) = A\xi(t)\,dt + \frac{1}{2}D_\xi\sigma(\xi(t))\,\sigma(\xi(t))\,dt + \sigma(\xi(t))d\mathcal{W}(t), \quad \xi(0) = \xi_0. \tag{A.22}$$

Here, $D_\xi(\sigma(\xi))$ denotes the Frechet derivative of σ with respect to ξ. In our case, the Wiener process is infinite dimensional, but can be written as a sum of infinitely many scalar Wiener processes with $\sum_{k\in\mathbb{Z}} \sigma_k d\beta_k(t)$, where $\{\beta_k : k \in \mathbb{Z}\}$ is a family of independent scalar valued Wiener processes, and σ_k is the multiplication operator given by $\xi\psi_k\delta_k$. Straightforward calculations reveal

$$\sum_{k\in\mathbb{Z}} D_\xi\sigma_k(\xi)\,\sigma_k(\xi) = \sum_{k\in\mathbb{Z}} \xi\psi_k^2\delta_k^2.$$

Taking into account that $\delta_k = \delta_{-k}$ and $\psi_k^2 + \psi_{-k}^2 = 2$, we have

$$\sum_{k\in\mathbb{Z}} D_\xi(\sigma_k(\xi))\,\sigma_k(\xi) = S(\gamma)\,\xi, \tag{A.23}$$

where

$$S(\gamma) := \sum_{k\in\mathbb{Z}^d} (1-\lambda_k)^{-\gamma}.$$

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