Nonlinear damped wave equations arising in high-intensity ultrasonics: Analytical and numerical aspects

Mechthild Thalhammer
Leopold–Franzens Universität Innsbruck, Austria

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Nonlinear acoustics

Nonlinear acoustics. Field of nonlinear acoustics concerned with propagation of sound waves in thermoviscous fluids. Applications in high-intensity ultrasonics include

- medical treatment (lithotripsy, thermotherapy) and
- industrial applications (ultrasound cleaning, welding).

Simulations. Numerical simulations provide valuable tools for design and improvement of high-intensity ultrasound devices.

Kidney stones, Lithotripsy. Quotation from https://www.healthline.com/

> *Kidney stones, or renal calculi, are solid masses made of crystals.*
>
> *Kidney stones are known to cause severe pain.*
>
> *Extracorporeal shock wave lithotripsy uses sound waves to break up large stones so they can more easily pass down the ureters into your bladder. This procedure can be uncomfortable and may require light anesthesia. It can cause bruising on the abdomen and back and bleeding around the kidney and nearby organs.*

Mathematical models

Mathematical models. Propagation of high-intensity ultrasound waves in thermoviscous fluids described by nonlinear damped wave equations. Blackstock–Crighton–Brunnhuber–Jordan–Kuznetsov equation has form

$$\begin{cases} \Big(\partial_{ttt} - \beta_1^{(a)}\,\Delta\partial_{tt} + \beta_2^{(a)}(\sigma_0)\,\Delta^2\partial_t - \beta_3\,\Delta\partial_t + \beta_4^{(a)}(\sigma_0)\,\Delta^2\Big)\,\psi^{(a)}(t) \\ \quad + \partial_{tt}\Big(\tfrac{1}{2}\,\beta_5(\sigma)\,\big(\partial_t\psi^{(a)}(t)\big)^2 + \beta_6(\sigma)\,\big|\nabla\psi^{(a)}(t)\big|^2\Big) = 0\,, \quad t \in (0,T)\,, \\ \psi^{(a)}(0) = \psi_0\,, \quad \partial_t\psi^{(a)}(0) = \psi_1\,, \quad \partial_{tt}\psi^{(a)}(0) = \psi_2\,. \end{cases}$$

Reduced models. Commonly used Kuznetsov and Westervelt equations result when neglecting thermal and local nonlinear effects

$$\begin{cases} \Big(\partial_{tt} - \beta_1^{(0)}\,\Delta\partial_t - \beta_3\,\Delta\Big)\,\psi(t) + \partial_t\Big(\tfrac{1}{2}\,\beta_5(\sigma)\,\big(\partial_t\psi(t)\big)^2 + \beta_6(\sigma)\,|\nabla\psi(t)|^2\Big) = 0\,, \quad t \in (0,T)\,, \\ \psi(0) = \psi_0\,, \quad \partial_t\psi(0) = \psi_1\,. \end{cases}$$

Numerical challenges. Use of transient numerical simulations within mathematical optimisation of high-intensity ultrasound devices still beyond scope of existing approaches.

Novel approach

Novel approach. Operator splitting methods known to be efficient time integration methods for nonlinear partial differential equations

$$\begin{cases} u'(t) = F\big(u(t)\big) = A\big(u(t)\big) + B\big(u(t)\big), \quad t \in (0,T), \\ u(0) \text{ given}, \end{cases}$$

$$u_n = \mathscr{S}_F(\tau_{n-1}, u_{n-1}) = \prod_{j=1}^{s} \mathrm{e}^{a_{s+1-j}\tau_{n-1}D_A}\, \mathrm{e}^{b_{s+1-j}\tau_{n-1}D_B}\, u_{n-1}$$

$$\approx u(t_n) = \mathscr{E}_F\big(\tau_{n-1}, u(t_{n-1})\big) = \mathrm{e}^{\tau_{n-1}D_F}\, u(t_{n-1}), \quad n \in \{1,\dots,N\}.$$

Motivates introduction and investigation of operator splitting methods for nonlinear damped wave equations arising in nonlinear acoustics.

Our contributions and plans

Former work.

Barbara Kaltenbacher, Vanja Nikolić, M. Th.
Efficient time integration methods based on operator splitting and application to the Westervelt equation.
IMA J. Numer. Anal. 35/3 (2015) 1092–1124.

Current work.

Barbara Kaltenbacher, M. Th.
Fundamental models in nonlinear acoustics. Part I. Analytical comparison.
Submitted to M3AS.

Future work.

Barbara Kaltenbacher, M. Th.
Part II. Numerical comparison.

Illustration (General model)

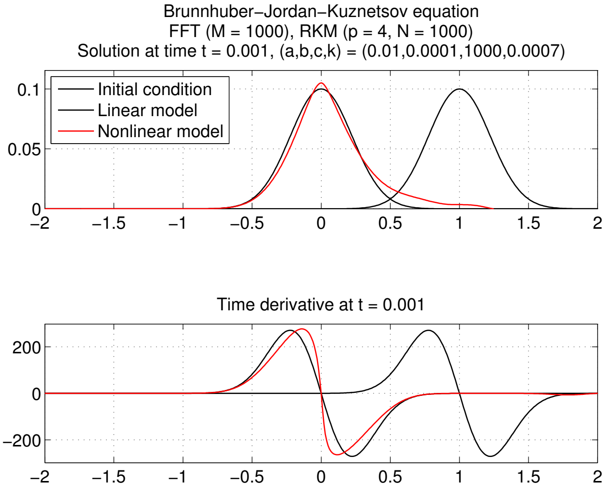
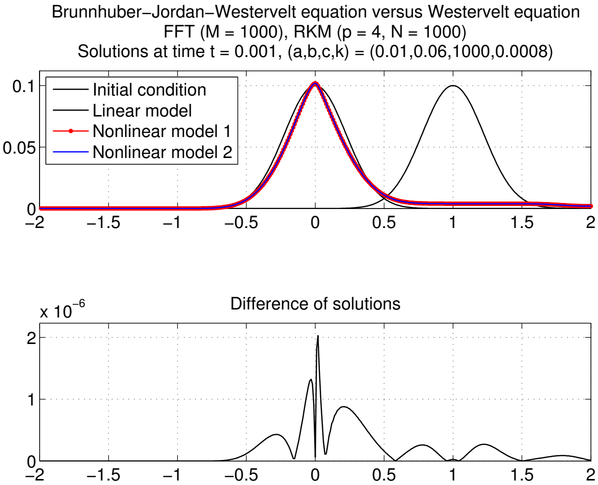


Illustration (General versus reduced model)



Numerical aspects

Westervelt equation
Operator splitting methods
Convergence analysis

Westervelt equation

Westervelt equation. Consider nonlinear damped wave equation for acoustic velocity potential

$$\begin{cases} \partial_{tt}\psi(t) - \alpha\,\partial_{xxt}\psi(t) - \beta\,\partial_{xx}\psi(t) \\ \quad = \gamma\,\partial_t\big(\partial_t\psi(t)\big)^2 = \delta\,\partial_t\psi(t)\,\partial_{tt}\psi(t)\,, \quad t\in(0,T)\,, \\ \psi(0) = \psi_0, \quad \partial_t\psi(0) = \psi_1\,, \end{cases}$$

involving constants $\alpha, \beta > 0$ and $\delta = 2\gamma \neq 0$.

Remarks.

- For notational simplicity, consider single space dimension.
- Focus on relevant case of homogeneous Dirichlet boundary conditions.
- Assume that prescribed initial data are sufficiently regular and small. Theoretical result ensures existence, non-degeneracy, and regularity of solution.
- Justifies reformulation of non-degenerate Westervelt equation

$$\partial_{tt}\psi(t) = \alpha\left(1-\delta\,\partial_t\psi(t)\right)^{-1}\partial_{xxt}\psi(t) + \beta\left(1-\delta\,\partial_t\psi(t)\right)^{-1}\partial_{xx}\psi(t)\,.$$

Associated subproblems (Decomposition I)

Reformulation. Regarding introduction and error analysis of operator splitting methods, rewrite Westervelt equation as first-order system for $\Psi = (\Psi_1, \Psi_2) = (\psi, \partial_t \psi)$

$$\begin{cases} \partial_t \Psi_1(t) = \Psi_2(t)\,, \\ \partial_t \Psi_2(t) = \alpha \big(1 - \delta\, \Psi_2(t)\big)^{-1} \partial_{xx} \Psi_2(t) + \beta \big(1 - \delta\, \Psi_2(t)\big)^{-1} \partial_{xx} \Psi_1(t)\,. \end{cases}$$

Abstract formulation and subproblems. Employ compact formulation as nonlinear evolution equation with nonlinear operators A, B

$$u'(t) = F\big(u(t)\big) = A\big(u(t)\big) + B\big(u(t)\big)\,, \quad t \in (0, T)\,,$$

$$A(v) = \begin{pmatrix} v_2 \\ \alpha \big(1 - \delta\, v_2\big)^{-1} \partial_{xx} v_2 \end{pmatrix}, \quad B(v) = \begin{pmatrix} 0 \\ \beta \big(1 - \delta\, v_2\big)^{-1} \partial_{xx} v_1 \end{pmatrix}.$$

Associated subproblems correspond to nonlinear diffusion equation and ordinary differential equation.

Associated subproblems (Decomposition I)

Abstract formulation. Employ compact formulation as nonlinear evolution equation with nonlinear operators A, B

$$u'(t) = F\big(u(t)\big) = A\big(u(t)\big) + B\big(u(t)\big)\,, \quad t \in (0, T)\,,$$

$$A(v) = \begin{pmatrix} v_2 \\ \alpha\,\big(1 - \delta\, v_2\big)^{-1}\,\partial_{xx}\, v_2 \end{pmatrix}, \quad B(v) = \begin{pmatrix} 0 \\ \beta\,\big(1 - \delta\, v_2\big)^{-1}\,\partial_{xx}\, v_1 \end{pmatrix}.$$

Subproblem (Nonlinear diffusion equation). Resolution of subproblem associated with A

$$\begin{cases} \partial_t \Psi_1(x,t) = \Psi_2(x,t)\,, \\ \partial_t \Psi_2(x,t) = \alpha\,\big(1 - \delta\,\Psi_2(x,t)\big)^{-1} \partial_{xx}\Psi_2(x,t)\,, \end{cases}$$

amounts to solution of nonlinear diffusion equation for second component $\Psi_2 = \partial_t \psi$

$$\partial_t \Psi_2(x,t) = \alpha\,\big(1 - \delta\,\Psi_2(x,t)\big)^{-1}\,\partial_{xx}\Psi_2(x,t)\,.$$

First component $\Psi_1 = \psi$ then retained by (pointwise) integration

$$\Psi_1(x,t) = \Psi_1(x,0) + \int_0^t \Psi_2(x,\tau)\,\mathrm{d}\tau\,.$$

Associated subproblems (Decomposition I)

Employ compact formulation as nonlinear evolution equation with nonlinear operators A, B

$$u'(t) = F\big(u(t)\big) = A\big(u(t)\big) + B\big(u(t)\big)\,, \quad t \in (0,T)\,,$$

$$A(v) = \begin{pmatrix} v_2 \\ \alpha\,\big(1-\delta\, v_2\big)^{-1}\,\partial_{xx}\, v_2 \end{pmatrix}, \quad B(v) = \begin{pmatrix} 0 \\ \beta\,\big(1-\delta\, v_2\big)^{-1}\,\partial_{xx}\, v_1 \end{pmatrix}.$$

Subproblem (Explicit representation). For subproblem associated with B

$$\begin{cases} \partial_t \Psi_1(x,t) = 0\,, \\ \partial_t \Psi_2(x,t) = \beta\,\big(1-\delta\,\Psi_2(x,t)\big)^{-1}\,\partial_{xx}\Psi_1(x,t)\,, \end{cases}$$

first component remains constant on considered time interval

$$\Psi_1(x,t) = \Psi_1(x,0)\,.$$

Consequently, second component is (pointwise) solution to ODE with explicit representation

$$\partial_t \Psi_2(x,t) = \beta\,\big(1-\delta\,\Psi_2(x,t)\big)^{-1}\partial_{xx}\Psi_1(x,0)\,,$$

$$\Psi_2(x,t) = \tfrac{1}{\delta}\,\Big(1-\sqrt{\varphi(x,t)}\Big), \quad \varphi(x,t) = \big(1-\delta\,\Psi_2(x,0)\big)^2 - 2\,\beta\,\delta\,t\,\partial_{xx}\Psi_1(x,0)\,.$$

Suitable choice of time increment $t>0$ ensures $\varphi(x,t)>0$ and hence $\Psi_2(x,t)\in\mathbb{R}$.

Operator splitting methods for Westervelt equation

Operator splitting methods

Operator splitting methods. Time integration of nonlinear evolution equation

$$u'(t) = F\big(u(t)\big) = A\big(u(t)\big) + B\big(u(t)\big), \quad t \in (0, T).$$

Approximations at time grid points $0 = t_0 < \cdots < t_N \leq T$ with increments $\tau_{n-1} = t_n - t_{n-1}$ are determined by recurrence of form

$$u_n = \mathscr{S}_F(\tau_{n-1}, u_{n-1}) \approx u(t_n) = \mathscr{E}_F\big(\tau_{n-1}, u(t_{n-1})\big) = \mathrm{e}^{\tau_{n-1} D_F}\, u(t_{n-1}), \quad n \in \{1, \dots, N\}.$$

Exponential operator splitting methods rely on suitable decomposition of right-hand side and presumption that associated subproblems are solvable in accurate and efficient manner

$$v'(t) = A\big(v(t)\big), \quad v(t) = \mathrm{e}^{tD_A}\, v(0), \qquad w'(t) = B\big(w(t)\big), \quad w(t) = \mathrm{e}^{tD_B}\, w(0).$$

Lower-order schemes. First-order Lie–Trotter splitting method and second-order Strang splitting method given by

$$\mathscr{S}_F(t, \cdot) = \mathrm{e}^{tD_B}\, \mathrm{e}^{tD_A}, \qquad \mathscr{S}_F(t, \cdot) = \mathrm{e}^{\frac{1}{2} tD_A}\, \mathrm{e}^{tD_B}\, \mathrm{e}^{\frac{1}{2} tD_A}.$$

Westervelt equation. Solution of subproblem associated with A requires resolution of nonlinear diffusion equation and (pointwise) integration. Explicit (pointwise) representation available for solution to subproblem associated with B.

Stability and error analysis of Lie–Trotter splitting method

Approach

Approach. Consider first-order Lie–Trotter splitting method

$$\mathscr{S}_F(t,\cdot) = \mathrm{e}^{tD_A}\,\mathrm{e}^{tD_B} = \mathscr{E}_A\big(t, \mathscr{E}_B(t,\cdot)\big)\,.$$

Employ compact local error expansion

$$\begin{aligned}\mathscr{L}_F(t,\cdot) = \int_0^t \int_0^{\tau_1} & \partial_2 \mathscr{E}_F\big(t-\tau_1, \mathscr{S}_F(\tau_1,\cdot)\big)\,\partial_2 \mathscr{E}_B\big(\tau_1-\tau_2, \mathscr{E}_A(\tau_1,\cdot)\big) \\ & \times \big[B,A\big]\Big(\mathscr{E}_B\big(\tau_2, \mathscr{E}_A(\tau_1,\cdot)\big)\Big)\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1\end{aligned}$$

deduced in Descombes, Th. (2010, 2012) and studied for Schrödinger equations in semi-classical regime.

Application to Westervelt equation

Challenge. Application to Westervelt equation requires derivation of auxiliary regularity results for Westervelt equation, associated subproblems and variational equations, as well as estimate for Lie-commutator

$$
\begin{aligned}
\|\mathscr{E}_F(t,v)\|_{H^{k+6}\times H^{k+5}} &\le \mathrm{e}^{Ct}\,\|v\|_{H^{k+6}\times H^{k+5}}\,, \quad k\in\mathbb{N}_{\ge 0}\,,\\
\|\mathscr{E}_A(t,v)\|_{H^{k+4}\times H^{k+2}} &\le \mathrm{e}^{Ct}\,\|v\|_{H^{k+4}\times H^{k+2}}\,, \quad k\in\mathbb{N}_{\ge 0}\,,\\
\|\mathscr{E}_B(t,v)\|_{H^{k+2}\times H^{k}} &\le \mathrm{e}^{Ct}\,\|v\|_{H^{k+2}\times H^{k}}\,, \quad k\in\mathbb{N}_{\ge 0}\,,\\
\|\partial_2\mathscr{E}_F(t,v)\,w\|_{H^{\ell+1}\times H^{\ell}} &\le \mathrm{e}^{C(\|v\|_{H^4\times H^4})\,t}\,\|w\|_{H^{\ell+1}\times H^{\ell}}\,, \quad \ell=0,1,2,3\,,\\
\|\partial_2\mathscr{E}_A(t,v)\,w\|_{H^{k+2}\times H^{k}} &\le \begin{cases} \mathrm{e}^{C(\|v\|_{H^5\times H^3})\,t}\,\|w\|_{H^{k+2}\times H^{k}}\,, & k=0,1,2\,,\\ \mathrm{e}^{C(\|v\|_{H^7\times H^5})\,t}\,\|w\|_{H^{k+2}\times H^{k}}\,, & k\in\mathbb{N}_{\ge 3}\,,\end{cases}\\
\|\partial_2\mathscr{E}_B(t,v)\,w\|_{H^{k+2}\times H^{k}} &\le \mathrm{e}^{C(\|v\|_{H^{k+4}\times H^{k+2}})\,t}\,\|w\|_{H^{k+2}\times H^{k}}\,, \quad k\in\mathbb{N}_{\ge 0}\,,\\
\|[A,B](v)\|_{H^{k+2}\times H^{k}} &\le C\big(\|v\|_{H^{k+4}\times H^{k+2}}\big)\,, \quad k\in\mathbb{N}_{\ge 0}\,.
\end{aligned}
$$

Remark. Obtained regularity results imply stability estimate for splitting methods. Global error estimate follows by standard approach (telescopic identity).

Main result (Lie–Trotter splitting)

Convergence result. Employ basic regularity assumption on initial state and additional compatibility conditions

$$u(0) = \big(\psi(\cdot,0), \partial_t \psi(\cdot,0)\big) \in H^6(\Omega) \times H^5(\Omega)\,,$$
$$\|u(0)\|_{H^6 \times H^5} = \|\psi(\cdot,0)\|_{H^6} + \|\partial_t \psi(\cdot,0)\|_{H^5} \leq C_0\,.$$

Apply auxiliary result that ensures regularity and boundedness of solution

$$u(t) \in H^6(\Omega) \times H^5(\Omega)\,, \quad \|u(t)\|_{H^6 \times H^5} \leq C\,, \quad t \in [0,T]\,.$$

Obtain global error estimate for Lie–Trotter splitting method applied to Westervelt equation.

Theorem (Kaltenbacher, Nikolić, Th., 2015)

Assume that initial state fulfills above requirements and that intial approximation u_0 remains bounded in $H^5(\Omega) \times H^3(\Omega)$. Then, Lie–Trotter splitting method applied to Westervelt equation satisfies global error estimate

$$\|u_N - u(t_N)\|_{H^3 \times H^1} \leq C\Big(\|u_0 - u(0)\|_{H^3 \times H^1} + \tau\Big)\,, \quad \tau = \max_{n \in \{0,1,\dots,N-1\}} \tau_n\,,$$

with constant depending on bounds for $\|u\|_{\mathscr{C}([0,t_N],H^6 \times H^5)}$, $\|u_0\|_{H^5 \times H^3}$, and final time T.

Illustration (Global error)

Situation.

- Consider Westervelt equation in single space dimension (facilitates computations)

$$a = 8, \quad \alpha = 1, \quad \beta = 1, \quad \gamma = \tfrac{1}{2}, \quad \delta = 2\gamma = 1,$$
$$\partial_{tt}\psi(x,t) - \alpha\,\partial_{xxt}\psi(x,t) - \beta\,\partial_{xx}\psi(x,t) = \delta\,\partial_t\psi(x,t)\,\partial_{tt}\psi(x,t),$$
$$\psi(x,0) = \mathrm{e}^{-x^2}, \quad \partial_t\psi(x,0) = -\,x\,\mathrm{e}^{-x^2}, \quad (x,t) \in [-a,a] \times [0,T],$$

  and impose homogeneous Dirichlet boundary conditions. Note that for prescribed initial data solution to Westervelt equation is regular.

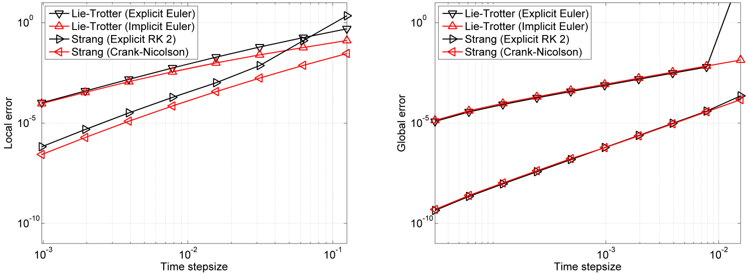
- Chose spatial grid width sufficiently fine such that global error dominated by time discretisation error ($M = 100$).

- Compare accuracy of Lie–Trotter and Strang splitting methods. For numerical solution of parabolic subproblem apply explicit and implicit time integrators of same order as underlying splitting method, i.e. combine Lie–Trotter splitting method with explicit and implicit Euler methods and Strang splitting method with second-order explicit Runge–Kutta method and Crank–Nicolson scheme. Note that use of explicit solvers requires sufficiently small time increments to avoid instabilities.

- Display local and global errors at time $T = 1$.

Illustration (Local and global errors)

Numerical results ($H^3 \times H^1$-norm). Time integration of Westervelt equation by Lie–Trotter and Strang splitting methods (Decomposition I). Comparison of different methods for numerical solution of subproblems. Computation of local (left) and global (right) errors with respect to $H^3 \times H^1$-norm. Nonstiff orders retained in accordance with convergence result.



Remark. Consider different ranges of time stepsizes for local error (include larger time stepsizes to study stability behaviour) and global error (include smaller time stepsizes to study attainable accuracy).

Analytical aspects

Derivation of general model
Existence and regularity result
Justification of limiting systems

Approach

Approach. Derivation of general model relies on physical and mathematical principles.

- Decompose basic state variables of acoustics into constant mean values and space-time-dependent fluctuations

$$\text{mass density} \quad \varrho = \varrho_0 + \varrho_\sim\,, \quad \text{acoustic particle velocity} \quad v = v_\sim\,,$$
$$\text{acoustic pressure} \quad p = p_0 + p_\sim\,, \quad \text{temperature} \quad T = T_0 + T_\sim\,.$$

- Use Helmholtz decomposition of acoustic particle velocity and assign irrotational part to gradient of acoustic velocity potential

$$v_\sim = \nabla\psi + \nabla \times S\,.$$

Approach

- Employ conservation laws for mass, momentum, energy

$$\partial_t \varrho + \nabla \cdot (\varrho\, v) = 0\,,$$
$$\partial_t (\varrho\, v) + v \nabla \cdot (\varrho\, v) + \varrho\,(v \cdot \nabla)\, v + \nabla p = \mu \Delta v + \left(\mu_B + \tfrac{1}{3}\,\mu\right) \nabla(\nabla \cdot v)\,,$$
$$\varrho \left(c_V\, \partial_t T + c_V\, v \cdot \nabla T + \frac{c_p - c_V}{\alpha_V}\, \nabla \cdot v\right)$$
$$= a\, \Delta T + \left(\mu_B - \tfrac{2}{3}\,\mu\right) (\nabla \cdot v)^2 + \tfrac{1}{2}\,\mu \left\| \nabla v + (\nabla v)^T \right\|_F^2\,,$$

as well as equation of state for acoustic pressure

$$p_\sim \approx A\, \frac{\varrho_\sim}{\varrho_0} + \frac{B}{2} \left(\frac{\varrho_\sim}{\varrho_0}\right)^2 + \hat{A}\, \frac{T_\sim}{T_0}\,.$$

Relations in particular involve thermal conductivity $a > 0$ and parameter of nonlinearity $\frac{B}{A} > 0$.

- Accordingly to Blackstock (1963) and Lighthill (1956), take first- and second-order contributions with respect to fluctuating quantities into account.

General model

General model. Above approach leads to general model

$$\begin{aligned}
&\partial_{ttt}\psi^{(a)}(t) - \beta_1^{(a)}\,\Delta\partial_{tt}\psi^{(a)}(t) + \beta_2^{(a)}(\sigma_0)\,\Delta^2\partial_t\psi^{(a)}(t)\\
&\quad - \beta_3\,\Delta\partial_t\psi^{(a)}(t) + \beta_4^{(a)}(\sigma_0)\,\Delta^2\psi^{(a)}(t)\\
&\quad + \partial_{tt}\Big(\tfrac{1}{2}\,\beta_5(\sigma)\,\big(\partial_t\psi^{(a)}(t)\big)^2 + \beta_6(\sigma)\,\big|\nabla\psi^{(a)}(t)\big|^2\Big) = 0\,, \qquad t \in (0,T)\,,
\end{aligned}$$

involving coefficients

$$\begin{gathered}
\beta_1^{(a)} = a\,\big(1 + \tfrac{B}{A}\big) + \nu\Lambda\,, \quad \beta_2^{(a)}(\sigma_0) = a\,\big(\nu\Lambda + a\,\tfrac{B}{A} + \sigma_0\,\tfrac{B}{A}\,(\nu\Lambda - a)\big)\,,\\
\beta_3 = c_0^2\,, \quad \beta_4^{(a)}(\sigma_0) = a\,\big(1 + \sigma_0\,\tfrac{B}{A}\big)\,c_0^2\,,\\
\beta_5(\sigma) = \tfrac{1}{c_0^2}\,\big(2\,(1-\sigma) + \tfrac{B}{A}\big)\,, \quad \beta_6(\sigma) = \sigma\,, \quad \sigma, \sigma_0 \in \{0,1\}\,.
\end{gathered}$$

Hierarchy

Hierarchy. Overview of considered hierachy of nonlinear damped wave equations.

$$\begin{tikzcd}
\text{Brunnhuber–Jordan–Kuznetsov (BJK)} \arrow[r, "\sigma=0"] \arrow[d, "\sigma_0=0"] & \text{Brunnhuber–Jordan–Westervelt (BJW)} \arrow[d, "\sigma_0=0"] \\
\text{Blackstock–Crighton–Kuznetsov (BCK)} \arrow[r, "\sigma=0"] \arrow[d, "a\to 0_+"] & \text{Blackstock–Crighton–Westervelt (BCW)} \arrow[d, "a\to 0_+"] \\
\text{Kuznetsov (K)} \arrow[r, "\sigma=0"] & \text{Westervelt (W)}
\end{tikzcd}$$

Remarks.

- BJK cast into general formulation with $\sigma = \sigma_0 = 1$.
- BCK describes monatomic gases (quantity $(\nu\Lambda - a)\,\frac{B}{A}$ negligible).
- Kuznetsov equation results as limiting system.
- Westervelt-type equations additionally do not take into account local nonlinear effects (term $c_0^2\,|\nabla\psi|^2 - (\partial_t\psi)^2$ negligible).

Existence and regularity result

Initial-boundary value problem.

- Let $a \in (0, \overline{a}]$.
- Consider nonlinear damped wave equation

$$\begin{cases} \partial_{ttt}\psi^{(a)}(t) - \beta_1^{(a)}\,\Delta\partial_{tt}\psi^{(a)}(t) + \beta_2^{(a)}(\sigma_0)\,\Delta^2\partial_t\psi^{(a)}(t) \\ \quad - \beta_3\,\Delta\partial_t\psi^{(a)}(t) + \beta_4^{(a)}(\sigma_0)\,\Delta^2\psi^{(a)}(t) \\ \quad + \partial_{tt}\Big(\frac{1}{2}\,\beta_5(\sigma)\,\big(\partial_t\psi^{(a)}(t)\big)^2 + \beta_6(\sigma)\,\big|\nabla\psi^{(a)}(t)\big|^2\Big) = 0\,, \quad t \in (0,T)\,, \\ \psi^{(a)}(0) = \psi_0, \quad \partial_t\psi^{(a)}(0) = \psi_1\,, \quad \partial_{tt}\psi^{(a)}(0) = \psi_2\,. \end{cases}$$

- Impose homogeneous Dirichlet boundary conditions

$$\partial_{tt}\psi(t)\big|_{\partial\Omega} = 0\,, \quad \Delta\partial_t\psi(t)\big|_{\partial\Omega} = 0\,, \quad \Delta\psi(t)\big|_{\partial\Omega} = 0\,,$$
$$\partial_{ttt}\psi(t)\big|_{\partial\Omega} = 0\,, \quad \Delta\partial_{tt}\psi(t)\big|_{\partial\Omega} = 0\,.$$

Existence and regularity result

Assumptions.

- Suppose that prescribed initial data satisfy regularity and compatibility conditions

$$\psi_0, \psi_1 \in H^3(\Omega) \cap H_0^1(\Omega)\,, \quad \Delta\psi_0, \Delta\psi_1, \psi_2 \in H_0^1(\Omega)\,.$$

- Assume that for $\|\Delta\psi_0\|_{L_2}$, $\|\nabla\Delta\psi_0\|_{L_2}$, and upper bounds $\overline{e}_0, \overline{e}_1 > 0$ on initial energies

$$\begin{aligned}
\big\|\psi_2\big\|_{L_2}^2 + \beta_2^{(a)}(\sigma_0)\,\big\|\Delta\psi_1\big\|_{L_2}^2 + \big\|\nabla\psi_1\big\|_{L_2}^2 &\leq \overline{e}_0\,,\\
\big\|\nabla\psi_2\big\|_{L_2}^2 + \beta_2^{(a)}(\sigma_0)\,\big\|\nabla\Delta\psi_1\big\|_{L_2}^2 + \big\|\Delta\psi_1\big\|_{L_2}^2 &\leq \overline{e}_1\,,
\end{aligned}$$

following quantity is sufficiently small

$$\begin{aligned}
M\big(\overline{e}_0, \overline{e}_1\big) = \frac{C_{\mathrm{PF}}^2\, C_{L_4 \hookleftarrow H^1}^2\, \beta_5(\sigma)}{\underline{\beta}_1}\,\sqrt{\overline{e}_0} + C_0\,\overline{e}_1 \\
+ \frac{C_2}{\underline{\beta}_1}\Big(\big\|\Delta\psi_0\big\|_{L_2}^2 + C_3\,T^2\,\overline{e}_1\Big) + C_4\Big(\tfrac{1}{2}\,\big\|\nabla\Delta\psi_0\big\|_{L_2} + \sqrt{\overline{e}_1}\Big).
\end{aligned}$$

Existence and regularity result

Theorem (Kaltenbacher, Th., 2018)

Under the above assumptions, there exists a weak solution

$$\psi \in X = H^2\big([0,T], H^2_\diamond(\Omega)\big) \cap W^2_\infty\big([0,T], H^1_0(\Omega)\big) \cap W^1_\infty\big([0,T], H^3_\diamond(\Omega)\big),$$
$$H^2_\diamond(\Omega) = \big\{\chi \in H^2(\Omega) : \chi \in H^1_0(\Omega)\big\}, \quad H^3_\diamond(\Omega) = \big\{\chi \in H^3(\Omega) : \chi, \Delta\chi \in H^1_0(\Omega)\big\},$$

to the associated equation

$$\begin{aligned}
&\partial_{tt}\psi(t) - \psi_2 - \beta_1^{(a)}\,\Delta\big(\partial_t\psi(t) - \psi_1\big) + \beta_2^{(a)}(\sigma_0)\,\Delta^2\big(\psi(t) - \psi_0\big) - \beta_3\,\Delta\big(\psi(t) - \psi_0\big) \\
&\quad + \beta_4^{(a)}(\sigma_0)\int_0^t \Delta^2\psi(\tau)\,\mathrm{d}\tau + \beta_5(\sigma)\big(\partial_{tt}\psi(t)\,\partial_t\psi(t) - \psi_2\,\psi_1\big) \\
&\quad + 2\,\beta_6(\sigma)\big(\nabla\partial_t\psi(t)\cdot\nabla\psi(t) - \nabla\psi_1\cdot\nabla\psi_0\big) = 0,
\end{aligned}$$

obtained by integration with respect to time.

Existence and regularity result

Theorem (Kaltenbacher, Th., 2018)

This solution satisfies a priori energy estimates of the form

$$\mathscr{E}_0\big(\psi(t)\big) = \big\|\partial_{tt}\psi(t)\big\|_{L_2}^2 + \beta_2^{(a)}(\sigma_0)\,\big\|\Delta\partial_t\psi(t)\big\|_{L_2}^2 + \big\|\nabla\partial_t\psi(t)\big\|_{L_2}^2\,,$$

$$\mathscr{E}_1\big(\psi(t)\big) = \big\|\nabla\partial_{tt}\psi(t)\big\|_{L_2}^2 + \beta_2^{(a)}(\sigma_0)\,\big\|\nabla\Delta\partial_t\psi(t)\big\|_{L_2}^2 + \big\|\Delta\partial_t\psi(t)\big\|_{L_2}^2\,,$$

$$\sup_{t\in[0,T]}\,\mathscr{E}_0\big(\psi(t)\big) \leq \overline{E}_0\,, \quad \sup_{t\in[0,T]}\,\mathscr{E}_1\big(\psi(t)\big) \leq \overline{E}_1\,, \quad \int_0^T \big\|\Delta\partial_{tt}\psi(t)\big\|_{L_2}^2\,\mathrm{d}t \leq \overline{E}_2\,,$$

which hold uniformly for $a \in (0,\overline{a}]$. *In particular, the quantity* $M(\overline{E}_0,\overline{E}_1)$ *remains sufficiently small to ensure uniform boundedness and hence non-degeneracy of the first time derivative*

$$0 < \underline{\alpha} = \tfrac{1}{2} \leq \big\|1 + \beta_5(\sigma)\,\partial_t\psi\big\|_{L_\infty([0,T],L_\infty(\Omega))} \leq \overline{\alpha} = \tfrac{3}{2}\,,$$

$$0 < \tfrac{1}{\overline{\alpha}} = \tfrac{2}{3} \leq \Big\|\big(1 + \beta_5(\sigma)\,\partial_t\psi\big)^{-1}\Big\|_{L_\infty([0,T],L_\infty(\Omega))} \leq \tfrac{1}{\underline{\alpha}} = 2\,.$$

Existence and regularity result

Main tools. Introduction of higher-order energy functional

$$\mathscr{E}_1\big(\psi^{(a)}(t)\big) = \big\|\nabla\partial_{tt}\psi^{(a)}(t)\big\|_{L_2}^2 + \beta_2^{(a)}(\sigma_0)\,\big\|\nabla\Delta\partial_t\psi^{(a)}(t)\big\|_{L_2}^2 + \big\|\Delta\partial_t\psi^{(a)}(t)\big\|_{L_2}^2 .$$

Derivation of a priori bound of form

$$\sup_{t\in[0,T]} \mathscr{E}_1\big(\psi^{(a)}(t)\big) + \int_0^T \big\|\Delta\partial_{tt}\psi^{(a)}(t)\big\|_{L_2}^2 \,\mathrm{d}t \leq C .$$

Application of fixed point theorem by Schauder (weak formulation).

Remark. Second term in energy functional associated with Bochner–Sobolev space

$$W_\infty^1\big([0,T], H^3(\Omega)\big) .$$

Due to fact that $\beta_2^{(a)}(\sigma_0) \to 0$ as $a \to 0_+$, only convergence in weaker sense

$$\psi^{(a)} \overset{*}{\rightharpoonup} \psi^{(0)} \text{ in } H^2\big([0,T], H^2(\Omega)\big)$$

can be achieved.

Justification of limiting systems

Additional assumption. In above situation, assume in addition that prescribed initial data satisfy consistency condition

$$\psi_2 - \beta_1^{(0)}\,\Delta\psi_1 - \beta_3\,\Delta\psi_0 + \beta_5(\sigma)\,\psi_2\,\psi_1 + 2\,\beta_6(\sigma)\,\nabla\psi_1\cdot\nabla\psi_0 = 0\,.$$

For any $a \in (0,\overline{a}]$, let $\psi^{(a)} : [0,T] \to L_2(\Omega)$ denote solution to nonlinear damped wave equation or of reformulation obtained by integration

$$\begin{aligned}
&\partial_{tt}\psi^{(a)}(t) - \beta_1^{(0)}\,\Delta\partial_t\psi^{(a)}(t) - \big(\beta_1^{(a)} - \beta_1^{(0)}\big)\big(\Delta\partial_t\psi^{(a)}(t) - \Delta\psi_1\big)\\
&\quad + \beta_2^{(a)}(\sigma_0)\big(\Delta^2\psi^{(a)}(t) - \Delta^2\psi_0\big) - \beta_3\,\Delta\psi^{(a)}(t) + \beta_4^{(a)}(\sigma_0)\int_0^t \Delta^2\psi^{(a)}(\tau)\,\mathrm{d}\tau\\
&\quad + \beta_5(\sigma)\,\partial_{tt}\psi^{(a)}(t)\,\partial_t\psi^{(a)}(t) + 2\,\beta_6(\sigma)\,\nabla\partial_t\psi^{(a)}(t)\cdot\nabla\psi^{(a)}(t) = 0\,.
\end{aligned}$$

Justification of limiting systems

Theorem

Under the above assumptions, as $a \to 0_+$, the family $(\psi^{(a)})_{a\in(0,\overline{a}]}$ converges to the solution $\psi^{(0)} : [0,T] \to L_2(\Omega)$ of the limiting system

$$\begin{aligned}&\partial_{tt}\psi^{(0)}(t) - \beta_1^{(0)}\,\Delta\partial_t\psi^{(0)}(t) - \beta_3\,\Delta\psi^{(0)}(t)\\&\quad + \beta_5(\sigma)\,\partial_{tt}\psi^{(0)}(t)\,\partial_t\psi^{(0)}(t) + 2\,\beta_6(\sigma)\,\nabla\partial_t\psi^{(0)}(t)\cdot\nabla\psi^{(0)}(t) = 0\,.\end{aligned}$$

More precisely, for the solution to the associated weak formulation, obtained by testing with $v \in L_1([0,T], H_0^1(\Omega))$ and performing integration-by-parts, convergence is ensured in the following sense

$$\psi^{(a)} \overset{*}{\rightharpoonup} \psi^{(0)} \text{ in } X_0 \text{ as } a \to 0_+\,.$$

Conclusions and future work

Summary.

- Rigorous justification of Kuznetsov and Westervelt equations as limiting systems.
- Efficient time integration of Westervelt equation by operator splitting methods.
  Rigorous stability and error analysis for Lie–Trotter splitting method.

Relevant open questions.

- Numerical methods for more involved models arising in nonlinear acoustics.
- Application of higher-order splitting methods involving complex coefficients.
- Reliable and efficient time integration based on adaptive time stepsize control.

Thank you!