CONVERGENCE OF IMPLICIT RUNGE–KUTTA TIME DISCRETISATION METHODS FOR FUNDAMENTAL MODELS IN NONLINEAR ACOUSTICS

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Dedicated to Joachim Gwinner on the occasion of his 70th birthday.

Abstract. For the first time, a class of implicit Runge–Kutta time discretisation methods is studied for nonlinear damped wave equations arising in nonlinear acoustics. The analysis in particular applies to the Westervelt, Jordan–Moore–Gibson–Thompson, and Blackstock–Crighton–Brunnhuber–Jordan–Kuznetsov equations. Under appropriate regularity, consistency, and smallness requirements on the time-continuous solutions, global error bounds are obtained from energy estimates for the time-discrete solutions. Existence and uniqueness of time-discrete solutions as well as their convergence is proven under weaker conditions on the initial data, based on energy estimates that are established in a continuous setting and then transferred to the time discretisations.

Keywords. Nonlinear acoustics; High-intensity ultrasonics; Mathematical models; Nonlinear damped wave equations; Time discretisation; Implicit Runge–Kutta methods; Stability; Error; Convergence.

1. INTRODUCTION

In this contribution, we introduce stiffly accurate Runge–Kutta methods for the time integration of nonlinear damped wave equations representing classical and advanced models of nonlinear acoustics. This class of implicit one-step methods includes as simplest instance the backward Euler method, and it is seen to be particularly suited for a study based on variational formulations and energy estimates.

Proceeding former work on stiffly accurate Runge–Kutta methods for nonlinear evolutionary problems, see [EMMRICH, THALHAMMER (2010)] and [GWINNER, THALHAMMER (2014)], we establish existence, boundedness, and convergence of the time-discrete solutions as well as global error bounds under regularity assumptions on the solutions, that identify the stage order as decisive quantity.

The present manuscript is organised as follows. In Section 2, we specify the considered fundamental models, the Westervelt equation, the Jordan–Moore–Gibson–Thompson equation, and the Blackstock–Crighton–Brunnhuber–Jordan–Kuznetsov equation. In Section 3, we state the general format of stiffly accurate Runge–Kutta methods and a basic condition on the coefficients that is essential in view of our variational approach. Section 4 is devoted to the derivation of global error bounds by means of suitable Taylor series expansions and higher-order energy estimates. Sections 5 and 6 provide energy estimates for the time-continuous and time-discrete

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systems in a more general framework. The latter also enables to establish existence and uniqueness of time-discrete solutions as well as their convergence as the time stepsize tends to zero.

2. Fundamental models

The field of nonlinear acoustics is concerned with the propagation of sound waves in thermoviscous fluids. This includes a wide range of applications, in particular in high-intensity ultrasonics [Abramov (1999), Dreyer et al. (2000), Kaltenbacher et al. (2002)]. In this context, the underlying mathematical models are generally based on nonlinear partial differential equations, and reliable as well as efficient numerical solvers are essential tools in view of arising design and monitoring tasks. As relevant application, we mention the shape design of an acoustic lens for focusing high-intensity ultrasound in medical treatment such as lithotripsy.

In the following, we state the fundamental models investigated in this work and introduce compact reformulations as first-order evolutionary systems. We begin with the Westervelt (W) equation, a well-known and oftentimes examined strongly damped wave equation that takes into account nonlinear effects but neglects thermal losses. A generalisation that additionally comprises a quadratic velocity term to reflect a local nonlinearity is the Kuznetsov equation. There exist numerous extensions of these two classical models avoiding the infinite signal speed paradox or incorporating thermal effects, respectively. We exemplarily study the Jordan–Moore–Gibson–Thompson (JMGT) equation and the Blackstock–Crighton–Brunnhuber–Jordan–Kuznetsov (BCBJK) equation. Detailed information on the underlying physics is found in the by now classical references [Crighton (1979), Enflo, Hedberg (2006), Hamilton, Blackstock (1998), Kuznetsov (1971), Lesser, Seebass (1968), Makarov, Ochmann (1996), Makarov, Ochmann (1997a), Makarov, Ochmann (1997b), Pierce (1989), Westervelt (1963)]. For a review of recent results on the analyis of nonlinear damped wave equations and further references to significant contributions, we refer to [Kaltenbacher (2015)].

In essence, the employed setting and notation accord to our former work [Kaltenbacher, Thalhammer (2018)]. Throughout, we consider a bounded spatial domain $\Omega \subset \mathbb{R}^3$ with sufficiently regular boundary and a finite time interval $[0,T]$. Regarding convenient reformulations as evolution equations, we set

$$\mathscr{A} = -\Delta,$$

where Δ denotes the Laplacian with respect to the spatial variables. For simplicity, we restrict ourselves to homogeneous Dirichlet boundary conditions. We use standard notation for Lebesgue and Sobolev spaces.

Westervelt equation. The Westervelt equation [Westervelt (1963)] can be cast into the form

$$\begin{cases} \partial_{tt} p(x,t) - b\,\Delta\,\partial_t p(x,t) - c^2 \Delta\, p(x,t) \\ \quad = \frac{\beta_a}{\rho c^2}\,\partial_{tt} \big(p(x,t)\big)^2, \quad (x,t) \in \Omega \times (0,T)\,, \end{cases} \tag{2.1a}$$

with $p : \Omega \times [0,T] \to \mathbb{R}$ the acoustic pressure, $b > 0$ the diffusivity of sound, $c > 0$ the speed of sound, $\beta_a > 0$ the parameter of nonlinearity, and $\rho > 0$ the mean mass density. Performing differentiation on the right hand side, it is seen that degeneracy is exhibited when the multiplier of the second time derivative vanishes or becomes negative

$$\Big(1 - \tfrac{2\beta_a}{\rho c^2}\, p(x,t)\Big)\, \partial_{tt} p(x,t) - b\, \Delta\, \partial_t p(x,t) - c^2 \Delta\, p(x,t) = \tfrac{2\beta_a}{\rho c^2}\, \big(\partial_t p(x,t)\big)^2 .$$

Under certain regularity, consistency, and smallness requirements on the initial data, using a suitable linearisation of the equation that defines an analytic semigroup and maximal parabolic regularity in appropriate function spaces, results on well-posedness and asymptotic behavior are deduced in [KALTENBACHER, LASIECKA (2009), MEYER, WILKE (2011)]. In this situation, it is justified to investigate the following reformulation as abstract evolution equation

$$\begin{cases} u''(t) + \mathrm{b}\big(u(t), u'(t)\big)\, \mathscr{A} u'(t) + \mathrm{c}_1\big(u(t), u'(t)\big)\, \mathscr{A} u(t) \\ \quad = \mathscr{B}\big(u(t), u'(t)\big)\big[u(t), u'(t)\big], \quad t \in (0,T), \end{cases}$$
$$\alpha(v) = 1 - \tfrac{2\beta_a}{\rho c^2}\, v, \quad \widetilde{\alpha}(v) = \big(\alpha(v)\big)^{-1}, \tag{2.1b}$$
$$\mathrm{b}(v_0, v_1) = b\, \widetilde{\alpha}(v_0), \quad \mathrm{c}_1(v_0, v_1) = c^2\, \widetilde{\alpha}(v_0), \quad \mathscr{B}(v_0, v_1)[u_0, u_1] = \tfrac{2\beta_a}{\rho c^2}\, \widetilde{\alpha}(v_0)\, v_1 u_1 .$$

With regard to the introduction and global error analysis of the implicit Euler method and, more generally, stiffly accurate Runge–Kutta methods, it is helpful to rewrite the Westervelt equation as first-order evolutionary system

$$\mathbf{u}'(t) + \mathbf{A}\big(\mathbf{u}(t)\big)\, \mathbf{u}(t) = \mathbf{B}\big(\mathbf{u}(t)\big)\big[\mathbf{u}(t)\big], \quad t \in (0,T),$$
$$\mathbf{A}\big(\mathbf{u}(t)\big) = \begin{pmatrix} 0 & -I \\ \mathrm{c}_1\big(\mathbf{u}(t)\big)\, \mathscr{A} & \mathrm{b}\big(\mathbf{u}(t)\big)\, \mathscr{A} \end{pmatrix}, \quad \mathbf{B}\big(\mathbf{u}(t)\big)\big[\mathbf{u}(t)\big] = \begin{pmatrix} 0 \\ \mathscr{B}\big(\mathbf{u}(t)\big)\big[\mathbf{u}(t)\big] \end{pmatrix}, \tag{2.1c}$$

where $\mathbf{u} = (u_0, u_1)^T$ represents the solution u and its time derivative u'.

Jordan–Moore–Gibson–Thompson equation. Replacing in the derivation of the Westervelt equation the Fourier law for the heat flux by a Maxwell–Cattaneo law avoids the infinite signal speed paradox [JORDAN (2014)] and leads to the Jordan–Moore–Gibson–Thompson equation

$$\begin{cases} T_{rel}\, \partial_{ttt} p(x,t) + \partial_{tt} p(x,t) - b\, \Delta\, \partial_t p(x,t) - c^2 \Delta\, p(x,t) \\ \quad = \tfrac{\beta_a}{\rho c^2}\, \partial_{tt}\big(p(x,t)\big)^2, \quad (x,t) \in \Omega \times (0,T), \end{cases} \tag{2.2a}$$

with $T_{rel} > 0$ denoting the relaxation time. Compared to (2.1), the appearance of the third-order time derivative considerably changes the character of the equation. It prevents analyticity of the semigroup defined by the linearised equation. For parameter ranges $b < T_{rel}\, c^2$ even continuity of the semigroup is lost, and the problem is ill-posed. As significant contributions in this context, we mention [KALTENBACHER, NIKOLIĆ (2019), KALTENBACHER ET AL. (2011), KALTENBACHER ET AL. (2012), LASIECKA, WANG (2015), LASIECKA, WANG (2016), LIU, TRIGGIANI (2013), MARCHAND ET AL. (2012)]. We study the reformulation

$$\begin{cases} u'''(t) + \mathrm{b}\big(u(t), u'(t), u''(t)\big)\, u''(t) + \mathrm{c}_1\big(u(t), u'(t), u''(t)\big)\, \mathscr{A} u'(t) \\ \quad = \mathscr{B}\big(u(t), u'(t), u''(t)\big)\big[u(t), u'(t), u''(t)\big], \quad t \in (0,T), \end{cases}$$
$$\alpha(v) = 1 - \tfrac{2\beta_a}{\rho c^2}\, v, \quad \mathrm{b}(v_0, v_1, v_2) = \tfrac{1}{T_{rel}}\, \alpha(v_0), \quad \mathrm{c}_1(v_0, v_1, v_2) = \tfrac{b}{T_{rel}}, \tag{2.2b}$$
$$\mathscr{B}(v_0, v_1, v_2)[u_0, u_1, u_2] = -\tfrac{c^2}{T_{rel}}\, \mathscr{A} u_0 + \tfrac{2\beta_a}{\rho c^2 T_{rel}}\, v_1 u_1 .$$

The associated first-order evolutionary system is given by

$$\begin{gathered}\mathbf{u}'(t) + \mathbf{A}\big(\mathbf{u}(t)\big)\,\mathbf{u}(t) = \mathbf{B}\big(\mathbf{u}(t)\big)\big[\mathbf{u}(t)\big]\,, \quad t \in (0,T)\,, \\ \mathbf{A}\big(\mathbf{u}(t)\big) = \begin{pmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ 0 & \mathrm{c}_1\big(\mathbf{u}(t)\big)\,\mathscr{A} & \mathrm{b}\big(\mathbf{u}(t)\big) \end{pmatrix}, \quad \mathbf{B}\big(\mathbf{u}(t)\big)\big[\mathbf{u}(t)\big] = \begin{pmatrix} 0 \\ 0 \\ \mathscr{B}\big(\mathbf{u}(t)\big)\big[\mathbf{u}(t)\big] \end{pmatrix}, \end{gathered} \tag{2.2c}$$

with $\mathbf{u} = (u_0, u_1, u_2)$ comprising the solution as well as the first and second time derivatives.

Blackstock–Crighton–Brunnhuber–Jordan–Kuznetsov equation. The Blackstock–Crighton–Brunnhuber–Jordan–Kuznetsov equation [BLACKSTOCK (1963), BRUNNHUBER, JORDAN (2016), CRIGHTON (1979)] is an extension of the Westervelt and Kuznetsov equations that takes into account thermal effects. Accordingly to [KALTENBACHER, THALHAMMER (2018)], we consider the formulation

$$\begin{cases} \partial_{ttt}\,\psi(x,t) - \beta_1\,\Delta\,\partial_{tt}\,\psi(x,t) + \beta_2\,\Delta^2\partial_t\,\psi(x,t) - \beta_3\,\Delta\,\partial_t\,\psi(x,t) + \beta_4\,\Delta^2\,\psi(x,t) \\ \quad = -\,\partial_{tt}\Big(\frac{1}{2}\,\beta_5\,\big(\partial_t\,\psi(x,t)\big)^2 + \beta_6\,\big|\nabla\psi(x,t)\big|^2\Big)\,, \quad (x,t) \in \Omega \times (0,T)\,, \end{cases} \tag{2.3a}$$

with acoustic velocity potential $\psi : \Omega \times [0,T] \to \mathbb{R}$ and certain constants $\beta_i > 0$ for $i \in \{1,\dots,6\}$. In [KALTENBACHER (2017), KALTENBACHER, THALHAMMER (2018)], well-posedness is studied and a rigorous justification of the Westervelt and Kuznetsov equations as limiting models is given. We employ the reformulation

$$\begin{gathered} \begin{cases} u'''(t) + \mathrm{b}\big(u(t),u'(t),u''(t)\big)\,\mathscr{A}\,u''(t) \\ \qquad + \mathrm{c}_1\big(u(t),u'(t),u''(t)\big)\,\mathscr{A}\,u'(t) + \mathrm{c}_2\big(u(t),u'(t),u''(t)\big)\,\mathscr{A}^2 u'(t) \\ \quad = \mathscr{B}\big(u(t),u'(t),u''(t)\big)\big[u(t),u'(t),u''(t)\big]\,, \quad t \in (0,T)\,, \end{cases} \\ \alpha(v) = 1 + \beta_5\,v\,, \quad \widetilde{\alpha}(v) = \big(\alpha(v)\big)^{-1}\,, \\ \mathrm{b}(v_0,v_1,v_2) = \beta_1\,\widetilde{\alpha}(v_1)\,, \quad \mathrm{c}_1(v_0,v_1,v_2) = \beta_3\,\widetilde{\alpha}(v_1)\,, \quad \mathrm{c}_2(v_0,v_1,v_2) = \beta_2\,\widetilde{\alpha}(v_1)\,, \\ \mathscr{B}(v_0,v_1,v_2)[u_0,u_1,u_2] = -\,\widetilde{\alpha}(v_1)\,\Big(\beta_4\,\mathscr{A}^2 u_0 + \beta_5\,v_2\,u_2 + 2\,\beta_6\,\nabla v_1 \cdot \nabla u_1 \\ \qquad + 2\,\beta_6\,\nabla v_0 \cdot \nabla u_2\Big)\,. \end{gathered} \tag{2.3b}$$

The corresponding first-order evolutionary system reads as

$$\begin{gathered}\mathbf{u}'(t) + \mathbf{A}\big(\mathbf{u}(t)\big)\,\mathbf{u}(t) = \mathbf{B}\big(\mathbf{u}(t)\big)\big[\mathbf{u}(t)\big]\,, \quad t \in (0,T)\,, \\ \mathbf{A}\big(\mathbf{u}(t)\big) = \begin{pmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ 0 & \mathrm{c}_1\big(\mathbf{u}(t)\big)\,\mathscr{A} + \mathrm{c}_2\big(\mathbf{u}(t)\big)\,\mathscr{A}^2 & \mathrm{b}\big(\mathbf{u}(t)\big)\,\mathscr{A} \end{pmatrix}, \\ \mathbf{B}\big(\mathbf{u}(t)\big)\big[\mathbf{u}(t)\big] = \begin{pmatrix} 0 \\ 0 \\ \mathscr{B}\big(\mathbf{u}(t)\big)\big[\mathbf{u}(t)\big] \end{pmatrix}, \end{gathered} \tag{2.3c}$$

where $\mathbf{u} = (u_0, u_1, u_2)$ again represents the solution as well as the first and second time derivatives.

Compact formulation. Setting

$$\begin{aligned}
&\text{(W)} && m=1: \ k_1=1\,,\ r=1: \ \ell_1=1\,,\\
&\text{(JMGT)} && m=2: \ k_2=0\,,\ r=1: \ \ell_1=1\,,\\
&\text{(BCBJK)} && m=2: \ k_2=1\,,\ r=2: \ \ell_1=1\,,\ \ell_2=2\,,
\end{aligned}$$

we can cast the Westervelt equation (2.1), the Jordan–Moore–Gibson–Thompson equation (2.2), and the Blackstock–Crighton–Brunnhuber–Jordan–Kuznetsov equation (2.3) into the form of a higher-order evolution equation

$$\begin{cases} u^{(m+1)}(t) + \mathrm{b}\big(u(t),u'(t),\dots,u^{(m)}(t)\big)\,\mathscr{A}^{k_m}\,u^{(m)}(t) \\ \qquad\qquad + \displaystyle\sum_{i=1}^{r} \mathrm{c}_i\big(u(t),u'(t),\dots,u^{(m)}(t)\big)\,\mathscr{A}^{\ell_i}\,u^{(m-1)}(t) \\ \quad = \mathscr{B}\big(u(t),u'(t),\dots,u^{(m)}(t)\big)\big[u(t),u'(t),\dots,u^{(m)}(t)\big]\,, \quad t\in(0,T)\,, \end{cases} \tag{2.4a}$$

with non-negative integers $k_m \geq 0$ and $0 \leq \ell_1 < \ell_2 < \cdots < \ell_r$. Accordingly, the associated first-order evolutionary system is given by

$$\begin{gathered}
\mathbf{u}'(t) = \mathbf{F}\big(\mathbf{u}(t)\big) = -\,\mathbf{A}\big(\mathbf{u}(t)\big)\,\mathbf{u}(t) + \mathbf{B}\big(\mathbf{u}(t)\big)\big[\mathbf{u}(t)\big]\,, \quad t\in(0,T)\,,\\
\mathbf{u}(t) = \begin{pmatrix} u_0(t)\\ u_1(t)\\ \vdots\\ u_m(t)\end{pmatrix}, \quad \mathbf{B}\big(\mathbf{u}(t)\big)\big[\mathbf{u}(t)\big] = \begin{pmatrix} 0\\ \vdots\\ 0\\ \mathscr{B}\big(\mathbf{u}(t)\big)\big[\mathbf{u}(t)\big]\end{pmatrix},\\
\mathbf{A}\big(\mathbf{u}(t)\big) = \begin{pmatrix} 0 & -I & 0 & \cdots & \cdots & 0\\ \vdots & 0 & -I & 0 & \cdots & \vdots\\ \vdots & & \ddots & \ddots & 0 & \vdots\\ \vdots & & & \ddots & -I & 0\\ 0 & \cdots & \cdots & \cdots & \sum_{i=1}^{r}\mathrm{c}_i\big(\mathbf{u}(t)\big)\,\mathscr{A}^{\ell_i} & \mathrm{b}\big(\mathbf{u}(t)\big)\,\mathscr{A}^{k_m}\end{pmatrix},
\end{gathered} \tag{2.4b}$$

where the components of $\mathbf{u}$ represent the solution u and its time derivatives up to order m.

Hilbert space setting. In view of the subsequent sections, it is convenient to denote by $(\mathscr{H},(\cdot|\cdot),|\cdot|)$ the underlying Lebesgue space $L^2(\Omega)$. As common, the inner product of elements in $\mathscr{H}^{m+1}$ is defined componentwise

$$\big(\mathbf{h}\big|\mathbf{k}\big) = \sum_{i=0}^{m}\big(h_i\big|k_i\big)\,, \quad \mathbf{h}=(h_0,h_1,\dots,h_m)\,,\ \mathbf{k}=(k_0,k_1,\dots,k_m)\in\mathscr{H}^{m+1}\,.$$

We make use of the fact that the negative Laplacian subject to homogeneous Dirichlet boundary conditions is selfadjoint and satisfies a Poincaré–Friedrichs type inequality

$$\begin{gathered}
\mathscr{A} = -\Delta : \mathscr{D}(\mathscr{A}) = H^2(\Omega)\cap H_0^1(\Omega) \subset \mathscr{H} \longrightarrow \mathscr{H}\,,\\
|v| \leq C_{PF}\left(\big|\mathscr{A}^{1/2}v\big| + |\mathscr{A}v|\right), \quad v\in\mathscr{D}(\mathscr{A})\,.
\end{gathered} \tag{2.5}$$

The unbounded operator

$$\mathscr{B} : \mathscr{D}_1(\mathscr{B}) \subset \mathscr{H}^{m+1} \longrightarrow L\big(\mathscr{D}_2(\mathscr{B}),\mathscr{H}\big)\,, \quad \mathscr{D}_2(\mathscr{B}) \subset \mathscr{H}^{m+1}\,,$$

in particular comprises the arising nonlinear terms.

3. Stiffly accurate Runge–Kutta methods

For the time discretisation of the evolution equation (2.4), we study stiffly accurate Runge–Kutta methods, see [Emmrich, Thalhammer (2010), Gwinner, Thalhammer (2014)] and references given therein. We choose time grid points

$$0 = t_0 < t_1 < \cdots < t_N = T\,, \quad \tau_n = t_{n+1} - t_n\,, \quad n \in \{0,1,\dots,N-1\}\,,$$

where the integer number $N > 0$ is fixed. Approximations to the values of the exact solution are henceforth denoted by

$$\mathbf{u}^{(n)} = \begin{pmatrix} u_0^{(n)} \\ u_1^{(n)} \\ \vdots \\ u_m^{(n)} \end{pmatrix} \approx \mathbf{u}(t_n) = \begin{pmatrix} u_0(t_n) \\ u_1(t_n) \\ \vdots \\ u_m(t_n) \end{pmatrix} = \begin{pmatrix} u_0(t_n) \\ u_0'(t_n) \\ \vdots \\ u_0^{(m)}(t_n) \end{pmatrix}, \quad n \in \{0,1,\dots,N\}\,. \tag{3.1}$$

Implicit Euler method. A well-known instance of a stiffly accurate Runge–Kutta method is the implicit Euler method. For a prescribed initial approximation $\mathbf{u}^{(0)}$, the time-discrete solution to (2.4) is defined by the recurrence

$$\tfrac{1}{\tau_n}\big(\mathbf{u}^{(n+1)} - \mathbf{u}^{(n)}\big) = \mathbf{F}\big(\mathbf{u}^{(n+1)}\big)\,, \quad n \in \{0,1,\dots,N-1\}\,. \tag{3.2a}$$

In the derivation of energy estimates and global error bounds, we make use of the elementary relation

$$(x_1 - x_0)\,x_1 = \tfrac{1}{2}\,\big(x_1^2 - x_0^2\big) + \tfrac{1}{2}\,(x_1 - x_0)^2\,, \quad x_0\,, x_1 \in \mathbb{R}\,, \tag{3.2b}$$

which carries over to elements of the underlying Hilbert space. In particular, it implies

$$\begin{gathered}\big(\mathbf{u}^{(n+1)} - \mathbf{u}^{(n)}\big|\mathbf{u}^{(n+1)}\big) = \tfrac{1}{2}\,\Big(\big|\mathbf{u}^{(n+1)}\big|^2 - \big|\mathbf{u}^{(n)}\big|^2\Big) + \tfrac{1}{2}\,\big|\mathbf{u}^{(n+1)} - \mathbf{u}^{(n)}\big|^2\,, \\ n \in \{0,1,\dots,N-1\}\,.\end{gathered} \tag{3.2c}$$

General format. More generally, we consider a consistent stiffly accurate Runge–Kutta method of nonstiff order p with associated Butcher tableau

$$\begin{gathered}\begin{array}{c|c} \mathfrak{c} & \mathfrak{A} \\ \hline & \mathfrak{b} \end{array} \\ \mathfrak{A} = (\mathfrak{a}_{ij})_{i,j=1}^{s} \in \mathbb{R}^{s\times s}\,, \quad \mathfrak{b} = \mathfrak{e}^T \mathfrak{A} \in \mathbb{R}^s\,, \quad \mathfrak{e} = (0,\dots,0,1)^T \in \mathbb{R}^s\,, \quad \mathfrak{c} \in [0,1]^s\,.\end{gathered} \tag{3.3a}$$

The stages are determined by a nonlinear system and yield approximations to the exact solution values at the nodes

$$\begin{gathered}\tfrac{1}{\tau_n}\big(\mathbf{U}_i^{(n)} - \mathbf{u}^{(n)}\big) = \sum_{j=1}^{s} \mathfrak{a}_{ij}\,\mathbf{F}\big(\mathbf{U}_j^{(n)}\big)\,, \\ \mathbf{U}_i^{(n)} \approx \mathbf{u}(t_n + \mathfrak{c}_i\,\tau_n)\,, \quad i \in \{1,2,\dots,s\}\,, \quad n \in \{0,1,\dots,N-1\}\,.\end{gathered} \tag{3.3b}$$

We recall that the stage order q is characterised as the largest integer number such that

$$\sum_{j=1}^{s} \mathfrak{a}_{ij}\,\mathfrak{c}_j^k = \tfrac{1}{k+1}\,\mathfrak{c}_i^{k+1}\,, \quad i \in \{1,\dots,s\}\,, \quad k \in \{0,1,\dots,q-1\}\,. \tag{3.3c}$$

Compared to general implicit Runge–Kutta methods, a pecularity of stiffly accurate Runge–Kutta methods is that the weights coincide with the last row of the coefficient matrix and hence the time-discrete solution values are given by the last stages

$$\mathbf{u}^{(n+1)} = \mathbf{U}_s^{(n)} \approx \mathbf{u}(t_{n+1}), \quad n \in \{0,1,\dots,N-1\}. \tag{3.3d}$$

Under the assumption that the matrix

$$\begin{gathered} \mathfrak{B}\,\mathfrak{A} + \mathfrak{A}^T \mathfrak{B} - \mathfrak{b}\,\mathfrak{b}^T - \mathfrak{A}^T \mathfrak{C}\,\mathbf{1}\mathbf{1}^T \mathfrak{C}^T \mathfrak{A}\,, \\ \mathfrak{B} = \operatorname{diag}(\mathfrak{b})\,, \quad \mathfrak{C} = \mathfrak{B}\,\mathfrak{A}^{-1}\,, \quad \mathbf{1} = (1,\dots,1)^T \in \mathbb{R}^s\,, \end{gathered} \tag{3.4a}$$

is well-defined and positive semidefinite, the inequality

$$(x_1,\dots,x_s)\,\mathfrak{C} \begin{pmatrix} x_1 - x_0 \\ \vdots \\ x_s - x_0 \end{pmatrix} \geq \tfrac{1}{2}\left(x_s^2 - x_0^2\right), \quad x_0, x_1, \dots, x_s \in \mathbb{R}, \tag{3.4b}$$

is valid, see [EMMRICH, THALHAMMER (2010), Lemma 3.4]. This in particular ensures

$$\sum_{i,j=1}^{s} \mathfrak{C}_{ij} \left(\mathbf{U}_j^{(n)} - \mathbf{u}^{(n)} \middle| \mathbf{U}_i^{(n)} \right) \geq \tfrac{1}{2}\left(\left|\mathbf{u}^{(n+1)}\right|^2 - \left|\mathbf{u}^{(n)}\right|^2 \right), \quad n \in \{0,1,\dots,N-1\}. \tag{3.4c}$$

Third-order two-stage scheme. As an instance of a higher-order stiffly accurate Runge–Kutta method fulfilling (3.4), we refer to the Radau IIA method

$$\begin{gathered} \mathfrak{a}_{11} = \tfrac{5}{12}\,, \quad \mathfrak{a}_{12} = -\tfrac{1}{12}\,, \quad \mathfrak{a}_{21} = \tfrac{3}{4}\,, \quad \mathfrak{a}_{22} = \tfrac{1}{4}\,, \\ \mathfrak{b}_1 = \mathfrak{a}_{21} = \tfrac{3}{4}\,, \quad \mathfrak{b}_2 = \mathfrak{a}_{22} = \tfrac{1}{4}\,, \quad \mathfrak{c}_1 = \tfrac{1}{3}\,, \quad \mathfrak{c}_2 = 1\,, \end{gathered}$$

which has nonstiff order $p = 2s - 1 = 3$ and stage order $q = s = 2$. Indeed, the conditions

$$\begin{gathered} \sum_{i=1}^{2} \mathfrak{b}_i = 1\,, \quad \sum_{i=1}^{2} \mathfrak{b}_i\,\mathfrak{c}_i = \tfrac{1}{2}\,, \quad \sum_{i=1}^{2} \mathfrak{b}_i\,\mathfrak{c}_i^2 = \tfrac{1}{3}\,, \quad \sum_{i=1}^{2}\sum_{j=1}^{2} \mathfrak{b}_i\,\mathfrak{a}_{ij}\,\mathfrak{c}_j = \tfrac{1}{6}\,, \\ \sum_{j=1}^{2} \mathfrak{a}_{ij} = \mathfrak{c}_i\,, \quad \sum_{j=1}^{2} \mathfrak{a}_{ij}\,\mathfrak{c}_j = \tfrac{1}{2}\,\mathfrak{c}_i^2\,, \quad i \in \{1,2\}\,, \end{gathered}$$

hold true.

4. GLOBAL ERROR BOUNDS

In this section, we deduce global error bounds for stiffly accurate Runge–Kutta methods applied to nonlinear damped wave equations. For this purpose, we presume that the time-continuous and time-discrete solutions fulfill suitable regularity assumptions. In Sections 5 and 6 we provide existence and boundedness under substantially weaker regularity, consistency, and smallness requirements on the initial data.

Time-discrete and time-continuous solutions. Throughout, we assume that the considered stiffly accurate Runge–Kutta method (3.3) satisfies the fundamental condition (3.4). For a nonlinear damped wave equation cast into the form (2.4), the stages and hence the values of time-discrete solution are determined by the nonlinear system

$$\tfrac{1}{\tau_n}\left(\mathbf{U}_i^{(n)} - \mathbf{u}^{(n)}\right) = \sum_{j=1}^{s} \mathfrak{a}_{ij}\,\mathbf{F}\left(\mathbf{U}_j^{(n)}\right), \quad i \in \{1,2,\dots,s\}, \quad n \in \{0,1,\dots,N-1\}. \tag{4.1}$$

The corresponding relations for the time-continuous solution read as

$$t = t_n + \mathfrak{c}_i\,\tau_n\,, \quad \mathbf{u}'(t) = \mathbf{F}\big(\mathbf{u}(t)\big)\,, \quad i \in \{1,2,\dots,s\}\,, \quad n \in \{0,1,\dots,N-1\}\,.$$

Errors and defects. The differences between the values of the time-discrete and time-continuous solutions are denoted by

$$\begin{gathered}\mathbf{E}_i^{(n)} = \mathbf{U}_i^{(n)} - \mathbf{u}(t_n + \mathfrak{c}_i\,\tau_n)\,, \quad i \in \{1,2,\dots,s\}\,, \quad n \in \{0,1,\dots,N-1\}\,,\\ \mathbf{e}^{(n)} = \mathbf{u}^{(n)} - \mathbf{u}(t_n)\,, \quad n \in \{0,1,\dots,N\}\,.\end{gathered} \tag{4.2}$$

Inserting the exact solution values into the numerical scheme defines the defects

$$\begin{gathered}\tfrac{1}{\tau_n}\big(\mathbf{u}(t_n + \mathfrak{c}_i\,\tau_n) - \mathbf{u}(t_n)\big) = \sum_{j=1}^{s} \mathfrak{a}_{ij}\,\mathbf{F}\big(\mathbf{u}(t_n + \mathfrak{c}_j\,\tau_n)\big) - \mathbf{r}_i^{(n)}\,,\\ i \in \{1,2,\dots,s\}\,, \quad n \in \{0,1,\dots,N-1\}\,.\end{gathered} \tag{4.3}$$

In accordance with (3.1) and (4.4), we set

$$\begin{gathered}\mathbf{U}_i = \begin{pmatrix} U_{i0}^{(n)} \\ U_{i1}^{(n)} \\ \vdots \\ U_{im}^{(n)} \end{pmatrix}, \quad \mathbf{E}_i^{(n)} = \begin{pmatrix} E_{i0}^{(n)} \\ E_{i1}^{(n)} \\ \vdots \\ E_{im}^{(n)} \end{pmatrix}, \quad \mathbf{e}^{(n)} = \begin{pmatrix} e_0^{(n)} \\ e_1^{(n)} \\ \vdots \\ e_m^{(n)} \end{pmatrix}, \quad \mathbf{r}_i^{(n)} = \begin{pmatrix} r_{i0}^{(n)} \\ r_{i1}^{(n)} \\ \vdots \\ r_{im}^{(n)} \end{pmatrix},\\ i \in \{1,2,\dots,s\}\,, \quad n \in \{0,1,\dots,N-1\}\,.\end{gathered}$$

Taylor series expansions. Under appropriate regularity assumptions, Taylor series expansions with remainders in integral form yield the representation

$$\begin{gathered}\mathbf{u}^{(L)}(t_n + \mathfrak{c}_i\,\tau_n) = \sum_{k=0}^{K} \tfrac{1}{k!}\,\mathfrak{c}_i^k\,\tau_n^k\,\mathbf{u}^{(k+L)}(t_n) + \mathfrak{c}_i^{K+1}\,\tau_n^{K+1}\,\mathbf{R}_{K+1}\big(\mathbf{u}^{(K+L+1)}, t_n, \mathfrak{c}_i\,\tau_n\big)\,,\\ \mathbf{R}_{K+1}\big(\mathbf{u}^{(K+L+1)}, t_n, \mathfrak{c}_i\,\tau_n\big) = \tfrac{1}{K!}\int_0^1 (1-\sigma)^K\,\mathbf{u}^{(K+L+1)}(t_n + \sigma\,\mathfrak{c}_i\,\tau_n)\,\mathrm{d}\sigma\,,\\ i \in \{1,2,\dots,s\}\,, \quad n \in \{0,1,\dots,N-1\}\,, \quad K,L \in \mathbb{N}_{\geq 0}\,.\end{gathered}$$

Replacing in (4.3) the defining operator by the time-derivative of the solution and recalling the stage order conditions (3.3c), this implies

$$\begin{aligned}
\mathbf{r}_i^{(n)} &= \sum_{j=1}^{s} \mathfrak{a}_{ij}\, \mathbf{F}\big(\mathbf{u}(t_n + \mathfrak{c}_j\, \tau_n)\big) - \tfrac{1}{\tau_n}\big(\mathbf{u}(t_n + \mathfrak{c}_i\, \tau_n) - \mathbf{u}(t_n)\big) \\
&= \sum_{j=1}^{s} \mathfrak{a}_{ij}\, \mathbf{u}'(t_n + \mathfrak{c}_j\, \tau_n) - \tfrac{1}{\tau_n}\big(\mathbf{u}(t_n + \mathfrak{c}_i \tau_n) - \mathbf{u}(t_n)\big) \\
&= \sum_{k=0}^{q-1} \tfrac{1}{k!}\, \tau_n^k \left(\sum_{j=1}^{s} \mathfrak{a}_{ij}\, \mathfrak{c}_j^k - \tfrac{1}{k+1}\, \mathfrak{c}_i^{k+1} \right) \mathbf{u}^{(k+1)}(t_n) \\
&\qquad + \tau_n^q \left(\sum_{j=1}^{s} \mathfrak{a}_{ij}\, \mathfrak{c}_j^q\, \mathbf{R}_q\big(\mathbf{u}^{(q+1)}, t_n, \mathfrak{c}_j\, \tau_n\big) - \mathfrak{c}_i^{q+1}\, \mathbf{R}_{q+1}\big(\mathbf{u}^{(q+1)}, t_n, \mathfrak{c}_i\, \tau_n\big) \right) \\
&= \tau_n^q \left(\sum_{j=1}^{s} \mathfrak{a}_{ij}\, \mathfrak{c}_j^q\, \mathbf{R}_q\big(\mathbf{u}^{(q+1)}, t_n, \mathfrak{c}_j\, \tau_n\big) \right. \\
&\qquad \left. - \mathfrak{c}_i^{q+1}\, \mathbf{R}_{q+1}\big(\mathbf{u}^{(q+1)}, t_n, \mathfrak{c}_i\, \tau_n\big) \right), \quad i \in \{1,2,\dots,s\}, \quad n \in \{0,1,\dots,N-1\}.
\end{aligned}$$

Hence, boundedness of the defects is linked to boundedness of certain time derivatives of the solution in the underlying Hilbert space

$$\begin{gathered}
\big|\mathbf{r}_i^{(n)}\big| \leq C\, \tau_n^q\, |\mathbf{u}|_{W_\infty^{q+1}(0,T;\mathscr{H}^{m+1})}, \quad \big|r_{ij}^{(n)}\big| \leq C\, \tau_n^q\, |u_0|_{W_\infty^{q+j+1}(0,T;\mathscr{H})}, \quad j \in \{0,1,\dots,m\}, \\
i \in \{1,2,\dots,s\}, \quad n \in \{0,1,\dots,N-1\}.
\end{gathered} \tag{4.4}$$

The generic constant $C > 0$ depends on the coefficients of the considered stiffly accurate Runge–Kutta method.

Error relation. Our starting point for the derivation of a global error bound is the relation

$$\begin{gathered}
\tfrac{1}{\tau_n}\big(\mathbf{E}_i^{(n)} - \mathbf{e}^{(n)}\big) = \sum_{j=1}^{s} \mathfrak{a}_{ij} \Big(\mathbf{F}\big(\mathbf{U}_j^{(n)}\big) - \mathbf{F}\big(\mathbf{u}(t_n + \mathfrak{c}_j\, \tau_n)\big) \Big) + \mathbf{r}_i^{(n)}, \\
i \in \{1,2,\dots,s\}, \quad n \in \{0,1,\dots,N-1\},
\end{gathered} \tag{4.5}$$

which results from (4.1) and (4.3). In view of (3.4), it is essential to employ the reformulation

$$\begin{gathered}
\sum_{j=1}^{s} \mathfrak{C}_{ij}\big(\mathbf{E}_j^{(n)} - \mathbf{e}^{(n)}\big) = \tau_n\, \mathfrak{b}_i \Big(\mathbf{F}\big(\mathbf{U}_i^{(n)}\big) - \mathbf{F}\big(\mathbf{u}(t_n + \mathfrak{c}_i\, \tau_n)\big) \Big) + \tau_n \sum_{j=1}^{s} \mathfrak{C}_{ij}\, \mathbf{r}_j^{(n)}, \\
i \in \{1,2,\dots,s\}, \quad n \in \{0,1,\dots,N-1\}.
\end{gathered}$$

In addition, we make use of the integral representation

$$\begin{gathered}
\mathbf{F}\big(\mathbf{U}_i^{(n)}\big) - \mathbf{F}\big(\mathbf{u}(t_n + \mathfrak{c}_i\, \tau_n)\big) = \int_0^1 \mathbf{F}'\big(\sigma\, \mathbf{U}_i^{(n)} + (1-\sigma)\, \mathbf{u}(t_n + \mathfrak{c}_i\, \tau_n)\big)\, \mathrm{d}\sigma\, \mathbf{E}_i^{(n)}, \\
i \in \{1,2,\dots,s\}, \quad n \in \{0,1,\dots,N-1\}.
\end{gathered}$$

Altogether, this yields the error relation

$$\begin{gathered}
\sum_{j=1}^{s} \mathfrak{C}_{ij}\big(\mathbf{E}_j^{(n)} - \mathbf{e}^{(n)}\big) = \tau_n\, \mathfrak{b}_i \Big(\int_0^1 \mathbf{F}'\big(\sigma\, \mathbf{U}_i^{(n)} + (1-\sigma)\, \mathbf{u}(t_n + \mathfrak{c}_i\, \tau_n)\big)\, \mathrm{d}\sigma\, \mathbf{E}_i^{(n)} \Big) + \tau_n \sum_{j=1}^{s} \mathfrak{C}_{ij}\, \mathbf{r}_j^{(n)}, \\
i \in \{1,2,\dots,s\}, \quad n \in \{0,1,\dots,N-1\}.
\end{gathered}$$

Specifically, for the implicit Euler method with $p = q = s = \mathfrak{C}_{11} = \mathfrak{b}_1 = \mathfrak{c}_1 = 1$, this relation reduces to

$$\begin{gathered}\big(\mathbf{e}^{(n+1)} - \mathbf{e}^{(n)}\big) = \tau_n \Big(\int_0^1 \mathbf{F}'\big(\sigma\,\mathbf{u}^{(n+1)} + (1-\sigma)\,\mathbf{u}(t_{n+1}\big)\,\mathrm{d}\sigma\,\mathbf{e}^{(n+1)}\Big) + \tau_n\,\mathbf{r}_1^{(n)}\,, \\ i \in \{1,2,\dots,s\}\,, \quad n \in \{0,1,\dots,N-1\}\,.\end{gathered}$$

4.1. **Westervelt equation.** We first focus on the derivation of a global error bound for stiffly accurate Runge–Kutta methods applied to the simplest model, the Westervelt equation. For (2.1), the arising nonlinear operator and its Fréchet derivative are given by

$$\begin{gathered}\mathbf{v} = (v_0, v_1)\,, \quad \mathbf{F}(\mathbf{v}) = \begin{pmatrix} v_1 \\ F_2(\mathbf{v}) \end{pmatrix}, \quad \mathbf{F}'(\mathbf{v}) = \begin{pmatrix} 0 & I \\ F_{21}'(\mathbf{v}) & F_{22}'(\mathbf{v}) \end{pmatrix}, \\ F_2(\mathbf{v}) = (1-\beta_3\,v_0)^{-1}\,\big(-\beta_1\,\mathscr{A}\,v_0 - \beta_2\,\mathscr{A}\,v_1 + \beta_3\,v_1^2\big)\,, \\ F_{21}'(\mathbf{v}) = f_{11}(\mathbf{v}) - f_{12}(\mathbf{v})\,\mathscr{A}\,, \quad F_{22}'(\mathbf{v}) = f_{21}(\mathbf{v}) - f_{22}(\mathbf{v})\,\mathscr{A}\,, \\ f_{11}(\mathbf{v}) = \beta_3\,(1-\beta_3\,v_0)^{-2}\,\big(-\beta_1\,\mathscr{A}\,v_0 - \beta_2\,\mathscr{A}\,v_1 + \beta_3\,v_1^2\big)\,, \quad f_{12}(\mathbf{v}) = \beta_1\,(1-\beta_3\,v_0)^{-1}\,, \\ f_{21}(\mathbf{v}) = 2\,\beta_3\,(1-\beta_3\,v_0)^{-1}\,v_1\,, \quad f_{22}(\mathbf{v}) = \beta_2\,(1-\beta_3\,v_0)^{-1}\,,\end{gathered}$$

with $\mathscr{A}$ representing the negative Laplacian subject to homogeneous boundary conditions and $\beta_1, \beta_2, \beta_3 > 0$ denoting certain constants. An essential observation is that regularity and non-degeneracy ensure bounds of the form

$$\begin{aligned}&\max_{\ell\in\{11,12,21,22\}} \max_{\substack{i\in\{1,2,\dots,s\}\\ n\in\{1,2,\dots,N\}}} \Big| \int_0^1 f_\ell\big(\sigma\,\mathbf{U}_i^{(n)} + (1-\sigma)\,\mathbf{u}(t_n + \mathfrak{c}_i\,\tau_n)\big)\,\mathrm{d}\sigma \Big|_{L^\infty(\Omega)} \\ &\quad + \max_{\ell\in\{12,22\}} \max_{\substack{i\in\{1,2,\dots,s\}\\ n\in\{1,2,\dots,N\}}} \Big| \int_0^1 f_\ell\big(\sigma\,\mathbf{U}_i^{(n)} + (1-\sigma)\,\mathbf{u}(t_n + \mathfrak{c}_i\,\tau_n)\big)\,\mathrm{d}\sigma \Big|_{W^1_\infty(\Omega)} \leq C_u\,, \\ &C_u = C_u\Big(\max_{\substack{i\in\{1,2,\dots,s\}\\ n\in\{1,2,\dots,N\}}} \big|\mathbf{U}_i^{(n)}\big|_{(W^2_\infty(\Omega))^2}, |\mathbf{u}|_{L^\infty(0,T;(W^2_\infty(\Omega))^2)} \Big)\,, \\ &\min_{\substack{i\in\{1,2,\dots,s\}\\ n\in\{1,2,\dots,N\}}} \operatorname*{ess\,inf}_{x\in\Omega} \int_0^1 f_{22}\big(\sigma\,\mathbf{U}_i^{(n)} + (1-\sigma)\,\mathbf{u}(t_n + \mathfrak{c}_i\,\tau_n)\big)(x)\,\mathrm{d}\sigma \geq \underline{C_u}\,.\end{aligned}$$

Weak formulation. We test the error relation stated above with

$$\begin{gathered}\boldsymbol{\mathcal{A}}\mathbf{E}_i^{(n)}\,, \quad i \in \{1,2,\dots,s\}\,, \quad n \in \{0,1,\dots,N-1\}\,, \\ \boldsymbol{\mathcal{A}} = \operatorname{diag}\big(\mathscr{A}^{k+1}, \mathscr{A}^k\big)\,, \quad k \in \{0,1\}\,,\end{gathered}$$

perform integration-by-parts, and add all components to obtain

$$\begin{aligned}
&\sum_{i,j=1}^{s} \mathfrak{C}_{ij} \left(\boldsymbol{\mathcal{A}}^{1/2} \big(\mathbf{E}_j^{(n)} - \mathbf{e}^{(n)} \big) \Big| \boldsymbol{\mathcal{A}}^{1/2} \mathbf{E}_i^{(n)} \right) \\
&\quad = \tau_n \sum_{i=1}^{s} \mathfrak{b}_i \left(\int_0^1 \mathbf{F}' \big(\sigma \mathbf{U}_i^{(n)} + (1-\sigma)\, \mathbf{u}(t_n + \mathfrak{c}_i \tau_n) \big) \, \mathrm{d}\sigma \, \mathbf{E}_i^{(n)} \Big| \boldsymbol{\mathcal{A}} \mathbf{E}_i^{(n)} \right) \\
&\qquad + \tau_n \sum_{i,j=1}^{s} \mathfrak{C}_{ij} \big(\boldsymbol{\mathcal{A}}^{1/2} \mathbf{r}_j^{(n)} \big| \boldsymbol{\mathcal{A}}^{1/2} \mathbf{E}_i^{(n)} \big) \,, \quad n \in \{0, 1, \dots, N-1\} \,.
\end{aligned}$$

The elementary estimate given in (3.4), Young's inequality with additional weight

$$x_1\, x_2 \leq C_Y(\varepsilon)\, x_1^2 + C_Y\big(\tfrac{1}{\varepsilon}\big)\, x_2^2 \,, \quad x_1, x_2 \in \mathbb{R} \,, \quad C_Y(\varepsilon) = \tfrac{1}{2}\,\varepsilon \,, \quad C_Y\big(\tfrac{1}{\varepsilon}\big) = \tfrac{1}{2}\,\tfrac{1}{\varepsilon} \,, \quad \varepsilon > 0 \,,$$

and the expansion of the defects (4.4) imply

$$\begin{aligned}
&\big| \boldsymbol{\mathcal{A}}^{1/2} \mathbf{e}^{(n+1)} \big|^2 - \big| \boldsymbol{\mathcal{A}}^{1/2} \mathbf{e}^{(n)} \big|^2 \\
&\quad \leq \varepsilon_0 \, \tau_n \sum_{i=1}^{s} \big| \boldsymbol{\mathcal{A}}^{1/2} \mathbf{E}_i^{(n)} \big|^2 \\
&\qquad + 2\, \tau_n \sum_{i=1}^{s} \mathfrak{b}_i \left(\int_0^1 \mathbf{F}' \big(\sigma \mathbf{U}_i^{(n)} + (1-\sigma)\, \mathbf{u}(t_n + \mathfrak{c}_i \tau_n) \big) \, \mathrm{d}\sigma \, \mathbf{E}_i^{(n)} \Big| \boldsymbol{\mathcal{A}} \mathbf{E}_i^{(n)} \right) \\
&\qquad + C \, \tfrac{1}{\varepsilon_0} \, \big| \boldsymbol{\mathcal{A}}^{1/2} \mathbf{u} \big|_{W_\infty^{q+1}(0,T;(L^2(\Omega))^2)}^2 \, \tau_n^{2q+1} \,, \quad n \in \{0, 1, \dots, N-1\} \,,
\end{aligned} \tag{4.6}$$

with $\varepsilon_0 > 0$ and generic constant $C > 0$ depending on the method coefficients. We note that the special form of the defining operator implies

$$\begin{gathered}
\mathscr{A}^{k/2} E_{i0}^{(n)} = \mathscr{A}^{k/2} e_0^{(n)} + \tau_n \sum_{j=1}^{s} a_{ij} \, \mathscr{A}^{k/2} E_{j1}^{(n)} + \tau_n \, \mathscr{A}^{k/2} r_{i0}^{(n)} \,, \\
i \in \{1, 2, \dots, s\} \,, \quad n \in \{0, 1, \dots, N-1\} \,, \quad k \in \mathbb{N}_{\geq 0} \,.
\end{gathered}$$

see (4.5). By means of Young's inequality and the estimate for the defects (4.4), we thus obtain the bound

$$\begin{aligned}
&\big| \mathscr{A}^{k/2} E_{i0}^{(n)} \big|^2 \\
&\quad \leq 3 \, \big| \mathscr{A}^{k/2} e_0^{(n)} \big|^2 + C \, \tau_n \sum_{j=1}^{s} \big| \mathscr{A}^{k/2} E_{j1}^{(n)} \big|^2 + C \, \tau_n^{2q+1} \, \big| \mathscr{A}^{k/2} u_0 \big|_{W_\infty^{q+1}(0,T;\mathscr{H})}^2 \,, \\
&i \in \{1, 2, \dots, s\} \,, \quad n \in \{0, 1, \dots, N-1\} \,, \quad k \in \mathbb{N}_{\geq 0} \,,
\end{aligned} \tag{4.7}$$

with a generic constant $C > 0$ that depends on the coefficients of the considered stiffly accurate Runge–Kutta method. We next deduce an auxiliary estimate for the term

$$\begin{gathered}
\left(\int_0^1 \mathbf{F}' \big(\sigma \mathbf{U}_i^{(n)} + (1-\sigma)\, \mathbf{u}(t_n + \mathfrak{c}_i \tau_n) \big) \, \mathrm{d}\sigma \, \mathbf{E}_i^{(n)} \Big| \boldsymbol{\mathcal{A}} \mathbf{E}_i^{(n)} \right) , \\
i \in \{1, 2, \dots, s\} \,, \quad n \in \{0, 1, \dots, N-1\} \,,
\end{gathered}$$

Auxiliary estimate. Let $i \in \{1, 2, \dots, s\}$ and $n \in \{0, 1, \dots, N-1\}$. Our starting point is the equality

$$\begin{aligned}
&\Big(\int_0^1 \mathbf{F}'\big(\sigma\, \mathbf{U}_i^{(n)} + (1-\sigma)\, \mathbf{u}(t_n + \mathfrak{c}_i\, \tau_n)\big)\, \mathrm{d}\sigma\, \mathbf{E}_i^{(n)} \Big| \boldsymbol{\mathcal{A}} \mathbf{E}_i^{(n)} \Big) \\
&\quad = \Big(E_{i1}^{(n)} \Big| \mathscr{A}_0\, E_{i0}^{(n)} \Big) \\
&\qquad + \Big(\int_0^1 f_{11}\big(\sigma\, \mathbf{U}_i^{(n)} + (1-\sigma)\, \mathbf{u}(t_n + \mathfrak{c}_i\, \tau_n)\big)\, \mathrm{d}\sigma\, E_{i0}^{(n)} \Big| \mathscr{A}_1\, E_{i1}^{(n)} \Big) \\
&\qquad - \Big(\int_0^1 f_{12}\big(\sigma\, \mathbf{U}_i^{(n)} + (1-\sigma)\, \mathbf{u}(t_n + \mathfrak{c}_i\, \tau_n)\big)\, \mathrm{d}\sigma\, \mathscr{A}\, E_{i0}^{(n)} \Big| \mathscr{A}_1\, E_{i1}^{(n)} \Big) \\
&\qquad + \Big(\int_0^1 f_{21}\big(\sigma\, \mathbf{U}_i^{(n)} + (1-\sigma)\, \mathbf{u}(t_n + \mathfrak{c}_i\, \tau_n)\big)\, \mathrm{d}\sigma\, E_{i1}^{(n)} \Big| \mathscr{A}_1\, E_{i1}^{(n)} \Big) \\
&\qquad - \Big(\int_0^1 f_{22}\big(\sigma\, \mathbf{U}_i^{(n)} + (1-\sigma)\, \mathbf{u}(t_n + \mathfrak{c}_i\, \tau_n)\big)\, \mathrm{d}\sigma\, \mathscr{A}\, E_{i1}^{(n)} \Big| \mathscr{A}_1\, E_{i1}^{(n)} \Big)
\end{aligned}$$

and reformulations resulting from integration-by-parts.

(i) We first consider the case

$$k = 1\,, \quad \mathscr{A}_0 = \mathscr{A}^2\,, \quad \mathscr{A}_1 = \mathscr{A}\,,$$

where the above given relation rewrites as

$$\begin{aligned}
&\Big(\int_0^1 \mathbf{F}'\big(\sigma\, \mathbf{U}_i^{(n)} + (1-\sigma)\, \mathbf{u}(t_n + \mathfrak{c}_i\, \tau_n)\big)\, \mathrm{d}\sigma\, \mathbf{E}_i^{(n)} \Big| \boldsymbol{\mathcal{A}} \mathbf{E}_i^{(n)} \Big) \\
&\quad = \Big(\mathscr{A}\, E_{i1}^{(n)} \Big| \mathscr{A}\, E_{i0}^{(n)} \Big) \\
&\qquad + \Big(\int_0^1 f_{11}\big(\sigma\, \mathbf{U}_i^{(n)} + (1-\sigma)\, \mathbf{u}(t_n + \mathfrak{c}_i\, \tau_n)\big)\, \mathrm{d}\sigma\, E_{i0}^{(n)} \Big| \mathscr{A}\, E_{i1}^{(n)} \Big) \\
&\qquad - \Big(\int_0^1 f_{12}\big(\sigma\, \mathbf{U}_i^{(n)} + (1-\sigma)\, \mathbf{u}(t_n + \mathfrak{c}_i\, \tau_n)\big)\, \mathrm{d}\sigma\, \mathscr{A}\, E_{i0}^{(n)} \Big| \mathscr{A}\, E_{i1}^{(n)} \Big) \\
&\qquad + \Big(\int_0^1 f_{21}\big(\sigma\, \mathbf{U}_i^{(n)} + (1-\sigma)\, \mathbf{u}(t_n + \mathfrak{c}_i\, \tau_n)\big)\, \mathrm{d}\sigma\, E_{i1}^{(n)} \Big| \mathscr{A}\, E_{i1}^{(n)} \Big) \\
&\qquad - \Big(\int_0^1 f_{22}\big(\sigma\, \mathbf{U}_i^{(n)} + (1-\sigma)\, \mathbf{u}(t_n + \mathfrak{c}_i\, \tau_n)\big)\, \mathrm{d}\sigma\, \mathscr{A}\, E_{i1}^{(n)} \Big| \mathscr{A}\, E_{i1}^{(n)} \Big)\,.
\end{aligned}$$

By means of Young's inequality with weight $\varepsilon_1 > 0$ and the Poincaré–Friedrichs type inequality (2.5), we obtain

$$\begin{aligned}
&\Big(\int_0^1 \mathbf{F}'\big(\sigma\, \mathbf{U}_i^{(n)} + (1-\sigma)\, \mathbf{u}(t_n + \mathfrak{c}_i\, \tau_n)\big)\, \mathrm{d}\sigma\, \mathbf{E}_i^{(n)} \Big| \boldsymbol{\mathcal{A}} \mathbf{E}_i^{(n)} \Big) \\
&\quad \leq (1 + C\, C_u)\, \big|\mathscr{A}\, E_{i0}^{(n)}\big|\, \big|\mathscr{A}\, E_{i1}^{(n)}\big| + C_u\, \big|E_{i1}^{(n)}\big|\, \big|\mathscr{A}\, E_{i1}^{(n)}\big| - \underline{C_u}\, \big|\mathscr{A}\, E_{i1}^{(n)}\big|^2 \\
&\quad \leq \tfrac{1}{\varepsilon_1}\, (1 + C\, C_u)\, \big|\mathscr{A}\, E_{i0}^{(n)}\big|^2 + \tfrac{1}{\varepsilon_1}\, C_u\, \big|E_{i1}^{(n)}\big|^2 + \big(\varepsilon_1\, (1 + C\, C_u) - \underline{C_u}\big)\, \big|\mathscr{A}\, E_{i1}^{(n)}\big|^2
\end{aligned}$$

with generic constant $C > 0$ depending on $C_{PF} > 0$. Provided that the weight $\varepsilon_1 > 0$ is chosen sufficiently small such that

$$\varepsilon_1\, (1 + C\, C_u) - \underline{C_u} \leq -\,\delta < 0\,,$$

we arrive at the estimate

$$\begin{aligned} & k = 1\,, \quad \mathscr{A}_0 = \mathscr{A}^2\,, \quad \mathscr{A}_1 = \mathscr{A}\,, \\ & \Big(\int_0^1 \mathbf{F}'\big(\sigma\,\mathbf{U}_i^{(n)} + (1-\sigma)\,\mathbf{u}(t_n + \mathfrak{c}_i\,\tau_n)\big)\,\mathrm{d}\sigma\,\mathbf{E}_i^{(n)} \Big| \boldsymbol{\mathcal{A}}\mathbf{E}_i^{(n)} \Big) \\ & \quad \leq \tfrac{1}{\varepsilon_1}\,(1 + C\,C_u)\,\big|\mathscr{A}\,E_{i0}^{(n)}\big|^2 + \tfrac{1}{\varepsilon_1}\,C_u\,\big|E_{i1}^{(n)}\big|^2 - \delta\,\big|\mathscr{A}\,E_{i1}^{(n)}\big|^2\,. \end{aligned} \tag{4.8}$$

(ii) Similar arguments are employed in the case

$$k = 0\,, \quad \mathscr{A}_0 = \mathscr{A}\,, \quad \mathscr{A}_1 = I\,.$$

Here, applying integration-by-parts, Young's inequality with weight $\varepsilon_1 > 0$, and the Poincaré–Friedrichs type inequality (2.5), yields

$$\begin{aligned} & \Big(\int_0^1 \mathbf{F}'\big(\sigma\,\mathbf{U}_i^{(n)} + (1-\sigma)\,\mathbf{u}(t_n + \mathfrak{c}_i\,\tau_n)\big)\,\mathrm{d}\sigma\,\mathbf{E}_i^{(n)} \Big| \boldsymbol{\mathcal{A}}\mathbf{E}_i^{(n)} \Big) \\ & = \Big(\mathscr{A}^{1/2}\,E_{i1}^{(n)} \Big| \mathscr{A}^{1/2}\,E_{i0}^{(n)} \Big) \\ & \quad + \Big(\int_0^1 f_{11}\big(\sigma\,\mathbf{U}_i^{(n)} + (1-\sigma)\,\mathbf{u}(t_n + \mathfrak{c}_i\,\tau_n)\big)\,\mathrm{d}\sigma\,E_{i0}^{(n)} \Big| E_{i1}^{(n)} \Big) \\ & \quad - \Big(\mathscr{A}^{1/2}\,E_{i0}^{(n)} \Big| \mathscr{A}^{1/2} \Big(\int_0^1 f_{12}\big(\sigma\,\mathbf{U}_i^{(n)} + (1-\sigma)\,\mathbf{u}(t_n + \mathfrak{c}_i\,\tau_n)\big)\,\mathrm{d}\sigma\,E_{i1}^{(n)} \Big) \Big) \\ & \quad + \Big(\int_0^1 f_{21}\big(\sigma\,\mathbf{U}_i^{(n)} + (1-\sigma)\,\mathbf{u}(t_n + \mathfrak{c}_i\,\tau_n)\big)\,\mathrm{d}\sigma\,E_{i1}^{(n)} \Big| E_{i1}^{(n)} \Big) \\ & \quad - \Big(\mathscr{A}^{1/2}\,E_{i1}^{(n)} \Big| \mathscr{A}^{1/2} \Big(\int_0^1 f_{22}\big(\sigma\,\mathbf{U}_i^{(n)} + (1-\sigma)\,\mathbf{u}(t_n + \mathfrak{c}_i\,\tau_n)\big)\,\mathrm{d}\sigma\,E_{i1}^{(n)} \Big) \Big) \\ & \leq \big|\mathscr{A}^{1/2}\,E_{i0}^{(n)}\big|\,\big|\mathscr{A}^{1/2}\,E_{i1}^{(n)}\big| + C_u \Big(\big|E_{i0}^{(n)}\big|\,\big|E_{i1}^{(n)}\big| + \big|\mathscr{A}^{1/2}\,E_{i0}^{(n)}\big|\,\big|\mathscr{A}^{1/2}\,E_{i1}^{(n)}\big| \\ & \quad + \big|\mathscr{A}^{1/2}\,E_{i0}^{(n)}\big|\,\big|E_{i1}^{(n)}\big| + \big|E_{i1}^{(n)}\big|^2 + \big|E_{i1}^{(n)}\big|\,\big|\mathscr{A}^{1/2}\,E_{i1}^{(n)}\big| \Big) - \underline{C_u}\,\big|\mathscr{A}^{1/2}\,E_{i1}^{(n)}\big|^2 \\ & \leq \tfrac{1}{\varepsilon_1}\,(1 + C\,C_u)\,\big|\mathscr{A}^{1/2}\,E_{i0}^{(n)}\big|^2 + \big(1 + \tfrac{1}{\varepsilon_1}\big)\,C_u\,\big|E_{i1}^{(n)}\big|^2 \\ & \quad + \Big(\varepsilon_1\,(1 + C\,C_u) - \underline{C_u} \Big)\,\big|\mathscr{A}^{1/2}\,E_{i1}^{(n)}\big|^2 \end{aligned}$$

with generic constant $C > 0$ depending on $C_{PF} > 0$. Hence, by choosing the weight $\varepsilon_1 > 0$ sufficiently small such that

$$\varepsilon_1\,(1 + C\,C_u) - \underline{C_u} \leq -\delta < 0\,,$$

we get the estimate

$$\begin{aligned} & k = 0\,, \quad \mathscr{A}_0 = \mathscr{A}\,, \quad \mathscr{A}_1 = I\,, \\ & \Big(\int_0^1 \mathbf{F}'\big(\sigma\,\mathbf{U}_i^{(n)} + (1-\sigma)\,\mathbf{u}(t_n + \mathfrak{c}_i\,\tau_n)\big)\,\mathrm{d}\sigma\,\mathbf{E}_i^{(n)} \Big| \boldsymbol{\mathcal{A}}\mathbf{E}_i^{(n)} \Big) \\ & \quad \leq \tfrac{1}{\varepsilon_1}\,(1 + C\,C_u)\,\big|\mathscr{A}^{1/2}\,E_{i0}^{(n)}\big|^2 + \big(1 + \tfrac{1}{\varepsilon_1}\big)\,C_u\,\big|E_{i1}^{(n)}\big|^2 - \delta\,\big|\mathscr{A}^{1/2}\,E_{i1}^{(n)}\big|^2\,. \end{aligned} \tag{4.9}$$

Global error bound (Implicit Euler method). As illustration, we first consider the implicit Euler method, where it suffices to set $\varepsilon_0 = 1$ and the above stated arguments show

$$\begin{aligned}&\big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{e}^{(n+1)}\big|^2 - \big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{e}^{(n)}\big|^2\\&\quad\leq C\,C_1\big(\tfrac{1}{\varepsilon_1}, C_u\big)\,\tau_n\,\big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{e}^{(n+1)}\big|^2 + C\,\big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{u}\big|_{W^2_\infty(0,T;(L^2(\Omega))^2)}\,\tau_n^3\,,\quad n\in\{0,1,\dots,N-1\}\end{aligned}$$

with generic constants and C_1 depending on the indicated quantitites, see also (4.6), (4.8) and (4.9). Summation and a telescopic identity imply

$$\begin{aligned}&\big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{e}^{(N)}\big|^2\\&\quad\leq \big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{e}^{(0)}\big|^2 + C\,C_1\big(\tfrac{1}{\varepsilon_1}, C_u\big)\sum_{n=0}^{N-1}\tau_n\,\big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{e}^{(n+1)}\big|^2 + C\,\big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{u}\big|_{W^2_\infty(0,T;(L^2(\Omega))^2)}\sum_{n=0}^{N-1}\tau_n^3\,.\end{aligned}$$

Finally, by a Gronwall inequality, we obtain the global error bound

$$\begin{gathered}\big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{e}^{(N)}\big|^2 \leq C\,C_1\big(\tfrac{1}{\varepsilon_1}, C_u\big)\Big(\big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{e}^{(0)}\big|^2 + \tau_{\max}^2\Big)\,,\\ \tau_{\max} = \max\{\tau_n : n\in\{0,1,\dots,N-1\}\}\,,\end{gathered}$$

where the arising constant in addition depends on the final time.

Global error bound (Stiffly accurate Runge–Kutta method). More generally, combining (4.6), (4.7), (4.8), and (4.9), a bound of the form

$$\begin{aligned}&\big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{e}^{(n+1)}\big|^2 - \big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{e}^{(n)}\big|^2 + \tau_n\sum_{i=1}^{s}\big|\mathscr{A}^{(k+1)/2}\,E_{i1}^{(n)}\big|^2\\&\quad\leq C\,\tau_n\sum_{i=1}^{s}\big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{e}^{(n)}\big|^2\\&\qquad + C\,\tau_n\,(\varepsilon_0+\tau_n+C_u)\sum_{i=1}^{s}\big|\mathscr{A}^{(k+1)/2}\,E_{i1}^{(n)}\big|^2\\&\qquad + C\,\big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{u}\big|^2_{W^{q+1}_\infty(0,T;(L^2(\Omega))^2}\,\tau_n^{2q+1}\,,\quad n\in\{0,1,\dots,N-1\}\,,\end{aligned}$$

holds. Requiring the time stepsizes, the weight $\varepsilon_0 > 0$, the initial data and hence the bound C_u to be sufficiently small, we have

$$\begin{aligned}&\big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{e}^{(n+1)}\big|^2 - \big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{e}^{(n)}\big|^2\\&\quad\leq C\,\tau_n\sum_{i=1}^{s}\big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{e}^{(n)}\big|^2 + C\,\big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{u}\big|^2_{W^{q+1}_\infty(0,T;(L^2(\Omega))^2}\,\tau_n^{2q+1}\,,\quad n\in\{0,1,\dots,N-1\}\,.\end{aligned}$$

Thus, summation and a Gronwall inequality lead to the result

$$\begin{gathered}\big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{e}^{(N)}\big|^2_{(L^2(\Omega))^2} \leq C\Big(\big|\boldsymbol{\mathcal{A}}^{1/2}\,\mathbf{e}^{(0)}\big|^2_{(L^2(\Omega))^2} + \tau_{\max}^{2q}\Big)\,,\\ \tau_{\max} = \max\{\tau_n : n\in\{0,1,\dots,N-1\}\}\,.\end{gathered}$$

Altogether, we arrive at the following result.

Theorem 4.1 (Westervelt equation)**.** *Let* $\mathbf{u} = (u_0,u_1)^T = (u,u')^T$ *denote the solution to* (2.1)*, and set* $\boldsymbol{\mathcal{A}} = diag(\mathscr{A}^{k+1},\mathscr{A}^k)$ *for* $k\in\{0,1\}$*. Suppose that the considered stiffly accurate*

Runge–Kutta method of nonstiff order p and stage order q fulfills the fundamental condition (3.4)*. Provided that the time-continuous and time-discrete solutions satisfy the regularity requirements*

$$\begin{aligned} &|u_0|_{W^1_\infty(0,T;W^2_\infty(\Omega))} + |u_0|_{W^{r+1}_\infty(0,T;W^1_2(\Omega))} + |u_0|_{W^{r+2}_\infty(0,T;L^2(\Omega))} \leq C\,, \quad r = \min\{p,q\}\,, \\ &\max\Big\{\big|\mathbf{U}_i^{(n)}\big|_{(W^2_\infty(\Omega))^2} : i \in \{1,2,\dots,s\}, n \in \{1,2,\dots,N\}\Big\} \leq C\,, \end{aligned} \tag{4.10}$$

and that smallness of the time-continuous solution is ensured on the considered time interval, the global error bound

$$\big|\boldsymbol{\mathcal{A}}^{1/2}\left(\mathbf{u}^{(N)} - \mathbf{u}(T)\right)\big|^2_{(L^2(\Omega))^2} \leq C\big(\big|\boldsymbol{\mathcal{A}}^{1/2}\left(\mathbf{u}^{(0)} - \mathbf{u}(0)\right)\big|^2_{(L^2(\Omega))^2} + \tau_{\max}^{2r}\big)$$

holds for $\tau_{\max} = \max\{\tau_n : n \in \{0,1,\dots,N-1\}\}$ *sufficiently small. The arising constant in particular depends on the quantities in* (4.10).

4.2. **Jordan–Moore–Gibson–Thompson equation.** We next consider the Jordan–Moore–Gibson–Thompson equation (2.2), where the defining nonlinear operator and its Fréchet derivative are given by

$$\begin{gathered} \mathbf{v} = (v_0, v_1, v_2)\,, \quad \mathbf{F}(\mathbf{v}) = \begin{pmatrix} v_1 \\ v_2 \\ F_3(\mathbf{v}) \end{pmatrix}, \quad \mathbf{F}'(\mathbf{v}) = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ F'_{31}(\mathbf{v}) & F'_{32}(\mathbf{v}) & F'_{33}(\mathbf{v}) \end{pmatrix}, \\ F_3(\mathbf{v}) = \tfrac{1}{T_{rel}}\left(-\beta_1 \mathscr{A} v_0 - \beta_2 \mathscr{A} v_1 - (1 - \beta_3 v_0)\, v_2 + \beta_3 v_1^2\right), \\ F'_{31}(\mathbf{v}) = f_1(\mathbf{v}) - \tfrac{\beta_1}{T_{rel}} \mathscr{A}\,, \quad F'_{32}(\mathbf{v}) = f_2(\mathbf{v}) - \tfrac{\beta_2}{T_{rel}} \mathscr{A}\,, \quad F'_{33}(\mathbf{v}) = f_3(\mathbf{v})\,, \\ f_1(\mathbf{v}) = \tfrac{\beta_3}{T_{rel}} v_2\,, \quad f_2(\mathbf{v}) = \tfrac{2\beta_3}{T_{rel}} v_1\,, \quad f_3(\mathbf{v}) = -\tfrac{1}{T_{rel}}\,(1 - \beta_3 v_0)\,, \\ \beta_1 = c^2\,, \quad \beta_2 = b\,, \quad \beta_3 = \tfrac{2\beta_a}{\rho c^2}\,. \end{gathered}$$

Due to the fact that the analysis is significantly more involved, we include detailed arguments for the implicit Euler method satisfying

$$\mathbf{e}^{(n+1)} - \mathbf{e}^{(n)} = \tau_n\, \mathbf{F}'\big(\mathbf{u}(t_{n+1}) + \tfrac{1}{2}\,\mathbf{e}^{(n+1)}\big)\, \mathbf{e}^{(n+1)} + \tau_n\, \mathbf{r}_1^{(n)}\,, \quad n \in \{0,1,\dots,N-1\}\,.$$

We in particular employ the regularity requirements

$$\begin{aligned} &\max_{\ell \in \{1,2,3\}} \max_{n \in \{1,2,\dots,N\}} \Big| f_\ell\big(\mathbf{u}(t_{n+1}) + \tfrac{1}{2}\,\mathbf{e}^{(n+1)}\big)\Big|_{L^\infty(\Omega)} \leq C_u\,, \\ &C_u = C_u\Big(\max_{n \in \{1,2,\dots,N\}} \big|\mathbf{u}^{(n)}\big|_{(L^\infty(\Omega))^3}, |u_0|_{W^2_\infty(0,T;L^\infty(\Omega))}\Big)\,. \end{aligned}$$

(i) On the one hand, we test with

$$\begin{pmatrix} 0 \\ \frac{\beta_2}{T_{rel}} \mathscr{A}\, e_1^{(n+1)} \\ e_2^{(n+1)} \end{pmatrix}$$

and perform integration-by-parts. Together with the elementary relation in (3.2), this yields

$$\begin{aligned}
&\big|\mathscr{A}^{1/2}\, e_1^{(n+1)}\big|^2 - \big|\mathscr{A}^{1/2}\, e_1^{(n)}\big|^2 + \big|e_2^{(n+1)}\big|^2 - \big|e_2^{(n)}\big|^2 \\
&\quad \leq C\big(C_u\big)\,\tau_n \left(\big|\mathscr{A}\, e_0^{(n+1)}\big|^2 + \big|\mathscr{A}^{1/2}\, e_1^{(n+1)}\big|^2 + \big|e_2^{(n+1)}\big|^2\right) \\
&\qquad + C\left(\big|u_0\big|^2_{W^3_\infty(0,T;W^1_2(\Omega))} + \big|u_0\big|^2_{W^4_\infty(0,T;L^2(\Omega))}\right)\tau_n^3\,, \quad n \in \{0,1,\dots,N-1\}\,,
\end{aligned}$$

with a generic constant $C > 0$ that in particular depends on C_u.

(ii) On the other hand, testing with

$$\begin{pmatrix} 0 \\ \mathscr{A}\, e_1^{(n+1)} \\ \mathscr{A}\, e_1^{(n+1)} \end{pmatrix}$$

and employing smallness of the quantity C_u, we have

$$\begin{aligned}
&\big|\mathscr{A}^{1/2}\, e_1^{(n+1)}\big|^2 - \big|\mathscr{A}^{1/2}\, e_1^{(n)}\big|^2 + \tau_n\,\big|\mathscr{A}\, e_1^{(n+1)}\big|^2 \\
&\quad \leq C\big(C_u\big)\,\tau_n \left(\big|\mathscr{A}\, e_0^{(n+1)}\big|^2 + \big|e_2^{(n+1)}\big|^2\right) \\
&\qquad + C\left(\big|u_0\big|^2_{W^3_\infty(0,T;L^2(\Omega))} + \big|u_0\big|^2_{W^4_\infty(0,T;L^2(\Omega))}\right)\tau_n^3\,, \quad n \in \{0,1,\dots,N-1\}\,,
\end{aligned}$$

where the generic constant $C > 0$ again in particular depends on C_u.

(iii) Combining both bounds, summation and a Gronwall inequality show

$$\begin{gathered}
\big|\mathscr{A}^{1/2}\, e_1^{(N)}\big|^2 + \big|e_2^{(N)}\big|^2 \leq C\left(\big|\mathscr{A}^{1/2}\, e_1^{(0)}\big|^2 + \big|e_2^{(n)}\big|^2 + \tau_{\max}^2\right), \\
\tau_{\max} = \max\{\tau_n : n \in \{0,1,\dots,N-1\}\}\,.
\end{gathered}$$

More generally, we obtain the following result.

Theorem 4.2 (Jordan–Moore–Gibson–Thompson equation)**.** *Let* $\mathbf{u} = (u_0,u_1,u_2)^T = (u,u',u'')^T$ *denote the solution to* (2.2)*, and set* $\boldsymbol{\mathcal{A}} = diag(0,\mathscr{A},I)$*. Suppose that the considered stiffly accurate Runge–Kutta method of nonstiff order p and stage order q fulfills the fundamental condition* (3.4)*. Provided that the time-continuous and time-discrete solutions satisfy the regularity requirements*

$$\begin{gathered}
|u_0|_{W^2_\infty(0,T;W^2_\infty(\Omega))} + |u_0|_{W^{r+2}_\infty(0,T;W^1_2(\Omega))} + |u_0|_{W^{r+3}_\infty(0,T;L^2(\Omega))} \leq C\,, \quad r = \min\{p,q\}\,, \\
\max\left\{\big|\mathbf{U}_i^{(n)}\big|_{(W^2_\infty(\Omega))^3} : i \in \{1,2,\dots,s\}, n \in \{1,2,\dots,N\}\right\} \leq C\,,
\end{gathered} \tag{4.11}$$

and that non-degeneracy is ensured on the considered time interval, the global error bound

$$\big|\boldsymbol{\mathcal{A}}^{1/2}\left(\mathbf{u}^{(N)} - \mathbf{u}(T)\right)\big|^2_{(L^2(\Omega))^3} \leq C\big(\big|\boldsymbol{\mathcal{A}}^{1/2}\left(\mathbf{u}^{(0)} - \mathbf{u}(0)\right)\big|^2_{(L^2(\Omega))^3} + \tau_{\max}^{2r}\big)$$

holds for $\tau_{\max} = \max\{\tau_n : n \in \{0,1,\dots,N-1\}\}$ *sufficiently small. The arising constant in particular depends on the quantities in* (4.11)*.*

4.3. **Blackstock–Crighton–Brunnhuber–Jordan–Kuznetsov equation.** For (2.3), similar calculations yield

$$\mathbf{v} = (v_0, v_1, v_2)\,, \quad \mathbf{F}(\mathbf{v}) = \begin{pmatrix} v_1 \\ v_2 \\ F_3(\mathbf{v}) \end{pmatrix}, \quad \mathbf{F}'(\mathbf{v}) = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ F'_{31}(\mathbf{v}) & F'_{32}(\mathbf{v}) & F'_{33}(\mathbf{v}) \end{pmatrix},$$

$$\begin{aligned}
&F_3(\mathbf{v}) = -\big(1 + \beta_5\, v_1\big)^{-1} \\
&\qquad \times \Big(\beta_3\, \mathscr{A}\, v_1 + \beta_2\, \mathscr{A}^2\, v_1 + \beta_1\, \mathscr{A}\, v_2 + \beta_4\, \mathscr{A}^2 v_0 + \beta_5\, v_2^2 + 2\,\beta_6\, |\nabla v_1|^2 + 2\,\beta_6\, \nabla v_0 \cdot \nabla v_2\Big)\,, \\
&F'_{31}(\mathbf{v}) = -\, f_{11}(\mathbf{v})\, \mathscr{A}^2 - f_{12}(\mathbf{v}) \cdot \nabla\,, \\
&f_{11}(\mathbf{v}) = \beta_4\, \big(1 + \beta_5\, v_1\big)^{-1}, \quad f_{12}(\mathbf{v}) = 2\,\beta_6\, \big(1 + \beta_5\, v_1\big)^{-1}\, \nabla v_2\,, \\
&F'_{32}(\mathbf{v}) = f_{21}(\mathbf{v}) - f_{22}(\mathbf{v})\, \mathscr{A}^2 - f_{23}(\mathbf{v})\, \mathscr{A} - f_{24}(\mathbf{v}) \cdot \nabla\,, \\
&f_{21}(\mathbf{v}) = \beta_5\, \big(1 + \beta_5\, v_1\big)^{-2} \\
&\qquad \times \Big(\beta_3\, \mathscr{A}\, v_1 + \beta_2\, \mathscr{A}^2\, v_1 + \beta_1\, \mathscr{A}\, v_2 + \beta_4\, \mathscr{A}^2 v_0 + \beta_5\, v_2^2 + 2\,\beta_6\, |\nabla v_1|^2 + 2\,\beta_6\, \nabla v_0 \cdot \nabla v_2\Big)\,, \\
&f_{22}(\mathbf{v}) = \beta_2\, \big(1 + \beta_5\, v_1\big)^{-1}, \quad f_{23}(\mathbf{v}) = \beta_3\, \big(1 + \beta_5\, v_1\big)^{-1}, \quad f_{24}(\mathbf{v}) = 4\,\beta_6\, \big(1 + \beta_5\, v_1\big)^{-1}\, \nabla v_1\,, \\
&F'_{33}(\mathbf{v}) = -\, f_{31}(\mathbf{v}) - f_{32}(\mathbf{v})\, \mathscr{A} - f_{33}(\mathbf{v}) \cdot \nabla\,, \\
&f_{31}(\mathbf{v}) = 2\,\beta_5\, \big(1 + \beta_5\, v_1\big)^{-1}\, v_2\,, \quad f_{32}(\mathbf{v}) = \beta_1\, \big(1 + \beta_5\, v_1\big)^{-1}, \quad f_{33}(\mathbf{v}) = 2\,\beta_6\, \big(1 + \beta_5\, v_1\big)^{-1}\, \nabla v_0\,,
\end{aligned}$$

with certain constants $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 > 0$. Again, we include detailed calculations for the implicit Euler method

$$\begin{gathered}
\mathbf{e}^{(n+1)} - \mathbf{e}^{(n)} = \tau_n \int_0^1 \mathbf{F}'\big(\sigma\, \mathbf{u}^{(n+1)} + (1-\sigma)\, \mathbf{u}(t_{n+1})\big)\, \mathrm{d}\sigma\; \mathbf{e}^{(n+1)} + \tau_n\, \mathbf{r}_1^{(n)}\,, \\
n \in \{0, 1, \dots, N-1\}\,,
\end{gathered}$$

making use of the regularity requirements

$$\begin{gathered}
\max_{\ell \in \{11,12,21,22,23,24,31,32,33\}}\; \max_{n \in \{1,2,\dots,N\}} \Big| \int_0^1 f_\ell\big(\sigma\, \mathbf{u}^{(n+1)} + (1-\sigma)\, \mathbf{u}(t_{n+1})\big)\, \mathrm{d}\sigma \Big|_{L^\infty(\Omega)} \leq C_u\,, \\
C_u = C_u\Big(\max_{n \in \{1,2,\dots,N\}} \big|\mathbf{u}^{(n)}\big|_{(W_\infty^4(\Omega))^3}, |u_0|_{W_\infty^2(0,T;W_\infty^4(\Omega))}\Big)\,, \\
\min_{\ell \in \{22,32\}}\; \operatorname*{ess\,inf}_{x \in \Omega} \int_0^1 f_\ell\big(\sigma\, \mathbf{u}^{(n+1)} + (1-\sigma)\, \mathbf{u}(t_{n+1})\big)(x)\, \mathrm{d}\sigma \geq \underline{C_u}\,.
\end{gathered}$$

(i) On the one hand, we test with

$$\begin{pmatrix} 0 \\ \mathscr{A}^3\, e_1^{(n+1)} \\ \mathscr{A}\, e_2^{(n+1)} \end{pmatrix}$$

and perform integration-by-parts, which yields

$$\begin{aligned}
&\big|\mathscr{A}^{3/2}\, e_1^{(n+1)}\big|^2 - \big|\mathscr{A}^{3/2}\, e_1^{(n)}\big|^2 + \big|\mathscr{A}^{1/2}\, e_2^{(n+1)}\big|^2 - \big|\mathscr{A}^{1/2}\, e_2^{(n)}\big|^2 + \tau_n \big|\mathscr{A}\, e_2^{(n+1)}\big|^2 \\
&\quad \leq C\,\tau_n \left(\big|\mathscr{A}^2\, e_0^{(n+1)}\big|^2 + \big|\mathscr{A}^2\, e_1^{(n+1)}\big|^2\right) \\
&\qquad + C\left(|u_0|_{W_\infty^3(0,T;W_2^2(\Omega))} + |u_0|_{W_\infty^4(0,T;L^2(\Omega))}\right)\tau_n^3\,, \quad n \in \{0,1,\dots,N-1\}\,,
\end{aligned}$$

with a generic constant $C > 0$.

(ii) On the other hand, testing with

$$\begin{pmatrix} 0 \\ \mathscr{A}^3\, e_1^{(n+1)} \\ \mathscr{A}^2\, e_1^{(n+1)} \end{pmatrix},$$

we arrive at the bound

$$\begin{aligned}
&\big|\mathscr{A}^{3/2}\, e_1^{(n+1)}\big|^2 - \big|\mathscr{A}^{3/2}\, e_1^{(n)}\big|^2 + \tau_n \big|\mathscr{A}^2\, e_1^{(n+1)}\big|^2 \\
&\quad \leq C\,\tau_n \left(\big|\mathscr{A}^2\, e_0^{(n+1)}\big|^2 + \big|\mathscr{A}\, e_2^{(n+1)}\big|^2\right) \\
&\qquad + C\left(|u_0|_{W_\infty^3(0,T;W_2^2(\Omega))} + |u_0|_{W_\infty^4(0,T;L^2(\Omega))}\right)\tau_n^3\,, \quad n \in \{0,1,\dots,N-1\}\,.
\end{aligned}$$

(iii) Combining both bounds, summation and a Gronwall inequality show

$$\begin{gathered}
\big|\mathscr{A}^{3/2}\, e_1^{(N)}\big|^2 + \big|\mathscr{A}^{1/2}\, e_2^{(N)}\big|^2 \leq C\left(\big|\mathscr{A}^{3/2}\, e_1^{(0)}\big|^2 + \big|\mathscr{A}^{1/2}\, e_2^{(n)}\big|^2 + \tau_{\max}^2\right), \\
\tau_{\max} = \max\{\tau_n : n \in \{0,1,\dots,N-1\}\}\,.
\end{gathered}$$

More generally, we obtain the following result.

Theorem 4.3 (Blackstock–Crighton–Brunnhuber–Jordan–Kuznetsov equation)**.** *Let* $\mathbf{u} = (u_0,u_1,u_2)^T = (u,u',u'')^T$ *denote the solution to* (2.3)*, and set* $\boldsymbol{\mathcal{A}} = diag(0,\mathscr{A}^3,\mathscr{A})$*. Suppose that the considered stiffly accurate Runge–Kutta method of nonstiff order p and stage order q fulfills the fundamental condition* (3.4)*. Provided that the time-continuous and time-discrete solutions satisfy the regularity requirements*

$$\begin{gathered}
|u_0|_{W_\infty^2(0,T;W_\infty^4(\Omega))} + |u_0|_{W_\infty^{r+2}(0,T;W_2^2(\Omega))} + |u_0|_{W_\infty^{r+3}(0,T;L^2(\Omega))} \leq C\,, \quad r = \min\{p,q\}\,, \\
\max\left\{\big|\mathbf{U}_i^{(n)}\big|_{(W_\infty^4(\Omega))^3} : i \in \{1,2,\dots,s\}, n \in \{1,2,\dots,N\}\right\} \leq C\,,
\end{gathered} \tag{4.12}$$

and that non-degeneracy is ensured on the considered time interval, the global error bound

$$\big|\boldsymbol{\mathcal{A}}^{1/2}\left(\mathbf{u}^{(N)} - \mathbf{u}(T)\right)\big|_{(L^2(\Omega))^3}^2 \leq C\big(\big|\boldsymbol{\mathcal{A}}^{1/2}\left(\mathbf{u}^{(0)} - \mathbf{u}(0)\right)\big|_{(L^2(\Omega))^3}^2 + \tau_{\max}^{2r}\big)$$

holds for $\tau_{\max} = \max\{\tau_n : n \in \{0,1,\dots,N-1\}\}$ *sufficiently small. The arising constant in particular depends on the quantities in* (4.12)*.*

5. Energy estimates

In this section we deduce uniform energy estimates for the three nonlinear acoustic equations dicussed in the previous sections on a more abstract level, thus potentially comprising other nonlinear evolution equations. The purpose of these energy estimates, that in parts also differ from the energy estimates in the literature on nonlinear acoustics so far, is their transfer to time discretisation by stiffly accurate Runge Kutta methods.

Abstract higher order PDE model. As already indicated in (2.1b), (2.2b), (2.3b), we can write the equations under consideration in the framework on the abstract higher order PDE model

$$\begin{aligned}&\partial_t^{m+1}u+\mathsf{b}(u,\partial_t u,\ldots,\partial_t^m u)\mathscr{A}^{k_m}\partial_t^m u+\sum_{i=1}^r \mathsf{c}_i(u,\partial_t u,\ldots,\partial_t^m u)\mathscr{A}^{\ell_i}\partial_t^{m-1}u\\&=\mathscr{B}(u,\partial_t u,\ldots,\partial_t^m u)[(u,\partial_t u,\ldots,\partial_t^m u)]\end{aligned}\tag{5.1}$$

with $0\leq \ell_1<\ell_2<\cdots \ell_r$, $0\leq k_m$,

$$\mathscr{B}:\mathscr{D}_1(\mathscr{B})\to L(\mathscr{D}_2(\mathscr{B}),\mathscr{H}),\qquad \mathscr{D}_1(\mathscr{B}),\ \mathscr{D}_2(\mathscr{B})\ \subseteq \mathscr{H}^{m+1}$$

and a selfadjoint densely defined operator $\mathscr{A}$ on a Hilbert space $\mathscr{H}$

$$\mathscr{A}:\mathscr{D}(\mathscr{A})\to\mathscr{H},\qquad \mathscr{D}\subseteq\mathscr{H}$$

satisfying a Poincaré-Friedrichs type inequality

$$|v|\leq C_{PF}|\mathscr{A}v|,\quad v\in\mathscr{D}(\mathscr{A})\tag{5.2}$$

(e.g. $\mathscr{A}=-\Delta_D$, $\mathscr{D}(\mathscr{A})=H^2(\Omega)\cap H_0^1(\Omega)$, $\mathscr{H}=L^2(\Omega)$);
We will use the notation $|\cdot|$ and $(\cdot|\cdot)\ldots$ for norm and inner product on $\mathscr{H}$, respectively.

Recall that this applies to the above models with the following settings:

- Westervelt: $m=1$, $k_m=1$, $r=1$, $\ell_r=1$
- JMGT: $m=2$, $k_m=0$, $r=1$, $\ell_r=1$
- BCBJK: $m=2$, $k_m=1$, $r=2$, $\ell_1=1$, $\ell_r=2$

First order reformulation. We rewrite (5.1) as

$$\begin{aligned}&\mathbf{u}'(t)+\mathbf{A}\big(\mathbf{u}(t)\big)\,\mathbf{u}(t)=\mathbf{B}\big(\mathbf{u}(t)\big)\big[\mathbf{u}(t)\big],\quad t\in(0,T),\\&\mathbf{u}(t)=\begin{pmatrix}u_0(t)\\u_1(t)\\\vdots\\u_m(t)\end{pmatrix},\quad \mathbf{B}\big(\mathbf{u}(t)\big)\big[\mathbf{u}(t)\big]=\begin{pmatrix}0\\\vdots\\0\\\mathscr{B}\big(\mathbf{u}(t)\big)\big[\mathbf{u}(t)\big]\end{pmatrix},\\&\mathbf{A}\big(\mathbf{u}(t)\big)=\begin{pmatrix}0&-I&0&\cdots&\cdots&0\\\vdots&0&-I&0&\cdots&\vdots\\\vdots&&\ddots&\ddots&0&\vdots\\\vdots&&&\ddots&-I&0\\0&\cdots&\cdots&\cdots&\sum_{i=1}^r\mathsf{c}_i\big(\mathbf{u}(t)\big)\,\mathscr{A}^{\ell_i}&\mathsf{b}\big(\mathbf{u}(t)\big)\,\mathscr{A}^{k_m}\end{pmatrix},\end{aligned}\tag{5.3}$$

5.1. **Energy estimates in the linearised setting.** With coefficients $\mathsf{b},\mathsf{c}_i\in L^\infty(0,T;L(\mathscr{H},\mathscr{H}))$ (i.e., in case of $\mathscr{H}=L^2(\Omega)$ simply $\mathsf{b},\mathsf{c}_i\in L^\infty(0,T;L^\infty(\Omega)))$) that may depend on space and time and a one-homogeneosly bounded operator $\mathscr{G}$

$$\begin{aligned}&\mathscr{G}:(0,T)\times X_0\to\mathscr{H},\quad X_0=X_{0,0}\times X_{0,1}\times X_{0,m}\subseteq\mathscr{H}^{m+1}\\&\big|\mathscr{G}(t)[\vec{v}]\big|=\big|\mathscr{G}(t;v_0,\ldots,v_m)\big|\leq C_G\|\vec{v}\|_{X_0}\end{aligned}\tag{5.4}$$

with a uniform constant C_G (note that $\mathscr{G}$ does not necessarily need to be linear to satisfy this, but it will be in our application of the estimates to the models above) with the right space X_0 yet to be determined, and $f \in L^2(0,T;\mathscr{H})$, we consider

$$\partial_t^{m+1} u + \mathtt{b}\mathscr{A}^{k_m}\partial_t^m u + \sum_{i=1}^r \mathtt{c}_i \mathscr{A}^{\ell_i}\partial_t^{m-1} u = \mathscr{G}[u, \partial_t u, \dots, \partial_t^m u] + f \tag{5.5}$$

i.e., written as a first order system

$$\begin{aligned} &\mathbf{u}'(t) + \mathbf{A}(t)\mathbf{u}(t) = (0, \cdots, 0, \mathscr{G}(t)[\mathbf{u}(t)] + f(t))^T \\ &\text{with } \mathbf{A} = \begin{pmatrix} 0 & -I & 0 & \cdots & 0 \\ 0 & 0 & -I & 0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots \\ 0 & & & \sum_{i=1}^r \mathtt{c}_i \mathscr{A}^{\ell_i} & \mathtt{b}\mathscr{A}^{k_m} \end{pmatrix} \end{aligned} \tag{5.6}$$

First of all we demonstrate the ideas with constants $\mathtt{b} > 0$, $\mathtt{c}_i \geq 0$.
Testing with

$$(0, \dots, 0, \sum_{i=1}^r \mathscr{A}^{k_m}[\mathtt{c}_i \mathscr{A}^{\ell_i} u_{m-1}(t)], \mathscr{A}^{k_m} u_m(t))^T \tag{5.7}$$

yields

$$\begin{aligned} &\Big(u_{m-1}'(t) - u_m(t) \Big| \sum_{i=1}^r \mathscr{A}^{k_m}[\mathtt{c}_i \mathscr{A}^{\ell_i} u_{m-1}(t)]\Big) \\ &\qquad + \Big(u_m'(t) + \sum_{i=1}^r \mathtt{c}_i \mathscr{A}^{\ell_i} u_{m-1}(t)] + \mathtt{b}\mathscr{A}^{k_m} u_m(t) \Big| \mathscr{A}^{k_m} u_m(t)\Big) \\ &= \frac{1}{2}\frac{d}{dt}\Big|\mathscr{A}^{k_m/2} u_m\Big|^2(t) + \sum_{i=1}^r \frac{\mathtt{c}_i}{2}\frac{d}{dt}\Big|\mathscr{A}^{(k_m+\ell_i)/2} u_{m-1}\Big|^2(t) + \mathtt{b}\Big|\mathscr{A}^{k_m} u_m(t)\Big|^2 \\ &\quad = \Big(\mathscr{G}(t)[\mathbf{u}(t)] + f \Big| \mathscr{A}^{k_m} u_m(t)\Big) \ \leq \frac{1}{\mathtt{b}}\Big|\mathscr{G}(t)[\mathbf{u}(t)]\Big|^2 + \frac{1}{\mathtt{b}}\Big|f(t)\Big|^2 + \frac{\mathtt{b}}{2}\Big|\mathscr{A}^{k_m} u_m(t)\Big|^2, \end{aligned} \tag{5.8}$$

hence after integration and taking the supremum wrt. time, we get, for arbitrary $s \in [0,T]$, using $\sup_{t\in(0,s)}(a(t)+b(t)+c(t)) \geq \frac{1}{2}\sup_{t\in(0,T)} a(t) + \frac{1}{4}\sup_{t\in(0,T)} b(t) + \frac{1}{4}\sup_{t\in(0,T)} c(t)$,

$$\begin{aligned} &\|\mathscr{A}^{k_m/2} u_m\|^2_{L^\infty(0,s;\mathscr{H})} + \sum_{i=1}^r \frac{\mathtt{c}_i}{2}\|\mathscr{A}^{(k_m+\ell_i)/2} u_{m-1}\|^2_{L^\infty(0,T;\mathscr{H})} + \frac{\mathtt{b}}{2}\|\mathscr{A}^{k_m} u_m(t)\|^2_{L^2(0,s;\mathscr{H})} \\ &\leq \frac{4}{\mathtt{b}}\big(\|\mathscr{G}[\mathbf{u}]\|^2_{L^2(0,s;\mathscr{H})} + \|f\|^2_{L^2(0,s;\mathscr{H})}\big) + \big|\mathscr{A}^{k_m/2} u_m(0)\big|^2 + \sum_{i=1}^r \frac{\mathtt{c}_i}{2}\big|\mathscr{A}^{(k_m+\ell_i)/2} u_{m-1}(0)\big|^2, \end{aligned} \tag{5.9}$$

where due to the fact that $u_{m-2}(t) = u_{m-2}(0) + \int_0^t u_{m-1}(\tau)d\tau$, with

$$\bar{\mathbf{u}}(t) := (u_0(t), \dots, u_{m-3}(t), u_{m-2}(0) + \int_0^t u_{m-1}(\tau)d\tau, u_{m-1}(t), u_m(t))$$

we have

$$\begin{aligned} &\|\mathscr{G}[\mathbf{u}]\|^2_{L^2(0,s;\mathscr{H})} \leq C_G^2 \int_0^s \|\bar{\mathbf{u}}(t)\|^2_{X_0}\,dt \\ &\leq C_G^2 \int_0^s \Big(\|\mathbf{u}(t)\|_{X_0} + \|u_{m-2}(0)\|_{X_{0,m-2}} + \int_0^t \|u_{m-1}(\tau)\|_{X_{0,m-2}}\,d\tau\Big)\,dt\,. \end{aligned} \tag{5.10}$$

This extends to the case of space and time dependent multipliers

$$\begin{aligned}
&\mathtt{b}\,,\mathtt{c}_i \in L^\infty(0,T;L(\mathscr{H},\mathscr{H}))\,, \quad \mathtt{b} \geq \underline{\mathtt{b}} > 0 \quad \mathtt{c}_i \geq \underline{\mathtt{c}}_i > 0 \\
&\text{and either } \mathtt{b}'\,,\mathtt{c}_i' \in L^\infty(0,T;L(\mathscr{H},\mathscr{H})) \\
&\qquad \text{or } \|\mathtt{b}'\|_{L^1(0,T;L(\mathscr{H},\mathscr{H}))}\,,\ \|\mathtt{c}_i'\|_{L^1(0,T;L(\mathscr{H},\mathscr{H}))} \text{ small enough}
\end{aligned} \tag{5.11}$$

via the identity

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big|\sqrt{\mathtt{c}}\mathscr{A}^{(p+q)/2}v\Big|^2(t) = \frac{1}{2}\frac{d}{dt}\Big(\mathscr{A}^{(p+q)/2}v\Big|\mathtt{c}\mathscr{A}^{(p+q)/2}v)(t)\Big) \\
&= \Big(\mathscr{A}^{(p+q)/2}v'(t)\Big|\mathtt{c}(t)\mathscr{A}^{(p+q)/2}v(t)\Big) + \frac{1}{2}\Big(\mathscr{A}^{(p+q)/2}v(t)\Big|\mathtt{c}'(t)\mathscr{A}^{(p+q)/2}v(t)\Big) \\
&= \Big(v'(t)\Big|\mathscr{A}^{p}[\mathtt{c}(t)\mathscr{A}^{q}v(t)]\Big) \\
&\quad + \Big(v'(t)\Big|\mathscr{A}^{(p+q)/2}[\mathtt{c}(t)\mathscr{A}^{(p+q)/2}v(t)] - \mathscr{A}^{p}[\mathtt{c}(t)\mathscr{A}^{q}v(t)]\Big) \\
&\quad + \frac{1}{2}\Big(\mathtt{c}'(t)\mathscr{A}^{(p+q)/2}v(t)\Big|\mathscr{A}^{(p+q)/2}v(t)\Big)
\end{aligned} \tag{5.12}$$

which in case of $v = u_{m-1}$ and $\mathtt{c} = \mathtt{c}_i$, $p = k_m$, $q = \ell_i$, using the fact that $u_{m-1}' = u_m$, reads as

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big|\sqrt{\mathtt{c}}\mathscr{A}^{k_m+\ell_i}u_{m-1}\Big|^2(t) \;&= \Big(u_{m-1}'(t)\Big|\mathscr{A}^{k_m}[\mathtt{c}(t)\mathscr{A}^{\ell_i}u_{m-1}(t)]\Big) \\
&+ \Big(\mathscr{A}^{k_m}u_m(t)\Big|\mathscr{A}^{(\ell_i-k_m)/2}[\mathtt{c}(t)\mathscr{A}^{(k_m+\ell_i)/2}u_{m-1}(t)] - [\mathtt{c}(t)\mathscr{A}^{\ell_i}u_{m-1}(t)]\Big) \\
&+ \frac{1}{2}\Big(\mathtt{c}'(t)\mathscr{A}^{(k_m+\ell_i)/2}u_{m-1}(t)\Big|\mathscr{A}^{(k_m+\ell_i)/2}u_{m-1}(t)\Big)
\end{aligned}$$

This results in

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big|\mathscr{A}^{k_m/2}u_m\Big|^2(t) + \frac{1}{2}\sum_{i=1}^{r}\frac{d}{dt}\Big|\sqrt{\mathtt{c}_i}\mathscr{A}^{(k_m+\ell_i)/2}u_{m-1}\Big|^2(t) + \Big|\sqrt{\mathtt{b}(t)}\mathscr{A}^{k_m}u_m(t)\Big|^2 \\
&\quad = \Big(\mathscr{G}(t)[\mathbf{u}(t)] + f(t)\Big|\mathscr{A}^{k_m}u_m(t)\Big) \\
&\qquad + \frac{1}{2}\sum_{i=1}^{r}\Big(\mathtt{c}_i'(t)\mathscr{A}^{(k_m+\ell_i)/2}u_{m-1}(t)\Big|\mathscr{A}^{(k_m+\ell_i)/2}u_{m-1}(t)\Big) \\
&\qquad + \sum_{i=1}^{r}\Big(u_{m-1}'(t)\Big|\mathscr{A}^{(k_m+\ell_i)/2}[\mathtt{c}_i\mathscr{A}^{(k_m+\ell_i)/2}u_{m-1}(t)] - \mathscr{A}^{k_m}[\mathtt{c}_i\mathscr{A}^{\ell_i}u_{m-1}(t)]\Big) \\
&\qquad \leq \frac{1}{\underline{\mathtt{b}}}\Big|\mathscr{G}(t)[\mathbf{u}(t)] + f(t)\Big|^2 + \frac{1}{4}\Big|\sqrt{\mathtt{b}(t)}\mathscr{A}^{k_m}u_m(t)\Big|^2 \\
&\qquad\quad + \frac{1}{2}\sum_{i=1}^{r}\frac{1}{\underline{\mathtt{c}}_i}\Big|\mathtt{c}_i'(t)|_{L(\mathscr{H},\mathscr{H})}|\sqrt{\mathtt{c}_i(t)}\mathscr{A}^{(k_m+\ell_i)/2}u_{m-1}(t)\Big|^2 \\
&\qquad\qquad + \frac{1}{4}\Big|\sqrt{\mathtt{b}(t)}\mathscr{A}^{k_m}u_m(t)\Big|^2 \\
&\qquad\qquad\quad + \frac{1}{\underline{\mathtt{b}}}\sum_{i=1}^{r}\Big|\mathscr{A}^{(\ell_i-k_m)/2}[\mathtt{c}_i\mathscr{A}^{(k_m+\ell_i)/2}u_{m-1}(t)] - [\mathtt{c}_i\mathscr{A}^{\ell_i}u_{m-1}(t)]\Big|^2.
\end{aligned} \tag{5.13}$$

from which, as above, by time integration an energy estimate can be obtained.

A second energy estimate can be obtained by testing with

$$(0,\dots,0,\mathscr{A}^{k_m}[\mathsf{b}\mathscr{A}^{\ell_r}u_{m-1}(t)],\mathscr{A}^{\ell_r}u_{m-1}(t))^T, \tag{5.14}$$

which, by using (5.12) with $\mathsf{c}=\mathsf{b}$, $p=k_m$, $q=\ell_r$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big|\sqrt{\mathsf{b}}\mathscr{A}^{(k_m+\ell_r)/2}u_{m-1}\Big|^2(t)+\sum_{i=1}^{r}\Big|\sqrt{\mathsf{c}_i}\mathscr{A}^{(\ell_r+\ell_i)/2}u_{m-1}(t)\Big|^2\\
&=\Big(\mathscr{G}(t)[\mathbf{u}(t)]+f(t)\Big|\mathscr{A}^{\ell_r}u_{m-1}(t)\Big)\\
&\quad+\frac{1}{2}\Big(\mathsf{b}'(t)\mathscr{A}^{(k_m+\ell_r)/2}u_{m-1}(t)\Big|\mathscr{A}^{(k_m+\ell_r)/2}u_{m-1}(t)\Big)\\
&\quad+\Big(u_{m-1}'\Big|\mathscr{A}^{(k_m+\ell_r)/2}[\mathsf{b}\mathscr{A}^{(k_m+\ell_r)/2}u_{m-1}(t)]-\mathscr{A}^{k_m}[\mathsf{b}\mathscr{A}^{\ell_r}u_{m-1}(t)]\Big)\\
&\quad+\sum_{i=1}^{r}\Big(u_{m-1}\Big|\mathscr{A}^{(\ell_r+\ell_i)/2}[\mathsf{c}_i\mathscr{A}^{(\ell_r+\ell_i)/2}u_{m-1}(t)]-\mathscr{A}^{\ell_r}[\mathsf{c}_i\mathscr{A}^{\ell_i}u_{m-1}(t)]\Big)\\
&\le\frac{1}{\underline{\mathsf{c}}_r}\Big|\mathscr{G}(t)[\mathbf{u}(t)]+f(t)\Big|^2+\frac{1}{4}\Big|\sqrt{\mathsf{c}_r}\mathscr{A}^{\ell_r}u_{m-1}(t)\Big|^2\\
&\quad+\frac{1}{2\underline{\mathsf{c}}_r}\Big|\mathsf{b}'(t)|_{L(\mathscr{H},\mathscr{H})}|\sqrt{\mathsf{c}_r}\mathscr{A}^{(k_m+\ell_r)/2}u_{m-1}(t)\Big|^2\\
&\quad+\frac{1}{2}\Big|\sqrt{\mathsf{b}}\mathscr{A}^{k_m}u_m(t)\Big|^2+\frac{1}{2\underline{\mathsf{b}}}\Big|\mathscr{A}^{(\ell_r-k_m)/2}[\mathsf{b}\mathscr{A}^{(k_m+\ell_r)/2}u_{m-1}(t)]-[\mathsf{b}\mathscr{A}^{\ell_r}u_{m-1}(t)]\Big|^2\\
&\quad+\frac{1}{4}\Big|\sqrt{\mathsf{c}_r}\mathscr{A}^{\ell_r}u_{m-1}(t)\Big|^2\\
&\quad+\frac{1}{\underline{\mathsf{c}}_r}\left(\sum_{i=1}^{r}\Big|\mathscr{A}^{(\ell_i-\ell_r)/2}[\mathsf{c}_i\mathscr{A}^{(\ell_r+\ell_i)/2}u_{m-1}(t)]-[\mathsf{c}_i\mathscr{A}^{\ell_i}u_{m-1}(t)]\Big|\right)^2.
\end{aligned} \tag{5.15}$$

Therewith considering all left hand side terms in (5.13), (5.15), we expect – after time integration – to obtain bounds on

$$\begin{aligned}
&\|\mathscr{A}^{k_m/2}u_m\|_{L^\infty(0,T;\mathscr{H})},\quad \|\mathscr{A}^{k_m}u_m\|_{L^2(0,T;\mathscr{H})},\\
&\|\mathscr{A}^{k_m}u_{m-1}\|_{L^\infty(0,T;\mathscr{H})},\quad \|\mathscr{A}^{(k_m+\ell_r)/2}u_{m-1}\|_{L^\infty(0,T;\mathscr{H})},\quad \|\mathscr{A}^{\ell_r}u_{m-1}\|_{L^2(0,T;\mathscr{H})},\\
&\|\mathscr{A}^{(k_m+\ell_r)/2}u_{m-2}\|_{L^\infty(0,T;\mathscr{H})},\quad \|\mathscr{A}^{\ell_r}u_{m-2}\|_{L^\infty(0,T;\mathscr{H})},
\end{aligned}$$

and therefore, via the identity $u_j'=u_{j+1}$ and the estimate

$$\|u_j\|_{L^\infty(0,T;Z)}\le\|u_j(0)\|_Z+\sqrt{T}\|u_{j+1}\|_{L^2(0,T;Z)},$$

which allows to inherit the regularity of higher time derivatives,

$$\begin{aligned}
&\|\mathscr{A}^{k_m/2}u_m\|_{L^\infty(0,T;\mathscr{H})},\quad \|\mathscr{A}^{k_m}u_m\|_{L^2(0,T;\mathscr{H})},\\
&\|\mathscr{A}^{\max\{k_m,(k_m+\ell_r)/2\}}u_{m-1}\|_{L^\infty(0,T;\mathscr{H})},\quad \|\mathscr{A}^{\max\{k_m,\ell_r\}}u_{m-1}\|_{L^2(0,T;\mathscr{H})},\\
&\qquad\qquad \|\mathscr{A}^{k_m}u_{m-1}'\|_{L^2(0,T;\mathscr{H})},\\
&\|\mathscr{A}^{\max\{k_m,\ell_r\}}u_j\|_{L^\infty(0,T;\mathscr{H})},\quad \|\mathscr{A}^{\max\{k_m,\ell_r\}}u_j'\|_{L^2(0,T;\mathscr{H})},\quad j\in\{0,\dots,m-2\}.
\end{aligned} \tag{5.16}$$

Thus we define the energy induced spaces by

$$\begin{aligned} X = &\Big(L^\infty(0,T;\mathscr{D}(\mathscr{A}^{\max\{k_m,\ell_r\}}))\cap H^1(0,T;\mathscr{D}(\mathscr{A}^{\max\{k_m,\ell_r\}}))\Big)^{m-1} \\ &\times\ L^2(0,T;\mathscr{D}(\mathscr{A}^{\max\{k_m,\ell_r\}}))\cap L^\infty(0,T;\mathscr{D}(\mathscr{A}^{\max\{k_m,(k_m+\ell_r)/2\}})) \\ &\qquad\qquad\qquad\qquad\qquad \cap H^1(0,T;\mathscr{D}(\mathscr{A}^{k_m})) \\ &\times\ L^2(0,T;\mathscr{D}(\mathscr{A}^{k_m}))\cap L^\infty(0,T;\mathscr{D}(\mathscr{A}^{k_m/2})) \\ \subseteq\ & L^\infty(0,T;X_0)\cap L^2(0,T;X_1) \end{aligned} \tag{5.17}$$

with

$$\begin{aligned} X_0 &= X_{0,0}\times X_{0,1}\times\cdots\times X_{0,m} \\ &= \mathscr{D}(\mathscr{A}^{\max\{k_m,\ell_r\}})^{m-1}\ \times\ \mathscr{D}(\mathscr{A}^{\max\{k_m,(k_m+\ell_r)/2\}})\ \times\ \mathscr{D}(\mathscr{A}^{k_m/2}) \\ X_1 &= X_{1,0}\times X_{1,1}\times\cdots\times X_{1,m} \\ &=\ \mathscr{D}(\mathscr{A}^{\max\{k_m,\ell_r\}})^m\ \times\ \mathscr{D}(\mathscr{A}^{k_m})\,. \end{aligned} \tag{5.18}$$

Indeed, integrating with respect to time and applying Gronwall's inequality

$$\begin{aligned} &\eta(t)\le a(t)+\int_0^t b(s)\eta(s)\,ds \quad \text{for all } t\in(0,T) \\ &\Rightarrow\ \eta(t)\le a(t)+\int_0^t a(s)\,b(s)\,\exp\Big(\int_s^t b(\sigma)\,d\sigma\Big)\,ds \quad \text{for all } t\in(0,T) \end{aligned} \tag{5.19}$$

for $\eta,a,b\ge 0$ (see, e.g, [TESCHL (2012), Lemma 2.7]) to the energy estimates resulting from (5.13), (5.15), together with (5.2), $u_j(t)=u_j(0)+\int_0^t u_{j+1}(\tau)d\tau$, as well as (5.10) and the fact that $X_{1,m-1}$ continuosly embeds into $X_{0,m-2}$, yields uniform bounds on the energy terms (5.16), provided the initial data satisfy the regularity

$$(u_0(0),u_1(0),\ldots,u_m(0))\in X_0 \tag{5.20}$$

and the terms emerging from space dependence of the multipliers can be dominated by the corresponding energy terms:

$$\begin{aligned} &|\mathscr{A}^{(\ell_i-k_m)/2}[\mathsf{c}_i\mathscr{A}^{(k_m+\ell_i)/2}u_{m-1}(t)]-[\mathsf{c}_i\mathscr{A}^{\ell_i}u_{m-1}(t)]| \\ &+|\mathscr{A}^{(\ell_r-k_m)/2}[\mathsf{b}\mathscr{A}^{(k_m+\ell_r)/2}u_{m-1}(t)]-[\mathsf{b}\mathscr{A}^{\ell_r}u_{m-1}(t)]| \\ &+|\mathscr{A}^{(\ell_i-\ell_r)/2}[\mathsf{c}_i\mathscr{A}^{(\ell_r+\ell_i)/2}u_{m-1}(t)]-[\mathsf{c}_i\mathscr{A}^{\ell_i}u_{m-1}(t)]| \\ &\le C_{\mathsf{bc}}\Big(|\mathscr{A}^{k_m}u_{m-1}(t)|+|\mathscr{A}^{(k_m+\ell_r)/2}u_{m-1}(t)|\Big)+c_{\mathsf{bc}}|\mathscr{A}^{\ell_r}u_{m-1}(t)|\,. \end{aligned} \tag{5.21}$$

We therefore arrive at the following result.

Proposition 5.1. *Under conditions* (5.2)*,* (5.4)*,* (5.11)*,* (5.20)*,* (5.21)*, with c_{bc} small enough, any solution u to* (5.5) *satisfies the estimate*

$$\|(u,\partial_t u,\ldots,\partial_t^m u)\|_X\le C\Big(\|(u(0),\partial_t u(0),\ldots,\partial_t^m u(0))\|_{X_0}+\|f\|_{L^2(0,T;\mathscr{H})}\Big)$$

for some C depending only on T and the constants $\underline{bb}$, $\underline{cc}_i$, C_{PF}, C_{bc}, C_G.

Remark 5.1. Existence of a solution to (5.5) with given initial data of the regularity prescribed in Proposition 5.1 can be proven, e.g., by means of a Galerkin discretisation with eigenfunctions of $\mathscr{A}$ and taking limits as the discretisation gets finer, based on energy estimates like those in Proposition 5.1, cf., e.g., [KALTENBACHER, NIKOLIĆ (2019)]. Uniqueness follows from Proposition 5.1, since the difference between any two solutions satisfies (5.5) with homogeneous initial data and $f = 0$.

5.2. **Energy estimates and well-posendess in the nonlinear setting.** To establish well-posedness and energy estimates for the nonlinear equation (5.1), i.e., (5.3) with initial conditions

$$\mathbf{u}(0) = (u(0), \partial_t u(0), \dots, \partial_t^m u(0)) = \vec{u}^0 \in X_0\,, \tag{5.22}$$

we define the fixed point operator $\mathscr{T} : B_R^X(0) \to X$ by assigning to $\mathbf{v} \in B_R^X(0) = \{\mathbf{v} \in X \,:\, \|\mathbf{v}\|_X \leq R\}$ the solution of the linear initial value problem (5.5), i.e., (5.6) with

$$\mathsf{b}(t) = \mathsf{b}(\mathbf{v}(t))\,, \quad \mathsf{c}_i(t) = \mathsf{c}_i(\mathbf{v}(t))\,, \quad \mathscr{G}(t)[\vec{v}] = \mathscr{B}(\mathbf{v}(t))[\vec{v}]\,, \quad f = 0 \tag{5.23}$$

and (5.22).

We assume that we can choose $R > 0$ such that

$$B_R^X(0) \subseteq \{\mathbf{v} \in X \,:\, \mathsf{b} = \mathsf{b}(\mathbf{v})\,, \quad \mathsf{c}_i = \mathsf{c}_i(\mathbf{v})\,, \quad \mathscr{G} = \mathscr{B}(\mathbf{v}) \text{ satisfy (5.4),(5.11),(5.21)}\} \tag{5.24}$$

(we will verfiy this for the nonlinear acoustics models from Section 2 in Section 5.3 below). Proposition 5.1 then implies that $\mathscr{T}$ is a self-mapping on $B_R^X(0)$ provided $\|\vec{u}_0\|_{X_0} \leq \frac{R}{C}$.

To obtain contractivity of $\mathscr{T}$ we assume that

$$\begin{aligned} &\sum_{i=1}^r \|\mathsf{c}_i(\vec{v}) - \mathsf{c}_i(\vec{\tilde{v}})\|_{L(\mathscr{H},\mathscr{H})} C_{PF}^{r-i} \leq C_{\mathrm{c}} \|\vec{v} - \vec{\tilde{v}}\|_{X_0}\,, \quad \|\mathsf{b}(\vec{v}) - \mathsf{b}(\vec{\tilde{v}})\|_{L(\mathscr{H},\mathscr{H})} \leq C_{\mathrm{b}} \|\vec{v} - \vec{\tilde{v}}\|_{X_0} \\ &\|\mathscr{B}(\vec{v}) - \mathscr{B}(\vec{\tilde{v}})\|_{L(X_1,\mathscr{H})} \leq C_B \|\vec{v} - \vec{\tilde{v}}\|_{X_0} \qquad \text{for all } \vec{v}, , \vec{\tilde{v}} \in X_0 \end{aligned} \tag{5.25}$$

(again to be verified for the nonlinear acoustics models from Section 2 in Section 5.3 below) and the fact that $\hat{\mathbf{u}} := \mathbf{u} - \tilde{\mathbf{u}} := \mathscr{T}(\mathbf{v}) - \mathscr{T}(\tilde{\mathbf{v}})$ with $\mathbf{v}$, $\tilde{\mathbf{v}} \in B_R^X(0)$ (thus, by the already shown self-mapping property of $\mathscr{T}$, also $\mathbf{u}$, $\tilde{\mathbf{u}} \in B_R^X(0)$) satisfies (5.6) with b, c_i, $\mathscr{G}$ as in (5.23),

$$f = (\mathscr{B}(\mathbf{v}) - \mathscr{B}(\tilde{\mathbf{v}}))[\tilde{\mathbf{u}}] + (\mathbf{A}(\mathbf{v}) - \mathbf{A}(\tilde{\mathbf{v}}))_{m+1} \tilde{\mathbf{u}}$$

and homogeneous initial data. Proposition 5.1 and the estimate

$$\begin{aligned} \|f\|_{L^2(0,T;\mathscr{H})} &\leq \|(\mathscr{B}(\mathbf{v}) - \mathscr{B}(\tilde{\mathbf{v}}))[\tilde{\mathbf{u}}]\|_{L^2(0,T;\mathscr{H})} \\ &\quad + \|\sum_{i=1}^r (\mathsf{c}_i(\mathbf{v}) - \mathsf{c}_i(\tilde{\mathbf{v}}) \mathscr{A}^{\ell_i} \tilde{u}_{m-1}\|_{L^2(0,T;\mathscr{H})} + \|(\mathsf{b}(\mathbf{v}) - \mathsf{b}(\tilde{\mathbf{v}}) \mathscr{A}^{k_m} \tilde{u}_m\|_{L^2(0,T;\mathscr{H})} \\ &\leq \Big(\int_0^T \|\mathscr{B}(\mathbf{v}(t)) - \mathscr{B}(\tilde{\mathbf{v}})\|_{L(X_1,\mathscr{H})}^2 \|\tilde{\mathbf{u}}(t)\|_{X_1}^2 \, dt \Big)^{1/2} \\ &\quad + \Big(\int_0^T \|\sum_{i=1}^r (\mathsf{c}_i(\mathbf{v}(t)) - \mathsf{c}_i(\tilde{\mathbf{v}}(t)) C_{PF}^{r-i}\|_{L(\mathscr{H},\mathscr{H})}^2 |\mathscr{A}^{\ell_r} \tilde{u}_{m-1}(t)|^2 \, dt \Big)^{1/2} \\ &\quad + \Big(\int_0^T \|\mathsf{b}(\mathbf{v}(t)) - \mathsf{b}(\tilde{\mathbf{v}}(t))\|_{L(\mathscr{H},\mathscr{H})}^2 |\mathscr{A}^{k_m} \tilde{u}_m(t)|^2 \, dt \Big)^{1/2} \\ &\leq (C_{\mathrm{c}} + C_{\mathrm{b}} + C_B) R \|\mathbf{v} - \tilde{\mathbf{v}}\|_X \end{aligned}$$

yields contractivity for R small enough. Thus from Banach's Fixed Point Theorem we obtain the following result.

Theorem 5.1. *Under conditions* (5.2), (5.24), (5.25), *there exists* $R > 0$ *(sufficiently small) such that for any* $\|\vec{u}_0\|_{X_0} \leq \frac{R}{C}$ *the initial value problem* (5.1), (5.22) *has a unique solution* $\mathbf{u} = (u, \partial_t u, \ldots, \partial_t^m u) \in X$ *and this solution satisfies the estimate*

$$\|(u, \partial_t u, \ldots, \partial_t^m u)\|_X \leq C\|\vec{u}^0\|_{X_0}$$

with C as in Proposition 5.1.

5.3. **Application to models from nonlinear acoustics.** We now verify conditions (5.24), (5.25), i.e.,

$$\|\mathbf{v}\|_X \leq R \;\Rightarrow\; \mathrm{b}(\mathbf{v}), \mathrm{c}_i(\mathbf{v}) \in L^\infty(0,T;L(\mathscr{H},\mathscr{H})) \cap W^{1,1}(0,T;L(\mathscr{H},\mathscr{H}))\,, \tag{5.26}$$

$$\begin{aligned}
&\|\vec{v}\|_{X_0} \leq R \;\Rightarrow \\
&\Big(\mathrm{b}(\vec{v}) \geq \underline{\mathrm{b}} > 0 \quad \mathrm{c}_i(\vec{v}) \geq \underline{\mathrm{c}}_i > 0 \\
&\quad |\mathscr{A}^{(\ell_i-k_m)/2}[\mathrm{c}_i(\vec{v})\mathscr{A}^{(k_m+\ell_i)/2}w_{m-1}] - [\mathrm{c}_i(\vec{v})\mathscr{A}^{\ell_i}w_{m-1}]| \\
&\qquad + |\mathscr{A}^{(\ell_r-k_m)/2}[\mathrm{b}(\vec{v})\mathscr{A}^{(k_m+\ell_r)/2}w_{m-1}] - [\mathrm{b}(\vec{v})\mathscr{A}^{\ell_r}w_{m-1}]| \\
&\qquad + |\mathscr{A}^{(\ell_i-\ell_r)/2}[\mathrm{c}_i(\vec{v})\mathscr{A}^{(\ell_r+\ell_i)/2}w_{m-1}] - [\mathrm{c}_i(\vec{v})\mathscr{A}^{\ell_i}w_{m-1}]| \\
&\qquad \leq C_{\mathsf{bc}}\big(|\mathscr{A}^{k_m}w_{m-1}| + |\mathscr{A}^{(k_m+\ell_r)/2}w_{m-1}|\big) + c_{\mathsf{bc}}|\mathscr{A}^{\ell_r}w_{m-1}|\,, \text{ for all } \vec{w} \in X_1 \\
&\quad |\mathscr{B}(\vec{v})[\vec{w}]| \leq C_G\|\vec{w}\|_{X_0} \text{ for all } \vec{w} \in X_0\Big)
\end{aligned} \tag{5.27}$$

and

$$\begin{aligned}
&\sum_{i=1}^{r} \|\mathrm{c}_i(\vec{v}) - \mathrm{c}_i(\vec{\tilde{v}})\|_{L(\mathscr{H},\mathscr{H})} C_{PF}^{r-i} \leq C_{\mathrm{c}}\|\vec{v} - \vec{\tilde{v}}\|_{X_0}\,, \\
&\|\mathrm{b}(\vec{v}) - \mathrm{b}(\vec{\tilde{v}})\|_{L(\mathscr{H},\mathscr{H})} \leq C_{\mathrm{b}}\|\vec{v} - \vec{\tilde{v}}\|_{X_0}\,, \\
&\|\mathscr{B}(\vec{v}) - \mathscr{B}(\vec{\tilde{v}})\|_{L(X_1,\mathscr{H})} \leq C_B\|\vec{v} - \vec{\tilde{v}}\|_{X_0} \text{ for all } \vec{v}, \vec{\tilde{v}} \in X_0
\end{aligned} \tag{5.28}$$

for the models from Section 2.

BCBJK:. $m = 2$, $k_m = 1$, $r = 2$, $\ell_1 = 1$, $\ell_r = 2$,

$$\begin{aligned}
&X_0 = \mathscr{D}(\mathscr{A}^2) \times \mathscr{D}(\mathscr{A}^{3/2}) \times \mathscr{D}(\mathscr{A}^{1/2})\,, \quad X_1 = \mathscr{D}(\mathscr{A}^2) \times \mathscr{D}(\mathscr{A}^2) \times \mathscr{D}(\mathscr{A})\,, \\
&\mathrm{b}(\vec{v}) = \frac{\beta_1}{\alpha(v_1)}\,, \quad \mathrm{c}_1(\vec{v}) = \frac{\beta_3}{\alpha(v_1)}\,, \quad \mathrm{c}_2(\vec{v}) = \frac{\beta_2}{\alpha(v_1)}\,, \quad \alpha(v_1) = 1 + \beta_5 v_1\,, \\
&\mathscr{B}(\vec{v})[\vec{w}] = \frac{1}{\alpha(v_1)}\Big(-\beta_4\,\mathscr{A}^2 w_0 + \beta_5\,v_2\,w_2 + 2\nabla v_1 \cdot \nabla w_1 + 2\nabla v_0 \cdot \nabla w_2\Big)\,.
\end{aligned}$$

Conditions (5.26), (5.27) follow from

$$|\frac{d}{dt}\mathrm{b}(\mathbf{v})(t)|_{L(\mathscr{H},\mathscr{H})} = \beta_1\beta_5\Big|\frac{1}{\alpha(v_1(t))^2}\,v_2(t)\Big|_{L^\infty(\Omega)} \leq \beta_1\beta_5\Big|\frac{1}{\alpha(v_1(t))}\Big|^2_{L^\infty(\Omega)}|v_2(t)|_{L^\infty(\Omega)}$$

and likewise for c_1, c_2, as well as the estimates

$$\begin{aligned}
|v_2|_{L^1(0,T;L^\infty(\Omega))} &\le \sqrt{T} C_{\mathscr{D}(\mathscr{A})\to L^\infty(\Omega)} \|\mathscr{A} v_2\|_{L^2(0,T;\mathscr{H})} \le \sqrt{T} C_{\mathscr{D}(\mathscr{A})\to L^\infty(\Omega)} R \\
\Big|\frac{1}{\alpha(v_1)}\Big|_{L^\infty(0,T;L^\infty(\Omega)} &\le \frac{1}{1-|1-\alpha(v_1)|_{L^\infty(0,T;L^\infty(\Omega)}} \\
|1-\alpha(v_1)|_{L^\infty(0,T;L^\infty(\Omega)} &\le \beta_5 C_{\mathscr{D}(\mathscr{A}^{3/2})\to L^\infty(\Omega)} \|\mathscr{A} v_1\|_{L^\infty(0,T;\mathscr{H})} \\
&\le \beta_5 C_{\mathscr{D}(\mathscr{A}^{3/2})\to L^\infty(\Omega)} R
\end{aligned} \tag{5.29}$$

$$\begin{aligned}
&|\mathscr{A}^{1/2}[\tfrac{1}{\alpha(v_1)}\mathscr{A}^{3/2} w_{m-1}] - [\tfrac{1}{\alpha(v_1)}\mathscr{A}^2 w_{m-1}]| \\
&= |\nabla[\tfrac{1}{\alpha(v_1)}\nabla\Delta w_{m-1}] - [\tfrac{1}{\alpha(v_1)}\Delta^2 w_{m-1}]| = \tfrac{1}{c^2}\tfrac{B}{A}|\tfrac{1}{\alpha(v_1)^2}\nabla v_1\cdot\nabla\Delta w_{m-1}| \\
&\le \tfrac{1}{c^2}\tfrac{B}{A}|\tfrac{1}{\alpha(v_1)^2}|_{L(\mathscr{H},\mathscr{H})} C_{\mathscr{D}(\mathscr{A})\to L^\infty(\Omega)} |\mathscr{A}^{3/2} v_1|\,|\mathscr{A}^{3/2} w_{m-1}| \\
&\le \tfrac{1}{c^2}\tfrac{B}{A}|\tfrac{1}{\alpha(v_1)^2}|_{L(\mathscr{H},\mathscr{H})} C_{\mathscr{D}(\mathscr{A})\to L^\infty(\Omega)} R\,|\mathscr{A}^{(k_m+\ell_r)/2} w_{m-1}|
\end{aligned}$$

$$\begin{aligned}
&|\mathscr{A}^{-1/2}[\tfrac{1}{\alpha(v_1)}\mathscr{A}^{3/2} w_{m-1}] - [\tfrac{1}{\alpha(v_1)}\mathscr{A} w_{m-1}]| \\
&= |\mathscr{A}^{-1/2}\Big([\tfrac{1}{\alpha(v_1)}\nabla\Delta w_{m-1}] - \nabla[\tfrac{1}{\alpha(v_1)}\Delta w_{m-1}]\Big)| \\
&= \tfrac{1}{c^2}\tfrac{B}{A}|\mathscr{A}^{-1/2}[\tfrac{1}{\alpha(v_1)^2}\nabla v_1\,\Delta w_{m-1}]| \\
&\le C_{PF}\tfrac{1}{c^2}\tfrac{B}{A}|\tfrac{1}{\alpha(v_1)^2}|_{L(\mathscr{H},\mathscr{H})} C_{\mathscr{D}(\mathscr{A})\to L^\infty(\Omega)} R\,|\mathscr{A}^{(k_m+\ell_r)/2} w_{m-1}|
\end{aligned}$$

$$\begin{aligned}
&|\mathscr{B}(\vec{v})[\vec{w}]| \\
&\le |\tfrac{1}{\alpha(v_1)}|_{L(\mathscr{H},\mathscr{H})}\Big(\beta_4|\mathscr{A}^2 w_0| + \beta_5|v_2 w_2| + 2|\nabla v_1\cdot\nabla w_1| + 2|\nabla v_0\cdot\nabla w_2|\Big) \\
&\le |\tfrac{1}{\alpha(v_1)}|_{L(\mathscr{H},\mathscr{H})}\Big(\beta_4|\mathscr{A}^2 w_0| + \beta_5 C^2_{\mathscr{D}(\mathscr{A}^{1/2})\to L^4(\Omega)}|\mathscr{A}^{1/2} v_2|\,|\mathscr{A}^{1/2} w_2| \\
&\qquad + 2C^2_{\mathscr{D}(\mathscr{A})\to W^{1,4}(\Omega)}|\mathscr{A} v_1|\,|\mathscr{A} w_1| + 2C_{\mathscr{D}(\mathscr{A})\to L^\infty(\Omega)}|\mathscr{A}^{3/2} v_0|\,|\mathscr{A}^{1/2} w_2|\Big) \\
&\le |\tfrac{1}{\alpha(v_1)}|_{L(\mathscr{H},\mathscr{H})}\|\vec{w}\|_{X_0} \\
&\quad \Big(\beta_4 + R\Big(\beta_5 C^2_{\mathscr{D}(\mathscr{A}^{1/2})\to L^4(\Omega)} + 2C^2_{\mathscr{D}(\mathscr{A})\to W^{1,4}(\Omega)} C_{PF} + 2C_{\mathscr{D}(\mathscr{A})\to L^\infty(\Omega)}|\Big)\Big).
\end{aligned} \tag{5.30}$$

Condition (5.28) follows from the estimates

$$\begin{aligned}
\Big|\frac{1}{\alpha(v_1)} - \frac{1}{\alpha(\tilde{v}_1)}\Big|_{L(\mathscr{H},\mathscr{H})} &= \beta_5\Big|\frac{1}{\alpha(v_1)\alpha(\tilde{v}_1)}(v_1-\tilde{v}_1)\Big|_{L^\infty(\Omega)} \\
&\le \beta_5\Big|\frac{1}{\alpha(v_1)}\Big|_{L^\infty(\Omega)}\Big|\frac{1}{\alpha(\tilde{v}_1)}\Big|_{L^\infty(\Omega)} C_{\mathscr{D}(\mathscr{A}^{3/2})\to L^\infty(\Omega)}\|\vec{v}-\vec{\tilde{v}}\|_{X_0}
\end{aligned} \tag{5.31}$$

as well as the identity

$$\begin{aligned}
\mathscr{B}(\vec{v})\vec{w} - \mathscr{B}(\vec{\tilde{v}})\vec{w} = \frac{1}{\alpha(\tilde{v}_1)}\Big(&(\alpha(\tilde{v}_1)-\alpha(v_1))\,\mathscr{B}(\vec{v})\vec{w} \\
&+ \beta_5(v_2-\tilde{v}_2)\,w_2 + 2\nabla(v_1-\tilde{v}_1)\cdot\nabla w_1 + 2\nabla(v_0-\tilde{v}_0)\cdot\nabla w_2\Big)
\end{aligned}$$

and the estimates (5.29), (5.30), (5.31), as well as (analogously to (5.30))

$$|(v_2-\tilde{v}_2)\,w_2| \le C^2_{\mathscr{D}(\mathscr{A}^{1/2})\to L^4(\Omega)}|\mathscr{A}^{1/2}(v_2-\tilde{v}_2)|\,|\mathscr{A}^{1/2}w_2|$$
$$|\nabla(v_1-\tilde{v}_1)\cdot\nabla w_1| \le C^2_{\mathscr{D}(\mathscr{A})\to W^{1,4}(\Omega)}|\mathscr{A}(v_1-\tilde{v}_1)|\,|\mathscr{A}w_1|$$
$$|\nabla(v_0-\tilde{v}_0)\cdot\nabla w_2| \le C_{\mathscr{D}(\mathscr{A})\to L^\infty(\Omega)}|\mathscr{A}^{3/2}(v_0-\tilde{v}_0)|\,|\mathscr{A}^{1/2}w_2|\,,$$

hence altogether

$$|\mathscr{B}(\vec{v})\vec{w}-\mathscr{B}(\vec{\tilde{v}})\vec{w}| \le C_B\|\vec{v}-\vec{\tilde{v}}\|_{X_0}\|\vec{w}\|_{X_0}\,.$$

JMGT:. $m=2$, $k_m=0$, $r=1$, $\ell_r=1$

$$X_0=\mathscr{D}(\mathscr{A})\times\mathscr{D}(\mathscr{A}^{1/2})\times\mathscr{H}\,,\quad X_1=\mathscr{D}(\mathscr{A})\times\mathscr{D}(\mathscr{A})\times\mathscr{H}\,,$$
$$\mathrm{b}(\vec{v})=\frac{\alpha(v_0)}{T_{rel}}\,,\quad \mathrm{c}(\vec{v})=\frac{b}{T_{rel}}\,,\quad \alpha(v_0)=1-\frac{2\beta_a}{\rho c^2}v_0\,,$$
$$\mathscr{B}(\vec{v})[\vec{w}]=-\frac{c^2}{T_{rel}}\mathscr{A}w_0+\frac{2\beta_a}{\rho c^2}v_1w_1$$

Conditions (5.26), (5.27) follow from

$$\|\tfrac{1}{T_{rel}}-\mathrm{b}(\mathbf{v})(t)\|_{L^\infty(0,T;L(\mathscr{H},\mathscr{H}))})=\frac{1}{T_{rel}}\|1-\alpha(v_0)\|_{L^\infty(0,T;L^\infty(\Omega))}$$
$$\le\frac{1}{T_{rel}}\frac{2\beta_a}{\rho c^2}\|v_0(t)|_{L^\infty(0,T;L^\infty(\Omega)}\le\frac{1}{T_{rel}}\frac{2\beta_a}{\rho c^2}C_{\mathscr{D}(\mathscr{A})\to L^\infty(\Omega)}R$$

$$\|\frac{d}{dt}\mathrm{b}(\mathbf{v})(t)\|_{L^1(0,T;L(\mathscr{H},\mathscr{H}))}=\frac{1}{T_{rel}}\frac{2\beta_a}{\rho c^2}\|v_1\|_{L^1(0,T;L^\infty(\Omega))}\le\frac{1}{T_{rel}}\frac{2\beta_a}{\rho c^2}\sqrt{T}C_{\mathscr{D}(\mathscr{A})\to L^\infty(\Omega)}R$$

$$|\mathscr{A}^{1/2}[\alpha(v_0)\mathscr{A}^{1/2}w_{m-1}-[\alpha(v_0)\mathscr{A}w_{m-1}|$$
$$=|\nabla[\alpha(v_0)\nabla w_1-[\alpha(v_0)\Delta w_{m-1}|=\tfrac{2\beta_a}{\rho c^2}|\nabla v_0\cdot\nabla w_{m-1}|$$
$$\le\tfrac{2\beta_a}{\rho c^2}C^2_{\mathscr{D}(\mathscr{A})\to W^{1,4}(\Omega)}|\mathscr{A}v_0|\,|\mathscr{A}w_{m-1}|\le\tfrac{2\beta_a}{\rho c^2}C^2_{\mathscr{D}(\mathscr{A})\to W^{1,4}(\Omega)}R\,|\mathscr{A}^{\ell_r}w_{m-1}|$$

$$|\mathscr{B}(\vec{v})[\vec{w}]|\le\frac{c^2}{T_{rel}}|\mathscr{A}w_0|+\frac{2\beta_a}{\rho c^2}|v_1w_1|$$
$$\le\frac{c^2}{T_{rel}}|\mathscr{A}w_0|+\frac{2\beta_a}{\rho c^2}C^2_{\mathscr{D}(\mathscr{A}^{1/2})\to L^4(\Omega)}|\mathscr{A}^{1/2}v_1|\,|\mathscr{A}^{1/2}w_1|$$
$$\le\Big(\frac{c^2}{T_{rel}}+\frac{2\beta_a}{\rho c^2}C^2_{\mathscr{D}(\mathscr{A}^{1/2})\to L^4(\Omega)}R\Big)\|\vec{w}\|_{X_0}\,,$$

and condition (5.28) from

$$|\alpha(v_0)-\alpha(\tilde{v}_0)|_{L(\mathscr{H},\mathscr{H})}=\tfrac{2\beta_a}{\rho c^2}|v_0-\tilde{v}_0|_{L^\infty(\Omega)}\le\tfrac{2\beta_a}{\rho c^2}C^2_{\mathscr{D}(\mathscr{A})\to L^\infty(\Omega)}|\mathscr{A}(v_0-\tilde{v}_0)|$$
$$\le\tfrac{2\beta_a}{\rho c^2}C^2_{\mathscr{D}(\mathscr{A})\to L^\infty(\Omega)}\|\vec{v}-\vec{\tilde{v}}\|_{X_0}$$

$$|\mathscr{B}(\vec{v})[\vec{w}]-\mathscr{B}(\vec{\tilde{v}})[\vec{w}]|=\frac{2\beta_a}{\rho c^2}|(v_1-\tilde{v}_1)w_1|\le C^2_{\mathscr{D}(\mathscr{A}^{1/2})\to L^4(\Omega)}|\mathscr{A}^{1/2}(v_1-\tilde{v}_1)|\,|\mathscr{A}^{1/2}w_1|$$
$$\le C^2_{\mathscr{D}(\mathscr{A}^{1/2})\to L^4(\Omega)}\|\vec{v}-\vec{\tilde{v}}\|_{X_0}\|\vec{w}\|_{X_0}\,.$$

Westervelt:. $m = 1$, $k_m = 1$, $r = 1$, $\ell_r = 1$

$$X_0 = \mathscr{D}(\mathscr{A}) \times \mathscr{D}(\mathscr{A}^{1/2}), \qquad X_1 = \mathscr{D}(\mathscr{A}) \times \mathscr{D}(\mathscr{A}),$$
$$\mathrm{b}(\vec{v}) = \frac{b}{\alpha(v_0)}, \quad \mathrm{c}(\vec{v}) = \frac{c^2}{\alpha(v_0)}, \quad \alpha(v_0) = 1 - \frac{2\beta_a}{\rho c^2} v_0,$$
$$\mathscr{B}(\vec{v})[\vec{w}] = \frac{2\beta_a}{\rho c^2} v_1 w_1.$$

Conditions (5.26), (5.27), (5.28) can be verified as for JMGT, just skipping the term $-\frac{c^2}{T_{rel}} \mathscr{A} w_0$ in B and taking into account the fact that $\frac{1}{\alpha(v_0)} - \frac{1}{\alpha(\tilde{v}_0)} = -\frac{1}{\alpha(v_0)\alpha(\tilde{v}_0)}(\alpha(v_0) - \alpha(\tilde{v}_0))$.

Corollary 5.1. *There exists $R > 0$ (sufficiently small) such that for any $\|\vec{u}_0\|_{X_0} \le \frac{R}{C}$ the initial value problems* (2.3)*,* (2.2)*, (with $m = 2$)* (2.1) *(with $m = 1$),* (5.22) *have unique solutions $u \in \tilde{X}$ and these solutions satisfy the estimate*

$$\|u\|_{\tilde{X}} \le C \|\vec{u}^0\|_{X_0}$$

with

$$\textbf{BCBJK:}\ \tilde{X} = W^{2,\infty}(0,T;H_0^1(\Omega)) \cap H^2(0,T;H_\diamondsuit^2(\Omega)) \cap W^{1,\infty}(0,T;H_\diamondsuit^3(\Omega)) \cap H^1(0,T;H_\diamondsuit^4(\Omega)),$$
$$X_0 = H_\diamondsuit^4(\Omega) \times H_\diamondsuit^3(\Omega) \times H_0^1(\Omega)$$

$$\textbf{JMGT:}\ \tilde{X} = W^{2,\infty}(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;H_0^1(\Omega)) \cap H^1(0,T;H_\diamondsuit^2(\Omega)),$$
$$X_0 = H_\diamondsuit^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$$

$$\textbf{Westervelt:}\ \tilde{X} = W^{1,\infty}(0,T;H_0^1(\Omega)) \cap H^1(0,T;H_\diamondsuit^2(\Omega)),$$
$$X_0 = H_\diamondsuit^2(\Omega) \times H_0^1(\Omega)$$

where $H_\diamondsuit^2(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$, $H_\diamondsuit^3(\Omega) = H^3(\Omega) \cap H_0^1(\Omega)$, $H_\diamondsuit^4(\Omega) = \{v \in H^4(\Omega) : v, \Delta v \in H_0^1(\Omega)\}$.

Remark 5.2. A comparison to [KALTENBACHER, LASIECKA (2009), KALTENBACHER, NIKOLIĆ (2019), KALTENBACHER, THALHAMMER (2018)] where the following regularity results have been established

$$u \in W^{2,\infty}(0,T;H_0^1(\Omega)) \cap H^2(0,T;H_\diamondsuit^2(\Omega)) \cap L^\infty(0,T;H_\diamondsuit^3(\Omega)),$$
$$(u_0,u_1,u_2) \in H_\diamondsuit^3(\Omega) \times H_\diamondsuit^3(\Omega) \times H_0^1(\Omega) \qquad \text{for BCBJK;}$$

$$u \in W^{2,\infty}(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;H_\diamondsuit^2(\Omega)),$$
$$(u_0,u_1,u_2) \in H_\diamondsuit^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \qquad \text{for JMGT;}$$

$$u \in H^2(0,T;H_0^1(\Omega)) \cap W^{2,\infty}(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;H_\diamondsuit^2(\Omega)),$$
$$(u_0,u_1) \in H_\diamondsuit^2(\Omega) \times H_\diamondsuit^2(\Omega) \qquad \text{for Westervelt;}$$

shows that our results here give stronger regularity under stronger (or the same) smoothness of the initial data for BCBJK and JMGT, and weaker regularity under weaker smoothness of the initial data for Westervelt. Beyond such a comparison, our aim is a unified approach that is amenable to implicit time stepping schemes, though, as we will carry out analogous energy estimates in the following section.

6. Energy estimates, well-posedness and convergence of time-discretised systems

In this section, we will transfer the energy estimates from Section 5 to the systems obtained by an implicit discretisation with stiffly accurate Runge Kutta methods. This will enable us to prove well- posedness of the (time) discretised problems. As the most transparent special case, we will first of all study the implicit Euler scheme.

6.1. **Euler scheme.** For a fixed time grid $t_0 < t_1 < \cdots < t_N$ which for simplicity of exposition we choose equidistant $t_i = i\tau$, $\tau = \frac{T}{N}$ we replace a time dependent function $u : [0,T] \to \mathbb{R}$ by a vector $\underline{u}_\tau = (u^{(0)}, \dots, u^{(N)})$ of approximations at the time instances. With some time discretisation $D_t^{(n)} \underline{u}_\tau \approx u'(t_n)$, e.g., according to a backward Euler scheme $D_t^{(n+1)} := \mathrm{d}_t^{(n+1)}$ where

$$\mathrm{d}_t^{(n+1)} \underline{u}_\tau := \frac{1}{\tau}(u^{(n+1)} - u^{(n)}), \tag{6.1}$$

we apply this component wise to the grid version $\underline{\mathbf{u}}_\tau = (\mathbf{u}^0, \dots, \mathbf{u}^N)$ of $\mathbf{u}$ to obtain an implicit time discretisation of (5.3)

$$D_t^{(n+1)} \underline{\mathbf{u}}_\tau + \mathbf{A}(\mathbf{u}^{(n+1)})\mathbf{u}^{(n+1)} = (0, \cdots, 0, \mathscr{B}(\mathbf{u}^{(n+1)})[\mathbf{u}^{(n+1)}])^T . \tag{6.2}$$

and of (5.6)

$$D_t^{(n+1)} \underline{\mathbf{u}}_\tau + \mathbf{A}^{(n+1)}\mathbf{u}^{(n+1)} = (0, \cdots, 0, \mathscr{G}^{(n+1)}[\mathbf{u}^{(n+1)}] + f^{(n+1)}))^T . \tag{6.3}$$

respectively.

Time discrete counterparts of the function spaces in Section 5 can be defined by setting, for some Hilbert space Z (of space dependent functions)

$$\|\underline{u}_\tau\|_{L^p_\tau(Z)} := \Big(\tau \sum_{n=0}^{N} \|u^{(n)}\|_Z^p\Big)^{1/p} \text{ for } 1 \le p < \infty, \quad \|\underline{u}_\tau\|_{L^\infty_\tau(Z)} := \max_{n\in\{0,\dots,N\}} \|u^{(n)}\|_Z ,$$
$$\|\underline{u}_\tau\|_{W^{1,p}_\tau(Z)} := \|\underline{\mathrm{d}_t \underline{u}_\tau}_\tau\|_{L^p_\tau(Z)}$$

that satisfy the τ independent estimates (using the crude estimate $(N+1)\tau \le 2T$)

$$\|\underline{u}_\tau\|_{L^p_\tau(Z)} \le (2T)^{1/p} \|\underline{u}_\tau\|_{L^\infty_\tau(Z)} , \quad \|\underline{u}_\tau\|_{L^\infty_\tau(Z)} \le \|u^0\|_Z + T^{(p-1)/p} \|\underline{u}_\tau\|_{W^{1,p}_\tau(Z)}$$

where the latter follows from Hölder's inequality and the inverse triangle inequality

$$\begin{aligned}\|\underline{\mathrm{d}_t \underline{u}_\tau}_\tau\|_{L^p_\tau(Z)} &= \Big(\tau^{1-p} \sum_{n=0}^{N-1} \|u^{(n+1)} - u^{(n)}\|_Z^p\Big)^{1/p} \ge \tau^{1/p-1} N^{1/p-1} \sum_{n=0}^{N-1} \|u^{(n+1)} - u^{(n)}\|_Z \\ &\ge T^{1/p-1} \sum_{n=0}^{N-1} \big(\|u^{(n+1)}\|_Z - \|u^{(n)}\|_Z\big) \ge T^{1/p-1} \big(\|u^N\|_Z - \|u^0\|_Z\big) .\end{aligned}$$

Therewith we define, analogously to (5.17) and using the identity $\frac{\mathrm{d}}{\mathrm{d}t}{}^{(n)}\underline{u_j}_\tau = u_{j+1}^{(n)}$

$$\begin{aligned} X_\tau = &\Big(L^\infty_\tau(\mathscr{D}(\mathscr{A}^{\max\{k_m,\ell_r\}})\cap H^1_\tau(\mathscr{D}(\mathscr{A}^{\max\{k_m,\ell_r\}}))\Big)^{m-1} \\ &\times\, L^2_\tau(\mathscr{D}(\mathscr{A}^{\max\{k_m,\ell_r\}}))\cap L^\infty_\tau(\mathscr{D}(\mathscr{A}^{\max\{k_m,(k_m+\ell_r)/2\}}))\cap H^1_\tau(\mathscr{D}(\mathscr{A}^{k_m})) \\ &\times\, L^2_\tau(\mathscr{D}(\mathscr{A}^{k_m}))\cap L^\infty_\tau(\mathscr{D}(\mathscr{A}^{k_m/2})) \\ \subseteq\, & L^\infty_\tau(X_0)\cap L^2_\tau(X_1) \end{aligned} \tag{6.4}$$

For obtaining energy estimates the inequality

$$\Big(D_t^{(n+1)}\underline{u}_\tau\Big|u^{(n+1)}\Big) \geq \frac{1}{2\tau}\Big(\big|u^{(n+1)}\big|^2 - \big|u^{(n)}\big|^2\Big) = \frac{1}{2}\mathrm{d}_t^{(n+1)}\underline{\big|u\big|^2}_\tau \tag{6.5}$$

substituting its continuous counterpart $\big(u'(t)\big|u(t)\big) = \frac{1}{2}\frac{d}{dt}|u|^2(t)$ will be crucial, which holds in the implicit Euler case $D_t^{(n+1)} = \mathrm{d}_t^{(n+1)}$ due to

$$\big(v-w\big|v\big) = \tfrac{1}{2}(\big|v\big|^2 - \big|w\big|^2 + \big|v-w\big|^2) \geq \tfrac{1}{2}(\big|v\big|^2 - \big|w\big|^2)$$

[EMMRICH (2004), Eqns. (7.5.9), (7.5.10) p. 197]) but also for certain Runge Kutta methods, see [EMMRICH, THALHAMMER (2010), GWINNER, THALHAMMER (2014)].

Energy estimates for (6.3) can be derived analogoulsy to those for (5.6) by testing with

$$(0,\ldots,0,\sum_{i=1}^r \mathscr{A}^{k_m}[\mathsf{c}_i^{(n+1)}\mathscr{A}^{\ell_i}u_{m-1}^{(n+1)}],\mathscr{A}^{k_m}u_m^{(n+1)})^T$$

and

$$(0,\ldots,0,\mathscr{A}^{k_m}[\mathsf{b}^{(n+1)}\mathscr{A}^{\ell_r}u_{m-1}^{(n+1)}],\mathscr{A}^{\ell_r}u_{m-1}^{(n+1)})^T,$$

respectively, cf. (5.7), (5.14), applying (6.5), substituting (5.12) by

$$\begin{aligned} &\frac{1}{2}\mathrm{d}_t^{(n+1)}\underline{\Big|\sqrt{\mathsf{c}}\mathscr{A}^{(p+q)/2}v\Big|^2}_\tau \\ &= \frac{1}{2\tau}\Big(\Big(\mathscr{A}^{(p+q)/2}v^{(n+1)}\Big|\mathsf{c}^{(n+1)}\mathscr{A}^{(p+q)/2}v^{(n+1)}\Big) - \Big(\mathscr{A}^{(p+q)/2}v^{(n)}\Big|\mathsf{c}^{(n)}\mathscr{A}^{(p+q)/2}v^{(n)}\Big)\Big) \\ &= \frac{1}{2}\Big(\mathrm{d}_t^{(n+1)}\underline{v}_\tau\Big|\mathscr{A}^{(p+q)/2}[\mathsf{c}^{(n+1)}\mathscr{A}^{(p+q)/2}(v^{(n+1)}+v^{(n)})]\Big) \\ &\quad + \frac{1}{2}\Big(v^{(n+1)}\Big|\mathscr{A}^{(p+q)/2}[(\mathrm{d}_t^{(n+1)}\underline{\mathsf{c}}_\tau)\mathscr{A}^{(p+q)/2}v^{(n)}]\Big) \\ &= \Big(D_t^{(n+1)}\underline{v}_\tau\Big|\mathscr{A}^p[\mathsf{c}^{(n+1)}\mathscr{A}^q v^{(n+1)}]\Big) \\ &\quad + \Big(D_t^{(n+1)}\underline{v}_\tau\Big|\mathscr{A}^{(p+q)/2}[\mathsf{c}^{(n+1)}\mathscr{A}^{(p+q)/2}v^{(n+1)}] - \mathscr{A}^p[\mathsf{c}^{(n+1)}\mathscr{A}^q v^{(n+1)}]\Big) \\ &\quad + \Big(\mathrm{d}_t^{(n+1)}\underline{v}_\tau - D_t^{(n+1)}\underline{v}_\tau\Big|\mathscr{A}^{(p+q)/2}[\mathsf{c}^{(n+1)}\mathscr{A}^{(p+q)/2}v^{(n+1)}]\Big) \\ &\quad - \tau\Big(\mathrm{d}_t^{(n+1)}\underline{v}_\tau\Big|\mathscr{A}^{(p+q)/2}[\mathsf{c}^{(n+1)}\mathscr{A}^{(p+q)/2}\mathrm{d}_t^{(n+1)}\underline{v}_\tau]\Big) \\ &\quad + \frac{1}{2}\Big(v^{(n+1)}\Big|\mathscr{A}^{(p+q)/2}[(\mathrm{d}_t^{(n+1)}\underline{\mathsf{c}}_\tau)\mathscr{A}^{(p+q)/2}v^{(n)}\Big) \end{aligned} \tag{6.6}$$

and using $D_t^{(n+1)}\underline{u}_{\tau,m-1} = u_m^{(n+1)}$.

The term $\tau\Big(\mathrm{d}_t^{(n+1)}\underline{v}_\tau\Big|\mathscr{A}^{(p+q)/2}[\mathsf{c}^{(n+1)}\mathscr{A}^{(p+q)/2}\mathrm{d}_t^{(n+1)}\underline{v}_\tau\Big) = \tau\Big|\sqrt{\mathsf{c}^{(n+1)}}\mathscr{A}^{(p+q)/2}\mathrm{d}_t^{(n+1)}\underline{v}_\tau\Big|^2$ is nonnegative and can therefore be skipped in the estimate.

The additional term containing $\mathrm{d}_t^{(n+1)}\underline{v}_\tau - D_t^{(n+1)}\underline{v}_\tau$ in (6.6) clearly vanishes in case an implicit Euler discretisation is used; otherwise it can be individually estimated.

Altogether this yields, in place of (5.13), (5.15)

$$\begin{aligned}
&\frac{1}{2}\mathrm{d}_t^{(n+1)}\underline{|\mathscr{A}^{k_m/2}u_m|^2}_\tau + \frac{1}{2}\sum_{i=1}^r \mathrm{d}_t^{(n+1)}\underline{|\sqrt{\mathsf{c}_i}\mathscr{A}^{(k_m+\ell_i)/2}u_{m-1}|^2}_\tau + |\sqrt{\mathsf{b}^{(n+1)}}\mathscr{A}^{k_m}u_m^{(n+1)}|^2\\
&\leq \frac{1}{\underline{\mathsf{b}}}|\mathscr{G}^{(n+1)}[\mathbf{u}^{(n+1)}] + f^{(n+1)}|^2 + \frac{1}{4}|\sqrt{\mathsf{b}^{(n+1)}}\mathscr{A}^{k_m}u_m^{(n+1)}|^2\\
&\quad + \frac{1}{2}\sum_{i=1}^r \frac{1}{\underline{\mathsf{c}}_i}|(\mathrm{d}_t^{(n+1)}\underline{\mathsf{c}_i}_\tau)|_{L(\mathscr{H},\mathscr{H})}|\sqrt{\mathsf{c}_i^{(n+1)}}\mathscr{A}^{(k_m+\ell_i)/2}u_{m-1}^{(n+1)}|^2\\
&\quad + \frac{1}{4}|\sqrt{\mathsf{b}^{(n+1)}}\mathscr{A}^{k_m}u_m^{(n+1)}|^2\\
&\quad + \frac{1}{\underline{\mathsf{b}}}\sum_{i=1}^r |\mathscr{A}^{(\ell_i-k_m)/2}[\mathsf{c}_i^{(n+1)}\mathscr{A}^{(k_m+\ell_i)/2}u_{m-1}^{(n+1)}] - [\mathsf{c}_i^{(n+1)}\mathscr{A}^{\ell_i}u_{m-1}^{(n+1)}]|^2,
\end{aligned} \tag{6.7}$$

$$\begin{aligned}
&\frac{1}{2}\mathrm{d}_t^{(n+1)}\underline{|\sqrt{\mathsf{b}}\mathscr{A}^{(k_m+\ell_r)/2}u_{m-1}|^2}_\tau + \sum_{i=1}^r |\sqrt{\mathsf{c}_i^{(n+1)}}\mathscr{A}^{(\ell_r+\ell_i)/2}u_{m-1}^{(n+1)}|^2\\
&\leq \frac{1}{\underline{\mathsf{c}}_r}|\mathscr{G}^{(n+1)}[\mathbf{u}^{(n+1)}] + f^{(n+1)}|^2 + \frac{1}{4}|\sqrt{\mathsf{c}_r^{(n+1)}}\mathscr{A}^{\ell_r}u_{m-1}^{(n+1)}|^2\\
&\quad + \frac{1}{2\underline{\mathsf{c}}_r}|(\mathrm{d}_t^{(n+1)}\underline{\mathsf{b}}_\tau)|_{L(\mathscr{H},\mathscr{H})}|\sqrt{\mathsf{c}_r^{(n+1)}}\mathscr{A}^{(k_m+\ell_r)/2}u_{m-1}^{(n+1)}|^2 + \frac{1}{2}|\sqrt{\mathsf{b}^{(n+1)}}\mathscr{A}^{k_m}u_m^{(n+1)}|^2\\
&\quad + \frac{1}{2\underline{\mathsf{b}}}|\mathscr{A}^{(\ell_r-k_m)/2}[\mathsf{b}^{(n+1)}\mathscr{A}^{(k_m+\ell_r)/2}u_{m-1}^{(n+1)}] - [\mathsf{b}^{(n+1)}\mathscr{A}^{\ell_r}u_{m-1}^{(n+1)}]|^2\\
&\quad + \frac{1}{4}|\sqrt{\mathsf{c}_r^{(n+1)}}\mathscr{A}^{\ell_r}u_{m-1}^{(n+1)}|^2\\
&\quad + \frac{1}{\underline{\mathsf{c}}_r}\left(\sum_{i=1}^r |\mathscr{A}^{(\ell_i-\ell_r)/2}[\mathsf{c}_i^{(n+1)}\mathscr{A}^{(\ell_r+\ell_i)/2}u_{m-1}^{(n+1)}] - [\mathsf{c}_i^{(n+1)}\mathscr{A}^{\ell_i}u_{m-1}^{(n+1)}]|\right)^2.
\end{aligned} \tag{6.8}$$

The time integration step between (5.8) and (5.9), i.e., bewteen (5.13), (5.15) and the respective energy estimates, is replaced by τ weigthed summation, so that we get, e.g. in place of $\int_0^{t_0}\frac{d}{dt}|\mathscr{A}^{k_m/2}u_m|^2(t)\,dt = |\mathscr{A}^{k_m/2}u_m(t_0)|^2 - |\mathscr{A}^{k_m/2}u_m(0)|^2$ the identity

$$\begin{aligned}
\tau\sum_{n=0}^{n_0-1}\mathrm{d}_t^{(n+1)}\underline{\Big|\mathscr{A}^{k_m/2}u_m\Big|^2}_\tau &= \sum_{n=0}^{n_0-1}\left(\Big|\mathscr{A}^{k_m/2}u_m^{(n+1)}\Big|^2 - \Big|\mathscr{A}^{k_m/2}u_m^{(n)}\Big|^2\right)\\
&= \Big|\mathscr{A}^{k_m/2}u_m^{(n_0)}\Big|^2 - \Big|\mathscr{A}^{k_m/2}u_m^{(0)}\Big|^2
\end{aligned}$$

Moreover, we use the following time discrete version of Gronwall's inequality:

$$\eta^{(n)} \le a^{(n)} + \tau \sum_{j=1}^{n} b^{(j)} \eta^{(j)} \quad \text{for all } n \in \{1,\dots,N\}$$
$$\Rightarrow \ \eta^{(n)} \le a^{(n)} + \tau \sum_{j=1}^{n} a^{(j)} b^{(j)} \exp\Big(\tau \sum_{i=j}^{n} b^{(i)}\Big) \quad \text{for all } n \in \{1,\dots,N\}$$

for $\underline{\eta}_\tau, \underline{a}_\tau, \underline{b}_\tau \ge 0$, which follows by application of its continuous counterpart (5.19) to the piecewise constant interpolants of $\underline{\eta}_\tau, \underline{a}_\tau, \underline{b}_\tau$.

For the linear equation (6.3) we therefore get the following result.

Proposition 6.1. *Under conditions* (5.2)*,* (5.4)*,* (5.11)*,* (5.21)*, with c_{bc} small enough, the time discretised initial value problem* (6.3)*,* (5.22) *with the implicit Euler scheme* (6.1) *has a unique solution $\underline{\mathbf{u}}_\tau \in X_\tau$ and this solution satisfies the estimate*

$$\|\underline{\mathbf{u}}_\tau\|_{X_\tau} \le C\Big(\|\vec{u}^0\|_{X_0} + \|f\|_{L^2_\tau(\mathscr{H})}\Big)$$

with C as in Proposition 5.1, in particular C independent of τ.

Proof. A Galerkin discretisation of (6.3), (5.22) with eigenfunctions ϕ_i of $\mathscr{A}$, i.e., an Ansatz $u_j^{(n)}(x) = \sum_{i=1}^{I} u_{i,j}^{(n)} \phi_i(x)$ after testing with ϕ_k, $k \in \{1,\dots,I\}$ yields I linear $m\cdot n \times m\cdot n$ systems of equations, one for each set of coefficients $(u_{i,j}^{(n)})_{j\in\{1,\dots,m\},n\in\{1,\dots,N\}}$. (Note that the coefficients $u_{i,j}^0 = \big(u_j^0 \big| \phi_i\big)$ are fixed by the initial data.) Due to mutual orthogonality of the eigenfunctions, system i_1 is decoupled from system i_2 for $i_1 \ne i_2$. System i reads as

$$D_t^{(n+1)} \underline{\mathbf{u}_i}_\tau = \mathbf{A}^{(n+1)} \mathbf{u}_i{}^{(n+1)} + (0,\cdots,0,\mathscr{G}^{(n+1)}[\mathbf{u}^{(n+1)}] + f_i^{(n)})^T ,$$
$$\text{with } \mathbf{A}^{(n)} = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots \\ 0 & & & -\sum_{\nu=1}^{r} \mathsf{c}_\nu(t_n)\lambda_i^{\ell_\nu} & -\mathsf{b}(t_n)\lambda_i^{k_m} \end{pmatrix}, \tag{6.9}$$

$f_i^{(n)} = \big(f^{(n)} \big| \phi_i\big)$, $\mathbf{u}_i^{(n)} = (u_{i,1}^{(n)},\dots,u_{i,m}^{(n)})$ and $\mathbf{u}_i^0 = (u_{i,1}^0,\dots,u_{i,m}^0)$, that is, the Galerkin discretisation simply replaces $\mathscr{A}$ by λ_i. Thus also the Galerkin approximation satisfies energy estimates analogous to those in Proposition 5.1, which implies uniqueness and, via Fredholm's alternative (since we are in finite dimensions now), also existence of a solution to (6.9). These energy estimates also yield uniform boundedness of the sequence $(\underline{\mathbf{u}_i}_\tau)_{i\in\mathbb{N}}$ defined by $\mathbf{u}^{(n)}(x) = \sum_{i=1}^{I} \mathbf{u}_i^{(n)} \phi_i(x)$ in X_τ and therefore its weak convergence to a limit $\underline{\mathbf{u}}_\tau$ that by linearity can easily be verified to be a solution to (6.3), to which the energy estimates transfer as well. Uniqueness again follows from the energy estimates.

$\diamondsuit$

This allows to transfer the well-posedness result and energy estimates from Theorem 5.1 to the time discretised equation (6.2) under the following time discretised versions of the conditions (5.26)

$$\|\underline{\mathbf{v}}_\tau\|_{X_\tau} \le R \ \Rightarrow$$
$$\Big(\underline{\mathsf{b}(\mathbf{v})}_\tau, \underline{\mathsf{c}_i(\mathbf{v})}_\tau \in L^\infty_\tau(L(\mathscr{H},\mathscr{H})), \quad \underline{\mathrm{d}_t \underline{\mathsf{b}(\mathbf{v})}_\tau}_\tau, \underline{\mathrm{d}_t \underline{\mathsf{c}_i(\mathbf{v})}_\tau}_\tau \in L^1_\tau(L(\mathscr{H},\mathscr{H}))\Big). \tag{6.10}$$

Theorem 6.1. *Under conditions* (5.2), (6.10), (5.27), (5.28), *there exists* $R > 0$ *(sufficiently small) such that for any* $\|\vec{u}_0\|_{X_0} \leq \frac{R}{C}$ *the time discretised initial value problem* (6.2), (5.22) *with the Euler scheme* (6.1) *has a unique solution* $\underline{\mathbf{u}}_\tau \in X_\tau$ *and this solution satisfies the estimate*

$$\|\underline{\mathbf{u}}_\tau\|_{X_\tau} \leq C\|\vec{u}^0\|_{X_0}$$

with C as in Proposition 5.1.

Verification of (6.10) for the models from Section 2 can be done by using the estimates (with $i = 1$, $\kappa = \beta_5$ for BCBJK and $i = 0$, $\kappa = \frac{2\beta_a}{c^2\rho}$ for JMGT, Westervelt)

$$\begin{aligned}
&\max_{n\in\{0,\dots,N\}} \|1-\alpha(v_i^{(n)})\|_{L(\mathscr{H},\mathscr{H})} = \kappa\|\underline{v_i}_\tau\|_{L^\infty_\tau(L^\infty(\Omega))} \\
&\tau\sum_{n=0}^{N} \Big\|\frac{\alpha(v_i^{(n+1)})-\alpha(v_i^{(n)})}{\tau}\Big\|_{L(\mathscr{H},\mathscr{H})} \leq \kappa\tau\sum_{n=0}^{N}\|\mathrm{d}_t^{(n+1)}\underline{v_i}_\tau\|_{L^\infty(\Omega)} \\
&\leq \kappa\|\Big(\|\underline{v_{i+1}}_\tau\|_{L^1_\tau(L^\infty(\Omega))} + \kappa\tau\sum_{n=0}^{N}\|\mathrm{d}_t^{(n+1)}\underline{v_i}_\tau - D_t^{(n+1)}\underline{v_i}_\tau\|_{L^\infty(\Omega)}\Big)
\end{aligned}$$

due to $D_t^{(n+1)}\underline{v_i}_\tau = v_{i+1}^{(n+1)}$, and can therefore be estimated analogously to Section 5.3 in the Euler case $D_t = d_t$.

The time discretised versions of the models from Section 2 read as follows

- **BCBJK:**

$$\begin{aligned}
D_t^{(n+1)}\underline{\psi_0}_\tau &= \psi_1^{(n+1)} \\
D_t^{(n+1)}\underline{\psi_1}_\tau &= \psi_2^{(n+1)} \\
D_t^{(n+1)}\underline{\psi_2}_\tau &= \frac{1}{1+\beta_5\psi_1^{(n+1)}}\Big(\beta_1\Delta\psi_2^{(n+1)} - \beta_2\Delta^2\psi_1^{(n+1)} + \beta_3\Delta\psi_1^{(n+1)} - \beta_4\Delta^2\psi_0^{(n+1)} \\
&\qquad - \beta_5\big(\psi_2^{(n+1)}\big)^2 - 2|\nabla\psi_1^{(n+1)}|^2 - 2\nabla\psi_0^{(n+1)}\cdot\nabla\psi_2^{(n+1)}\Big)
\end{aligned} \tag{6.11}$$

- **JMGT** with $\beta_0 = \frac{2\beta_a}{\rho c^2}$:

$$\begin{aligned}
D_t^{(n+1)}\underline{p_0}_\tau &= p_1^{(n+1)} \\
D_t^{(n+1)}\underline{p_1}_\tau &= p_2^{(n+1)} \\
D_t^{(n+1)}\underline{p_2}_\tau &= \frac{1}{T_{rel}}\Big(-(1-\beta_0 p_0^{(n+1)})\,p_2^{(n+1)} + b\Delta p_1^{(n+1)} + c^2\Delta p_0^{(n+1)} + \beta_0\big(p_1^{(n+1)}\big)^2\Big)
\end{aligned} \tag{6.12}$$

- **Westervelt** with $\beta_0 = \frac{2\beta_a}{\rho c^2}$:

$$\begin{aligned}
D_t^{(n+1)}\underline{p_0}_\tau &= p_1^{(n+1)} \\
D_t^{(n+1)}\underline{p_1}_\tau &= \frac{b}{1-\beta_0 p_0^{(n+1)}}\Delta p_1^{(n+1)} + \frac{c^2}{1-\beta_0 p_0^{(n+1)}}\Delta p_0^{(n+1)} + \frac{\beta_0}{1-\beta_0 p_0^{(n+1)}}\big(p_1^{(n+1)}\big)^2
\end{aligned} \tag{6.13}$$

Corollary 6.1. *There exists $R>0$ (sufficiently small) such that for any $\|\vec{u}_0\|_{X_0} \leq \frac{R}{C}$ the semidiscrete PDEs* (6.11), (6.12), *(with $m=2$)* (6.13) *(with $m=1$),* (5.22) *with* (6.1) *have unique solutions $\underline{\mathbf{u}}_\tau = (\underline{u_0}_\tau, \underline{u_1}_\tau, \ldots, \underline{u_m}_\tau) \in X_\tau$ and these solutions satisfy the estimate*

$$\|(\underline{u_0}_\tau, \underline{u_1}_\tau, \ldots, \underline{u_m}_\tau)\|_{X_\tau} \leq C\|\vec{u}^0\|_{X_0}$$

with

$$\begin{aligned}
\textbf{BCBJK: } X_\tau &= L^\infty_\tau(H^4_\diamondsuit(\Omega)) \cap L^2_\tau(H^4_\diamondsuit(\Omega)) \cap H^1_\tau(H^4_\diamondsuit(\Omega)) \\
&\quad \times\ L^\infty_\tau(H^3_\diamondsuit(\Omega)) \cap L^2_\tau(H^4_\diamondsuit(\Omega)) \cap H^1_\tau(H^2_\diamondsuit(\Omega))\ \times\ L^\infty_\tau(H^1_0(\Omega)) \cap L^2_\tau(H^2_\diamondsuit(\Omega)) \\
X_0 &= H^4_\diamondsuit(\Omega) \times H^3_\diamondsuit(\Omega) \times H^1_0(\Omega) \\
\textbf{JMGT: } X_\tau &= L^\infty_\tau(H^1_0(\Omega)) \cap L^2_\tau(H^2_\diamondsuit(\Omega)) \cap H^1_\tau(H^2_\diamondsuit(\Omega)) \\
&\quad \times\ L^\infty_\tau(H^1_0(\Omega)) \cap L^2_\tau(H^2_\diamondsuit(\Omega))\ \times\ L^\infty_\tau(L^2(\Omega)) \\
X_0 &= H^2_\diamondsuit(\Omega) \times H^1_0(\Omega) \times L^2(\Omega) \\
\textbf{Westervelt: } X_\tau &= L^\infty_\tau(H^1_0(\Omega)) \cap L^2_\tau(H^2_\diamondsuit(\Omega)) \cap H^1_\tau(H^2_\diamondsuit(\Omega))\ \times\ L^\infty_\tau(H^1_0(\Omega)) \cap L^2_\tau(H^2_\diamondsuit(\Omega)) \\
X_0 &= H^2_\diamondsuit(\Omega) \times H^1_0(\Omega)
\end{aligned} \tag{6.14}$$

6.2. **Runge-Kutta methods.** We consider a stiffly accurate Runge Kutta scheme with s stages and Butcher tableau $\begin{array}{c|c} \mathfrak{c} & \mathfrak{A} \\ \hline & \mathfrak{e}^T\mathfrak{A} \end{array}$ where $\mathfrak{A} = (\mathfrak{a}_{\mu\nu})_{1\leq\mu,\nu\leq s} \in \mathbb{R}^{s\times s}$, $\mathfrak{c}$, $\mathfrak{e} = (0,\ldots,1)^T \in \mathbb{R}^s$, and we additionally assume

$$\begin{gathered}
\sum_{\nu=1}^s a_{s\nu} = 1\,, \quad a_{s\nu} > 0\,, \quad \mathfrak{A} \text{ regular}\,, \quad \mathfrak{B} := \operatorname{diag}(a_{s\cdot})\,, \quad \mathfrak{C} := \mathfrak{B}\mathfrak{A}^{-1} \\
\mathfrak{B}\mathfrak{A} + \mathfrak{A}^T\mathfrak{B} - a_{s\cdot}a_{s\cdot}^T - \mathfrak{A}^T\mathfrak{C}\mathbf{1}\mathbf{1}^T\mathfrak{C}^T\mathfrak{A} \text{ positive semidefinite}
\end{gathered} \tag{6.15}$$

cf. [EMMRICH, THALHAMMER (2010)], which allows to conclude the following inequality cf. [EMMRICH, THALHAMMER (2010), Lemma 3.4]

$$(x_1,\ldots,x_s)\mathfrak{C}\begin{pmatrix} x_1 - x_0 \\ \vdots \\ x_s - x_0 \end{pmatrix} \geq \tfrac{1}{2}(x_s^2 - x_0^2)$$

for all $x_0, x_1, \ldots x_s \in \mathbb{R}$. The latter carries over to the Hilbert space $\mathscr{H}$ in place of $\mathbb{R}$ in the sense that

$$\sum_{\mu=1}^s \Big(\sum_{\nu=1}^s \mathfrak{C}_{\mu\nu}(u^\nu - u^0) \Big| u^\mu \Big) \geq \tfrac{1}{2}\left(\left|u^s\right|^2 - \left|u^0\right|^2 \right) \tag{6.16}$$

for all $u^0, u^1, \ldots u^s \in \mathscr{H}$. Note that (6.15) can be verfied under fairly general compatibility conditions, cf. [EMMRICH, THALHAMMER (2010), Theorem 3.1].

Inequality (6.16) is crucial for carrying over the energy estimates from Section 6.1 to the Runge Kutta discretisation of (5.3), (5.6), which according to [EMMRICH, THALHAMMER (2010), equations (1.1), (1.5), (4.1)], can be written as

$$\sum_{\nu=1}^s \mathfrak{C}_{\mu\nu}\mathrm{d}_{t\,\nu}^{(n)}\underline{\mathbf{u}}_\tau + \mathbf{A}(\mathbf{U}_\mu^{(n)})\mathbf{U}_\mu^{(n)} = (0,\cdots,0,\mathscr{B}(\mathbf{U}_\mu^{(n)})[\mathbf{U}_\mu^{(n)}])^T\,, \tag{6.17}$$

and of (5.6)

$$\sum_{\nu=1}^{s} \mathfrak{C}_{\mu\nu} \mathrm{d}_{t\,\nu}^{(n)} \underline{\mathbf{u}}_\tau + \mathbf{A}_\mu^{(n)} \mathbf{U}_\mu^{(n)} = (0,\cdots,0,\mathscr{G}_\mu^{(n)}[\mathbf{U}_\mu^{(n)}] + f_\mu^{(n)}))^T\,, \tag{6.18}$$

respectively, where in both cases

$$\mu = 1,\ldots,s\,, \quad \mathbf{u}_s^{(n)} =: \mathbf{u}^{(n+1)}$$

and we use the following abbreviations

$$v_\nu^{(n)} \approx v(t_n + \mathfrak{c}_\nu \tau)\,,\ v \in \{\mathbf{u}, u_j, \mathscr{G}, f\}\,, \quad \mathrm{d}_{t\,\nu}^{(n)} \underline{u}_\tau := \frac{1}{\tau}(u_\nu^{(n)} - u^{(n)})\,,$$

with the vector $\underline{u}_\tau = (u^{0,1},\ldots,u^{0,s},\ldots,u^{N,1},\ldots,u^{N,s})$ of approximations at the time instances (including the stages beween them).

Analogously to the previous section, cf. (5.7), (5.14) we test with

$$(0,\ldots,0,\sum_{i=1}^{r} \mathscr{A}^{k_m}[\mathsf{c}_{i,\mu}^{(n)} \mathscr{A}^{\ell_i} u_{m-1,\mu}^{(n)}], \mathscr{A}^{k_m} u_{m,\mu}^{(n)})^T$$

and

$$(0,\ldots,0,\mathscr{A}^{k_m}[\mathsf{b}_\mu^{(n)} \mathscr{A}^{\ell_r} u_{m-1,\mu}^{(n)}], \mathscr{A}^{\ell_r} u_{m-1,\mu}^{(n)})^T\,,$$

respectively, sum up over μ and apply (6.16). The identity (6.6) is substituted by an estimate as follows.

$$\begin{aligned}
\frac{1}{2}\mathrm{d}_t^{(n+1)} \underline{\left|\sqrt{\mathsf{c}}\mathscr{A}^{(p+q)/2} v\right|^2}_\tau &\le \sum_{\mu=1}^{s}\Big(\sum_{\nu=1}^{s} \mathfrak{C}_{\mu\nu} \mathrm{d}_{t\,\nu}^{(n)} \underline{\sqrt{\mathsf{c}}\mathscr{A}^{(p+q)/2} v}_\tau \Big| \sqrt{\mathsf{c}_\mu^{(n)}} \mathscr{A}^{(p+q)/2} v_\mu^{(n)}\Big) \\
&= \sum_{\mu=1}^{s}\Big(\sum_{\nu=1}^{s} \mathfrak{C}_{\mu\nu} \mathrm{d}_{t\,\nu}^{(n)} \underline{v}_\tau \Big| \mathscr{A}^{p}[\mathsf{c}_\mu^{(n)} \mathscr{A}^{q} v_\mu^{(n)}]\Big) + \sum_{\mu=1}^{s}\sum_{\nu=1}^{s} \mathfrak{C}_{\mu\nu} R_{\mu\nu}^{(n+1)}
\end{aligned}$$

where we have used (6.16) in the first inequality and

$$\begin{aligned}
R_{\mu\nu}^{(n+1)} &= \frac{1}{\tau}\Big(\sqrt{\mathsf{c}_\nu^{(n)}}\mathscr{A}^{(p+q)/2} v_\nu^{(n)} - \sqrt{\mathsf{c}^{(n)}}\mathscr{A}^{(p+q)/2} v^{(n)} \Big| \sqrt{\mathsf{c}_\mu^{(n)}}\mathscr{A}^{(p+q)/2} v_\mu^{(n)}\Big) \\
&\qquad - \frac{1}{\tau}\Big(v_\nu^{(n)} - v^{(n)} \Big| \mathscr{A}^{p}[\mathsf{c}_\mu^{(n)} \mathscr{A}^{q} v_\mu^{(n)}]\Big) \\
&= \frac{1}{\tau}\Big((\sqrt{\mathsf{c}_\nu^{(n)}} - \sqrt{c^{(n)}})\mathscr{A}^{(p+q)/2} v_\nu^{(n)} + \sqrt{\mathsf{c}^{(n)}}\mathscr{A}^{(p+q)/2}(v_\nu^{(n)} - v^{(n)}) \Big| \sqrt{\mathsf{c}_\mu^{(n)}}\mathscr{A}^{(p+q)/2} v_\mu^{(n)}\Big) \\
&\qquad - \frac{1}{\tau}\Big(v_\nu^{(n)} - v^{(n)} \Big| \mathscr{A}^{p}[\mathsf{c}_\mu^{(n)} \mathscr{A}^{q} v_\mu^{(n)}]\Big) \\
&= \frac{1}{\tau}\Big((\sqrt{\mathsf{c}_\nu^{(n)}} - \sqrt{c^{(n)}})\mathscr{A}^{(p+q)/2} v_\nu^{(n)} \Big| \sqrt{\mathsf{c}_\mu^{(n)}}\mathscr{A}^{(p+q)/2} v_\mu^{(n)}\Big) \\
&\quad + \frac{1}{\tau}\Big(v_\nu^{(n)} - v^{(n)} \Big| \mathscr{A}^{(p+q)/2}[\sqrt{\mathsf{c}^{(n)}}\sqrt{\mathsf{c}_\mu^{(n)}}\mathscr{A}^{(p+q)/2} v_\mu^{(n)}] - \mathscr{A}^{p}[\mathsf{c}_\mu^{(n)} \mathscr{A}^{q} v_\mu^{(n)}]\Big) \\
&= \Big((\mathrm{d}_{t\,\nu}^{(n)} \underline{\sqrt{\mathsf{c}}}_\tau)\mathscr{A}^{(p+q)/2} v_\nu^{(n)} \Big| \sqrt{\mathsf{c}_\mu^{(n)}}\mathscr{A}^{(p+q)/2} v_\mu^{(n)}\Big) \\
&\quad + \Big(\mathrm{d}_{t\,\nu}^{(n)} \underline{v}_\tau \Big| \mathscr{A}^{(p+q)/2}[\mathsf{c}_\mu^{(n)}\mathscr{A}^{(p+q)/2} v_\mu^{(n)}] - \mathscr{A}^{p}[\mathsf{c}_\mu^{(n)} \mathscr{A}^{q} v_\mu^{(n)}]\Big) \\
&\quad - \tau\Big(\mathrm{d}_{t\,\nu}^{(n)} \underline{v}_\tau \Big| \mathscr{A}^{(p+q)/2}[(\mathrm{d}_{t\,\mu}^{(n)} \underline{\sqrt{\mathsf{c}}}_\tau)\sqrt{\mathsf{c}_\mu^{(n)}}\mathscr{A}^{(p+q)/2} v_\mu^{(n)}]\Big)\,.
\end{aligned}$$

Thus with $v = u_{m-1}$ and $\mathsf{c} = \mathsf{c}_i$, $p = k_m$, $q = \ell_i$ (likewise for $\mathsf{c} = \mathsf{b}$, $p = k_m$, $q = \ell_r$ as needed in the second energy estimate) we get, using $\sum_{\nu=1}^s \mathfrak{C}_{\mu\nu} \mathrm{d}_{t\,\nu}^{(n)} \underline{u}_{\tau,m-1} = u_m^{(n+1)}$, that

$$\begin{aligned}
\frac{1}{2}\mathrm{d}_t^{(n+1)} \underline{\left|\sqrt{\mathsf{c}_i}\mathscr{A}^{(k_m+\ell_i)/2} u_{m-1}\right|^2}_\tau &\le \sum_{\mu=1}^s \Big(\sum_{\nu=1}^s \mathfrak{C}_{\mu\nu} \mathrm{d}_{t\,\nu}^{(n)} \underline{u_{m-1}}_\tau \Big| \mathscr{A}^{k_m} [\mathsf{c}_{i,\mu}^{(n)} \mathscr{A}^{\ell_i} u_{m-1,\mu}^{(n)}] \Big) \\
&+ \sum_{\mu=1}^s \sum_{\nu=1}^s \mathfrak{C}_{\mu\nu} \Big((\mathrm{d}_{t\,\nu}^{(n)} \underline{\sqrt{\mathsf{c}_i}}_\tau) \mathscr{A}^{(k_m+\ell_i)/2} u_{m-1,\nu}^{(n)} \Big| \sqrt{\mathsf{c}_\mu^{(n)}} \mathscr{A}^{(k_m+\ell_i)/2} u_{m-1,\mu}^{(n)} \Big) \\
&+ \sum_{\mu=1}^s \Big(\mathscr{A}^{k_m} u_m^{(n+1)} \Big| \mathscr{A}^{(\ell_i-k_m)/2} [\mathsf{c}_\mu^{(n)} \mathscr{A}^{(k_m+\ell_i)/2} u_{m-1,\mu}^{(n)}] - [\mathsf{c}_\mu^{(n)} \mathscr{A}^{\ell_i} u_{m-1,\mu}^{(n)}] \Big) \\
&- \tau \sum_{\mu=1}^s \Big(\mathscr{A}^{k_m} u_m^{(n+1)} \Big| \mathscr{A}^{(\ell_i-k_m)/2} [(\mathrm{d}_{t\,\mu}^{(n)} \underline{\sqrt{\mathsf{c}}}_\tau) \sqrt{\mathsf{c}_\mu^{(n)}} \mathscr{A}^{(k_m+\ell_i)/2} u_{m-1,\mu}^{(n)}] \Big) .
\end{aligned}$$

Altogether we therefore obtain the same results as for the implicit Euler method.

Theorem 6.2. *Under conditions* (5.2), (6.10), (5.27), (5.28), *there exists* $R > 0$ *(sufficiently small) such that for any* $\|\vec{u}_0\|_{X_0} \le \frac{R}{C}$ *the time discretised initial value problem* (6.17), (5.22) *with a stiffly accurate Runge Kutta method satisfying* (6.15) *has a unique solution* $\underline{\mathbf{u}}_\tau \in X_\tau$ *and this solution satisfies the estimate*

$$\|\underline{\mathbf{u}}_\tau\|_{X_\tau} \le C \|\vec{u}^0\|_{X_0}$$

with C *as in Proposition 5.1.*

Applying this to the models from Section 2 we get

- **BCBJK:**

$$\begin{aligned}
&\sum_{\nu=1}^s \mathfrak{C}_{\mu\nu} \mathrm{d}_{t\,\nu}^{(n)} \underline{\psi_0}_\tau = \psi_{1,\mu}^{(n)} \\
&\sum_{\nu=1}^s \mathfrak{C}_{\mu\nu} \mathrm{d}_{t\,\nu}^{(n)} \underline{\psi_1}_\tau = \psi_{2,\mu}^{(n)} \\
&\sum_{\nu=1}^s \mathfrak{C}_{\mu\nu} \mathrm{d}_{t\,\nu}^{(n)} \underline{\psi_2}_\tau = \frac{1}{1+\beta_5 \psi_{1,\mu}^{(n)}} \Big(\beta_1 \Delta \psi_{2,\mu}^{(n)} - \beta_2 \Delta^2 \psi_{1,\mu}^{(n)} + \beta_3 \Delta \psi_{1,\mu}^{(n)} - \beta_4 \Delta^2 \psi_{0,\mu}^{(n)} \\
&\qquad\qquad - \beta_5 \big(\psi_{2,\mu}^{(n)}\big)^2 - 2|\nabla \psi_{1,\mu}^{(n)}|^2 - 2\nabla \psi_{0,\mu}^{(n)} \cdot \nabla \psi_{2,\mu}^{(n)} \Big)
\end{aligned} \tag{6.19}$$

- **JMGT** with $\beta_0 = \frac{2\beta_a}{\rho c^2}$:

$$\begin{aligned}
&\sum_{\nu=1}^s \mathfrak{C}_{\mu\nu} \mathrm{d}_{t\,\nu}^{(n)} \underline{p_0}_\tau = p_{1,\mu}^{(n)} \\
&\sum_{\nu=1}^s \mathfrak{C}_{\mu\nu} \mathrm{d}_{t\,\nu}^{(n)} \underline{p_1}_\tau = p_{2,\mu}^{(n)} \\
&\sum_{\nu=1}^s \mathfrak{C}_{\mu\nu} \mathrm{d}_{t\,\nu}^{(n)} \underline{p_2}_\tau = \frac{1}{T_{rel}} \Big(-(1-\beta_0 p_{0,\mu}^{(n)})\, p_{2,\mu}^{(n)} + b\Delta p_{1,\mu}^{(n)} + c^2 \Delta p_{0,\mu}^{(n)} + \beta_0 \big(p_{1,\mu}^{(n)}\big)^2 \Big)
\end{aligned} \tag{6.20}$$

- **Westervelt** with $\beta_0 = \frac{2\beta_a}{\rho c^2}$:

$$\begin{aligned} &\sum_{\nu=1}^{s} \mathfrak{C}_{\mu\nu} \mathrm{d}_{t\,\nu}^{(n)} \underline{p_0}_\tau = p_{1,\mu}^{(n)} \\ &\sum_{\nu=1}^{s} \mathfrak{C}_{\mu\nu} \mathrm{d}_{t\,\nu}^{(n)} \underline{p_1}_\tau = \frac{b}{1-\beta_0 p_{0,\mu}^{(n)}} \Delta p_{1,\mu}^{(n)} + \frac{c^2}{1-\beta_0 p_{0,\mu}^{(n)}} \Delta p_{0,\mu}^{(n)} + \frac{\beta_0}{1-\beta_0 p_{0,\mu}^{(n)}} \left(p_{1,\mu}^{(n)}\right)^2 \end{aligned} \tag{6.21}$$

Corollary 6.2 (Existence of time-discrete solutions)**.** *There exists $R > 0$ (sufficiently small but independent of τ) such that for any $\|\vec{u}_0\|_{X_0} \leq \frac{R}{C}$ the semidiscrete PDEs* (6.11), (6.12), (6.13) *with initial data* (5.22) *and a stiffly accurate Runge Kutta method satisfying* (6.15) *have unique solutions $\underline{\mathbf{u}}_\tau = (\underline{u_0}_\tau, \underline{u_1}_\tau, \ldots, \underline{u_m}_\tau) \in X_\tau$ and these solutions satisfy the estimate*

$$\|(\underline{u_0}_\tau, \underline{u_1}_\tau, \ldots, \underline{u_m}_\tau)\|_{X_\tau} \leq C \|\vec{u}^0\|_{X_0}$$

with X_τ as in (6.14).

For the following convergence result we will consider the piecewise linear interpolants

$$v^\tau(t) = v_\nu^{(n)} + \frac{v_{\nu+1}^{(n)} - v_\nu^{(n)}}{(\mathfrak{c}_{\nu+1} - \mathfrak{c}_\nu)\tau} (t - t_n - \mathfrak{c}_\nu \tau), \quad t \in [t_n + \mathfrak{c}_\nu \tau, t_n + \mathfrak{c}_{\nu+1} \tau)$$

of grid functions $\underline{v}_\tau$.

Corollary 6.3 (Convergence)**.** *Let R be chosen as in Corollary 6.2, assume $\|\vec{u}_0\|_{X_0} \leq \frac{R}{C}$, and denote by $\mathbf{u}^\tau$ the piecewise linear interpolant of the solution $\underline{\mathbf{u}}_\tau = (\underline{u_0}_\tau, \underline{u_1}_\tau, \ldots, \underline{u_m}_\tau) \in X_\tau$ of one of the semidiscrete first order systems* (6.11), (6.12), (6.13), *with initial data* (5.22) *and a stiffly accurate Runge Kutta method satisfying* (6.15). *Then the family of these interpolants $(\mathbf{u}^\tau)_{\tau \in (0,\bar{\tau})}$ converges weakly* in X to the solution of the first order reformulation of the respective nonlinear initial value problem* (2.3), (2.2), (2.1), *with initial data* (5.22) *as τ tends to zero, that is,*

$$\mathbf{u}^\tau \stackrel{*}{\rightharpoonup} \mathbf{u} \text{ in } X \text{ as } \tau \to 0$$

In particular $\mathbf{u}^\tau$ converges strongly to $\mathbf{u}$ in any space that is compactly embedded into X.

Proof. According to Corollary 6.2, the family of interpolants $(\mathbf{u}^\tau)_{\tau \in (0,\bar{\tau})}$ is uniformly bounded in X as defined in (5.17). Therefore, there exist a weakly* convergent subsequence, which we denote by $(\mathbf{u}^k)_{k \in \mathbb{N}}$.

The fact that any wealy* convergent subsequence $(\mathbf{u}^k)_{k \in \mathbb{N}}$ tends to the unique solution $\mathbf{u}$ can be seen by considering, for arbitrary $\phi \in C_0^\infty(\Omega)$, $\theta \in C_0^\infty(0,T)$ and $j \in \{0, \ldots, m\}$ the integrals

$$L_j^k + N_j^k := \int_0^T \int_\Omega \Big((\mathscr{L}_j(\mathbf{u}^k - \mathbf{u})(x,t) + \big(\mathscr{N}_j(\mathbf{u}^k) - \mathscr{N}_j(\mathbf{u})\big)(x,t) \Big) \phi(x)\, \theta(t)\, \mathrm{d}x\, \mathrm{d}t$$

where $\mathscr{L}_j$ and $\mathscr{N}_j$ are the linear and nonlinear parts of the differential operators acting on the respective component (note that in case of BCBJK the linear part also contains a term in $\mathscr{B}$). It is straigthforward to see that $L_j^k \to 0$ as $k \to \infty$ due to the weak convergence $\mathbf{u}^k \stackrel{*}{\rightharpoonup} \mathbf{u}$ in X. For the nonlinear parts N_j^k we note that they vanish for $j \in \{0, \ldots, m-1\}$ and that N_m^k must be considered for the particular PDEs separately. We do so explicitely only for the Westervelt equation, where we have

$$\mathscr{N}_m(\mathbf{u}^k) - \mathscr{N}_j(\mathbf{u}) = \beta_0\Big(\frac{1}{1-\beta_0 u_0^k}\,\big(u_1^k\big)^2 - \frac{1}{1-\beta_0 u_0}\,\big(u_1\big)^2\Big)\,.$$

Hence with the functions $\omega^k = \frac{\beta_0^2\left(u_1^k\right)^2}{(1-\beta_0 u_0^k)(1-\beta_0 u_0)} + \frac{\beta_0(u_1^k+u_1)}{(1-\beta_0 u_0)}$ that are uniformly (with respect to k) bounded in $L^\infty(0,T;L^2(\Omega))$ due to nondegeneracy and boundedness of both the continuous and the discrete solution, we have

$$\begin{aligned} N_m^k &= \int_0^T \int_\Omega \Big(\omega^k(x,t)\,(u_0^k(x,t)-u_0(x,t))\,\phi(x)\,\theta(t)\,\mathrm{d}x\,\mathrm{d}t \\ &\le \|\omega^k\|_{L^\infty(0,T;L^2(\Omega))}\|u_0^k-u_0\|_{L^2(0,T;L^2(\Omega))}\|\phi\|_{L^\infty(\Omega)}\|\theta\|_{L^2(0,T)} \;\to\; 0 \text{ as } k\to\infty \end{aligned}$$

due to compactness of the embedding $H^1(0,T;H^2_\diamondsuit(\Omega)) \to L^2(0,T,L^2(\Omega))$. Similarly in principle, but with more involved computations, convergence to zero of the nonlinear terms can also be shown for JMGT and BCBJK. We point to [KALTENBACHER, NIKOLIĆ (2019), KALTENBACHER, THALHAMMER (2018)] and the fact that the same differences of nonlinear terms have to be tackled when studying convergence as one of the physical parameters in the PDE tends to zero. These estimates can be directly used here, since we have the same or higher regularity for BCBJK and JMGT, see Remark 5.2.

A subsequence-subsequence argument together with uniqeness of solutions to the respective limiting (time continuous) equations yields the assertion. □

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