Geometric Theory of Parabolic Problems

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Problem. Abstract parabolic evolution equation in Banach space under time discretisation

$$u' = F(u) \quad \xrightarrow[\text{discretisation}]{\text{time}} \quad \frac{u_{n+1} - u_n}{h} = F(u_{n+1}).$$

Classical approach. Constant $C(T)$ in error bound

$$\| u_n - u(nh) \| \le C(T)\,h, \qquad 0 \le nh \le T,$$

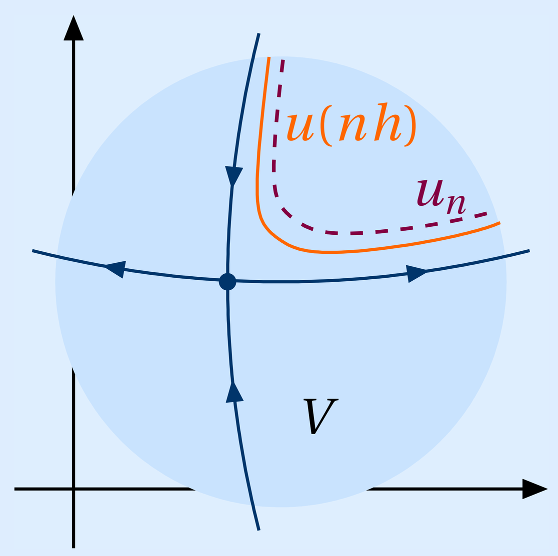
in general grows exponentially with time T.

Geometric theory. Analyse qualitative properties of continuous and corresponding discrete dynamical system.

Study invariant objects and discrete analogues.

- Equilibrium points, Periodic solutions
- Invariant manifolds, Bifurcations, etc.

Are stability properties captured by discretisation?



Saddle point. Study stable and unstable manifolds under discretisation.

Theorem. Error bound with constant C only depending on V

$$\| u_n - u(nh) \| \le C h, \qquad 0 \le nh \le T.$$

Numerical solution is shadowed by a true solution.

Problem classes.

- Non-autonomous problems
- Semilinear problems
- Fully nonlinear problems

Method classes.

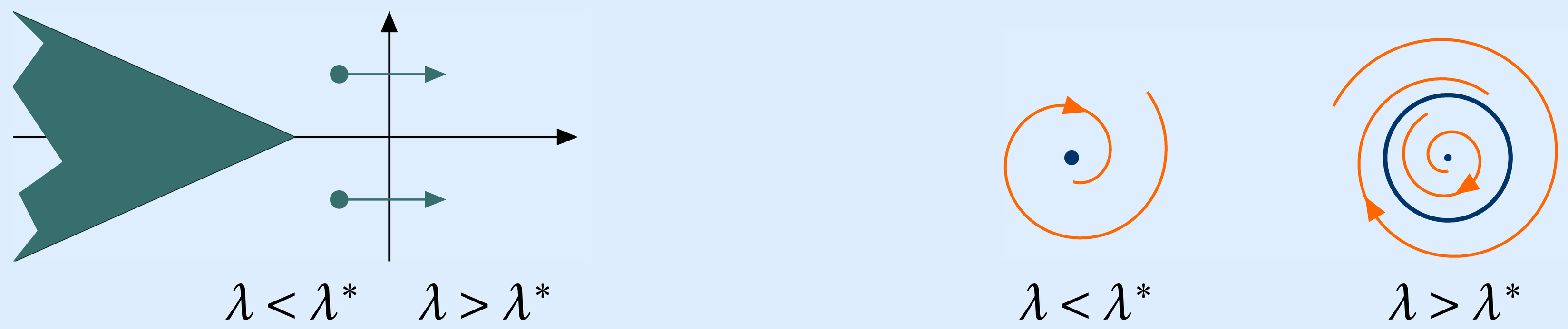
- Implicit Runge-Kutta methods
- Implicit multistep methods
- Linearly-implicit methods

Hopf bifurcation. Parameter-dependent parabolic equation

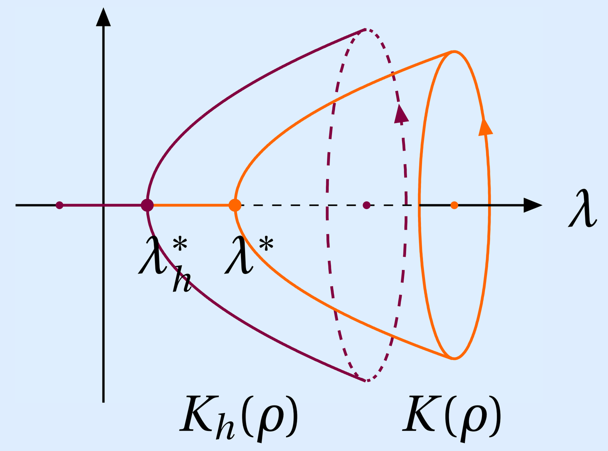
$$u' + A(\lambda)\,u = f(u, \lambda)$$

with equilibrium u^*, i.e., u^* satisfies $A(\lambda)\,u^* = f(u^*, \lambda)$.

Spectral assumption on $D_u f(u^*, \lambda) - A(\lambda)$



yields generalically supercritical Hopf bifurcation at $\lambda = \lambda^*$.



Define parameter ρ by

$$\lambda = \lambda^* + \rho^2$$
$$\lambda_h = \lambda_h^* + \rho^2$$

Theorem. Any strongly A(ϑ)-stable Runge-Kutta discretisation of order p and stage order q possesses a Naimark-Sacker bifurcation at λ_h^* for h sufficiently small. Further, the relations

$$|\lambda_h^* - \lambda^*| \le C h^p, \qquad \text{dist}_V\big(K(\rho), K_h(\rho)\big) \le C\,\rho\,h^r$$

are valid with $r = \min(p, q+1)$.

Example. Incompressible Navier-Stokes equations in 2D and 3D

$$u' - \lambda \Delta u = -(u \cdot \nabla)\,u - \nabla p, \qquad \text{div}\, u = 0.$$

Selected references.

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