# Exponential and Magnus Integrators for Parabolic Problems


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## Exponential Integrators

**Problem.** Semilinear parabolic evolution equation in Banach space

$$u' = Au + g(u), \qquad u(0) = u_0,$$

involving an unbounded linear operator $A$ generating an analytic semigroup $\big(\mathrm{e}^{tA}\big)_{t\geq 0}$. Solution represented by variation-of-constants formula

$$u(t) = \mathrm{e}^{tA}u_0 + \int_0^t \mathrm{e}^{(t-\tau)A} g\big(u(\tau)\big)\,\mathrm{d}\tau.$$

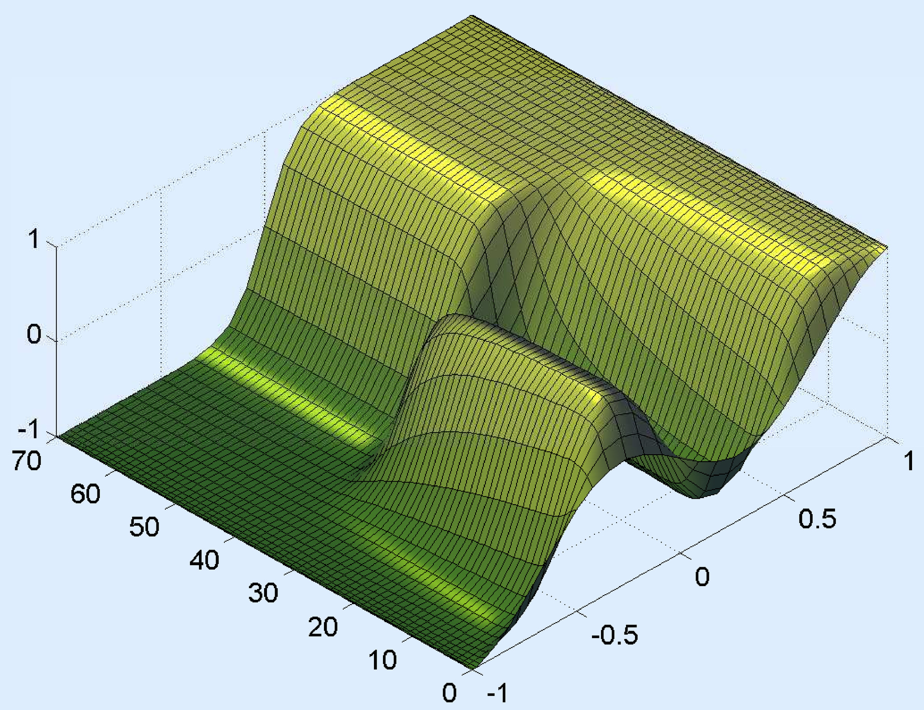
**Discretisation.** First-order approximation to $u(h)$ for $h > 0$ by exponential forward Euler method

$$u_1 = \mathrm{e}^{hA}u_0 + \int_0^h \mathrm{e}^{(h-\tau)A} g(u_0)\,\mathrm{d}\tau = \mathrm{e}^{hA}u_0 + h\,\varphi(hA)\,g(u_0)$$

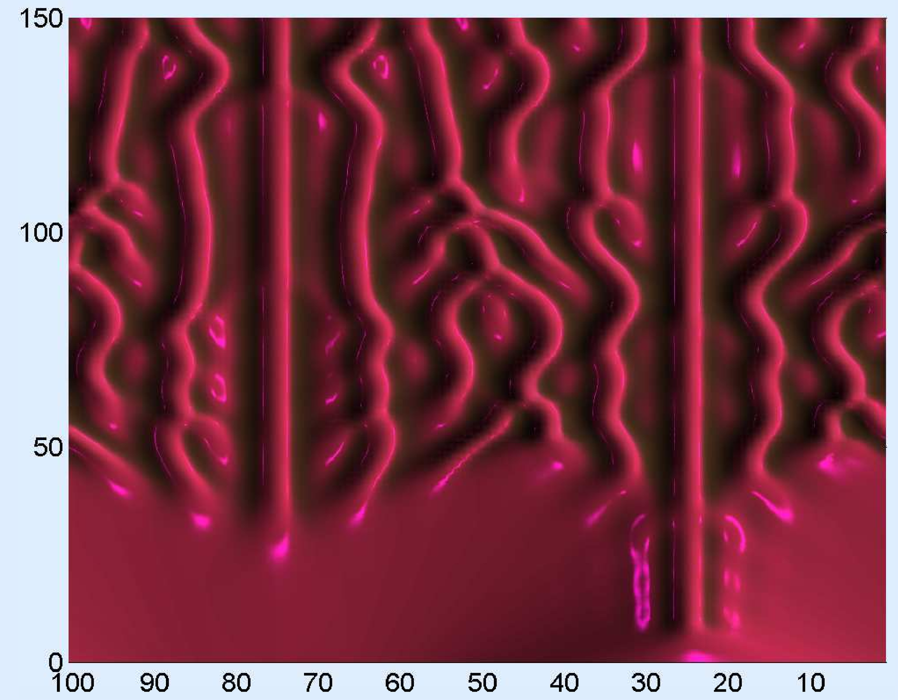
with $\varphi(z) = \big(\mathrm{e}^z - 1\big)/z$.

**Theorem.** There exist explicit exponential Runge-Kutta methods of arbitrarily high order.
*Proof.* Derive stiff order conditions.

**Special features.** Excellent stability properties. Conservation of positivity for second-order methods.




Solution of Allen-Cahn equation
$\partial_t u = 0.01\partial_x^2 u + u(1-u^2)$
by exponential Runge-Kutta method




Sol. of Kuramoto-Sivashinsky eq.
$\partial_t u = -\partial_x^4 u - \partial_x^2 u - u\partial_x u$
by exponential Runge-Kutta method

## Magnus Integrators

**Introduction.** Non-autonomous equation $u' = A(t)u$. Formal representation of solution $u(t) = \mathrm{e}^{\Omega(t)}u_0$ based on Magnus expansion

$$\Omega(t) = \int_0^t A(\tau)\,\mathrm{d}\tau - \tfrac{1}{2}\int_0^t \Big[\int_0^\tau A(\sigma)\,\mathrm{d}\sigma, A(\tau)\Big]\,\mathrm{d}\tau + \ldots$$

Second-order Magnus integrator $u_1 = \mathrm{e}^{hA(h/2)}u_0$ (truncate expansion, approximate integral by midpoint rule).

**Problem.** Linear abstract evolution equation involving time-dependent unbounded linear operator $A(t)$

$$u' = A(t)u + f(t), \qquad u(0) = u_0.$$

**Discretisation.** Combine second-order Magnus integrator and exponential midpoint rule

$$u_1 = \mathrm{e}^{hA(h/2)}u_0 + h\,\varphi\big(hA(h/2)\big)\,f(h/2).$$

**Theorem.** Order 2 in underlying Banach space. Order reduction in domain of linear operator $A$.

**Generalisations.** Magnus integrators for semilinear and quasilinear equations.

**Analysis of stability and convergence.** Formulate parabolic initial-boundary value problem as abstract initial value problem in the framework of sectorial operators and analytic semigroups on Banach spaces. Employ discrete variation-of-constants formula and perturbation techniques (Gronwall type inequalities). Efficient calculation of exponential feasible (Hochbruck, Lubich; SIAM J. Numer. Anal. 34, 1997).


**References (2004).**

**M. Hochbruck, A. O.**, *Exponential Runge-Kutta methods for parabolic problems.* To appear in Appl. Numer. Math.
**M. Hochbruck, A. O.**, *Explicit exponential Runge-Kutta methods for semilinear parabolic problems.* To appear in SIAM J. Numer. Anal.
**C. González, A. O., M. Th.**, *A second-order Magnus integrator for non-autonomous parabolic problems.* Submitted to J. Comp. Appl. Math.
**C. González, M. Th.**, *A second-order Magnus type integrator for quasilinear parabolic problems.* Submitted to Math. Comp.
**M. Th.**, *A second-order Magnus type integrator for non-autonomous semilinear parabolic problems.* Submitted to IMA J. Numer. Anal.



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