Novel approaches for the reliable and efficient numerical evaluation of Landau-type operators

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Main objectives

Evaluation of Landau operators (general integral kernels).
Time integration of associated Landau equations.

Future objectives

Future objectives. The numerical simulation of kinetic problems such as Vlasov–(Maxwell–)Landau-type equations remains a challenge, due to the structural complexity of the underlying partial differential equations and the related high computational effort.

- Our generally applicable and expedient approach relies on the application of operator splitting methods.
- Reliable and efficient solvers for Landau-type equations represent fundamental components of the entire algorithms.

Future objectives

Vlasov–Landau-type equation. Relevant models of plasma physics are given by inhomogeneous Vlasov–Landau-type equations

$$\partial_t f + v \cdot \nabla_x f - F \cdot \nabla_v f = Q(f, f)\,.$$

Here, $(x, v, t) \in \Omega^{(x)} \times \Omega^{(v)} \times [t_0, T] \subseteq \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ describe position, velocity, and time, $f : \Omega^{(x)} \times \Omega^{(v)} \times [t_0, T] \to \mathbb{R}$ the distribution of charged particles, $F : \Omega^{(x)} \times \Omega^{(v)} \times [t_0, T] \to \mathbb{R}$ a (given or self-consistent) force field including electromagnetic effects, and $Q(f, f)$ the Landau-type operator capturing collisions between particles.

Current objective

Current objective. We focus on the study of the collision integral, that is, we design reliable and efficient methods for

- the numerical evaluation of Landau-type operators and
- the time evolution of spatially homogeneous Landau-type equations.

Manuscript (in revision). J. A. Carrillo, M. Th. *Novel approaches for the reliable and efficient numerical evaluation of Landau-type operators.*

Remark. For simplicity, we refer to Landau-type operators as Landau operators.

Current objective

In the lines of CARRILLO ET AL. (2020).

> *The Landau equation represents a fundamental kinetic equation. It describes the evolution of the distribution of charged particles in a collisional plasma where grazing collisions are predominant.*
>
> *Together with the Boltzmann equation, the Landau equation is considered to be one of the most important equations in kinetic theory. Relevant applications related to fusion reactors and the International Thermonuclear Experimental Reactor (ITER) project gave rise to a renewed interest, amongst others in the field of computational plasma physics.*
>
> *In the special Maxwellian molecules case, the equation is reduced to a sort of degenerate linear Fokker–Planck equation which preserves the same moments as the Landau equation.*
>
> *The physically relevant Coulomb case can be derived from the Boltzmann equation in the grazing collision limit when particles interact via Coulomb forces.*

Model problem

Starting point. We consider spatially homogeneous Landau equations

$$\partial_t f = \nabla_v \cdot Q^{(c)}(f,f).$$

The arising integral operators can be cast into the general form

$$Q^{(c)}(f,f)(v) = C \int_\Omega \varphi(v-w)\, P(v-w) \big(f(w) \nabla_v f(v) - \nabla_w f(w)\, f(v) \big)\, \mathrm{d}w,$$
$$P(u) = u^T u\, I - u\, u^T, \quad u \in \Omega \subseteq \mathbb{R}^d.$$

Kernels. We study integral kernels with (a single) strong singularity (at the origin) and allow for a coupling of all velocity directions, e.g.

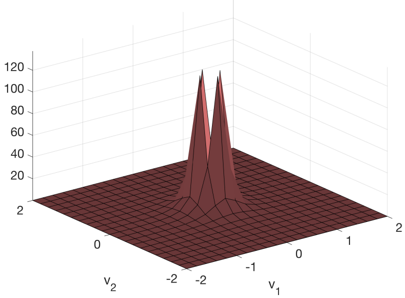
$$\varphi(u) = C\, |u|^\beta\, \mathrm{e}^{\gamma |u|}, \quad \beta, \gamma < 0, \quad u \in \mathbb{R}^d.$$

In the relevant special case of Coulomb interaction ($d = 3, \beta = -3, \gamma = 0$), simplifications apply to our approach, see Pareschi, Russo, Toscani (2000), Zhang, Gamba (2017).

Challenges

Computational issues. The numerical simulation of the physically most relevant case with Coulomb interaction requires

- computations in three dimensions (integrand involves $(v, w) \in \mathbb{R}^6$),
- a careful treatment of the singular integral kernel $\varphi(u) = C\,|u|^{-3}$.

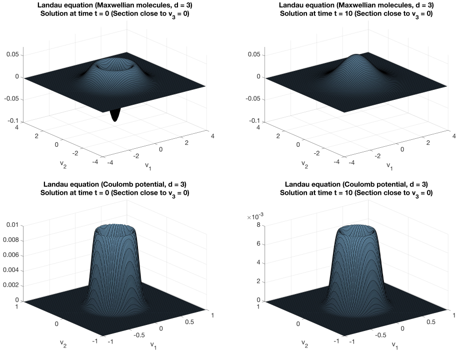


Integral kernel (intersection along $v_3 = 0$)

Guide line

Stepwise generalisation of the approach and validation of the implementation for test problems with known solutions.

- Constant integral kernel. Maxwellian molecules case with BKW solution on unbounded domain.
- Regular integral kernel. First test problem on bounded domain and second test problem with localised solution.
- Singular integral kernel.

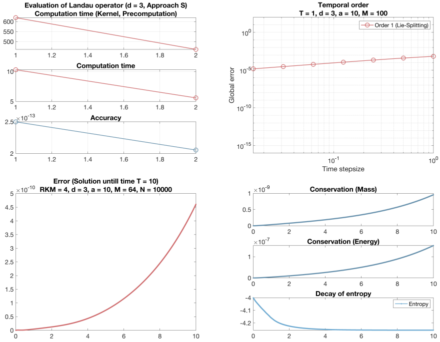


Sorry, certain technicalities will arise ...

Guide line

Focus on important aspects.

- Accuracy of results.
  Verification of approach on basis of several test problems.
- Efficiency of approach.
  Exploitation of savings by precomputations.
- Reliability in integration.
  Study of stability and conserved quantities.



Maxwellian molecules case (BKW solution)

Main strategies

First representation of Landau operator.
Fundamental means regarding implementation.

Our strategy

Our strategy. We consider Landau equations as nonlocal drift-diffusion equations and introduce numerical methods in the spirit of collocation (in view of future applications to kinetic equations)

$$Q(f,f) = \nabla_v \cdot Q^{(c)}(f,f), \quad Q^{(c)}(f,f)(v) = I(f)(v)\,\nabla_v f(v) + J(f)(v)\, f(v),$$

$$Q^{(c)}(f,f)(v) = C \int_\Omega \varphi(v-w)\, P(v-w)\, \big(f(w)\,\nabla_v f(v) - \nabla_w f(w)\, f(v)\big)\, \mathrm{d}w.$$

- We identify fundamental integrals involving the singular integral kernels $\varphi : \mathbb{R}^3 \to \mathbb{R}$, polynomials of degree two $p : \mathbb{R}^3 \to \mathbb{R}$, and regular functions $g : \mathbb{R}^3 \to \mathbb{R}$ reflecting the values of the density function or its derivatives

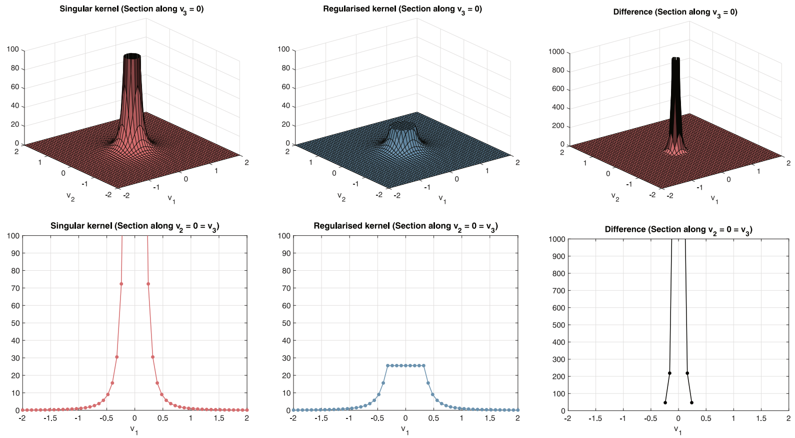
$$\int_{\mathbb{R}^3} \varphi(v-w)\, p(v-w)\, g(w)\, \mathrm{d}w.$$

Our strategy

- We use decompositions involving suitable regularisations of the kernels (obtained by interpolation nearby the singularity) such that the remaining difference vanishes on the main part of the velocity domain (see visualisation)

$$\begin{aligned}
&\int_{\mathbb{R}^3} \varphi(v-w)\, p(v-w)\, g(w)\, \mathrm{d}w \\
&\quad = \int_{\mathbb{R}^3} \psi(v-w)\, p(v-w)\, g(w)\, \mathrm{d}w \\
&\qquad + \int_{\mathbb{R}^3} (\varphi-\psi)(v-w)\, p(v-w)\, g(w)\, \mathrm{d}w\,.
\end{aligned}$$

Our strategy



For a kernel with isolated singularity at the origin, a regularised kernel is obtained by interpolation on a small neighbourhood of the origin. The remaining difference vanishes on the main part of the velocity domain.

Our strategy

- For the numerical computation of the integrals, we employ series expansions and quadrature approximations (for few grid points).
- We favour the Fourier spectral method

$$\sum_{m \in \mathcal{M}} g_m \mathcal{F}_m \approx g\,, \quad \sum_{m \in \mathcal{M}} \psi_m \mathcal{F}_m \approx \psi\,,$$

because of the particular properties of the Fourier functions (complex exponentials, multiplicativity)

$$\mathcal{F}_\kappa(\xi) = \frac{1}{\sqrt{2\,b}}\,\mathrm{e}^{\mu_\kappa(\xi+b)}\,, \quad \xi \in \mathbb{R}\,, \quad \kappa \in \mathbb{Z}\,,$$
$$\mathcal{F}_m(u) = \mathcal{F}_{m_1}(u_1)\,\mathcal{F}_{m_2}(u_2)\,\mathcal{F}_{m_3}(u_3)\,, \quad u \in \mathbb{R}^3\,, \quad m \in \mathbb{Z}^3\,,$$

and its highly efficient practical implementation by fast transforms.

- We recall that g represents f and $\nabla_v f(v)$. A novel aspect is the introduction and Fourier expansion of the regularised kernel ψ.

Our strategy

The numerical computation of the fundamental integrals relies on

- integrals with known explicit representations (1d, precomputation)

$$\int_{-b}^{b} \xi^i \, \mathscr{F}_\kappa(\xi) \, \mathrm{d}\xi \,, \quad i \in \{0,1,2\} \,, \quad \kappa \in \left\{-\tfrac{M}{2}, \dots, \tfrac{M}{2} - 1\right\},$$

- quadrature approximations of integrals (3d, precomputation)

$$\int_{\text{Small domain}} (\varphi - \psi)(w) \, p(w) \, \mathscr{F}_m(w) \, \mathrm{d}w \,, \quad m \in \left\{-\tfrac{M}{2}, \dots, \tfrac{M}{2} - 1\right\}^3,$$

- fast Fourier transforms, and summations along certain directions.

Computational effort

Computational effort. The numerical evaluation of the Landau operator in a substep of the time evolution relies on

- (inverse) Fourier transforms (computationally most elaborate components),
- summations, and pointwise multiplications (parallelisation).

Precomputation. Compared to a quadrature approximation on the whole domain, a suitable adjustment of the small neighbourhood of the singularity makes it possible to significantly reduce the precomputation time and effort for the same accuracy.

Numerical comparison. Test problem C (regular integral kernel, unbounded domain, known solution, two velocity dimensions). Numerical evaluation of the Landau operator based on 256×256 uniform grid points covering the truncated velocity domain $[-10, 10]^2$. Precomputation times observed for a quadrature approximation based on 5×5 grid points versus a quadrature approximation on the whole domain based on 256×256 grid points. In both cases, an overall relative accuracy of about $4 \cdot 10^{-11}$ is obtained.

Quadrature on a small neighbourhood	Precomputation time CT
Quadrature on the whole domain	88×CT

Alternative approaches

Approaches for the numerical evaluation of Landau collision operators.

Approach CST2
Based on the conservative formulation of the Landau operator.
Uses numerical differentiation of the integral operator.
Adapted to kernels with an isolated singularity at the origin.
The integral transform is applied to the singular kernel and its regularisation.

Approach CST1
Based on the conservative formulation of the Landau operator.
Uses numerical differentiation of the integral operator.
Adapted to kernels with an isolated singularity at the origin.
The integral transform is applied to the singular kernel.

Approach NST1
Based on the non-conservative formulation of the Landau operator.
Avoids numerical differentiation of the integral operator.
Adapted to kernels with an isolated singularity at the origin.
The integral transform is applied to the singular kernel and its derivatives.

C conservative formulation / N non-conservative formulation. S singular kernel / R simplifications for regular kernels.
T1 integral transform applied to first part / T2 integral transform applied to first and second part.

Some technical details ...

First representation

Approach. Specify the Landau operator in components and perform straightforward calculations to obtain a first representation involving derivatives of the density function and linear operators that comprise (quite a few) integrals

$$\big(Q(f,f)\big)(v) = \begin{pmatrix} \partial_{v_1} \\ \partial_{v_2} \\ \partial_{v_3} \end{pmatrix} \cdot \big(Q_c(f,f)\big)(v)\,, \quad v = (v_1, v_2, v_3) \in \Omega\,,$$

$$\begin{aligned}
&\big(Q_c(f,f)\big)(v) \\
&= \int_\Omega \varphi(v-w) \begin{pmatrix} (v_2-w_2)^2 + (v_3-w_3)^2 & -(v_1-w_1)(v_2-w_2) & -(v_1-w_1)(v_3-w_3) \\ -(v_1-w_1)(v_2-w_2) & (v_1-w_1)^2 + (v_3-w_3)^2 & -(v_2-w_2)(v_3-w_3) \\ -(v_1-w_1)(v_3-w_3) & -(v_2-w_2)(v_3-w_3) & (v_1-w_1)^2 + (v_2-w_2)^2 \end{pmatrix} \\
&\quad \times \begin{pmatrix} \partial_{v_1} f(v)\, f(w) - f(v)\, \partial_{w_1} f(w) \\ \partial_{v_2} f(v)\, f(w) - f(v)\, \partial_{w_2} f(w) \\ \partial_{v_3} f(v)\, f(w) - f(v)\, \partial_{w_3} f(w) \end{pmatrix} \mathrm{d}w \\
&= \big(Q_{000}(f)\big)(v)\, f(v) + \big(Q_{100}(f)\big)(v)\, \partial_{v_1} f(v) + \ldots
\end{aligned}$$

Fundamental means

Main task. Compute suitable approximations to the arising derivatives and the decisive integrals, e.g.

$$\partial_{v_1} f(v)\,, \quad I_{w000\varphi000f000}(v) = \int_\Omega \varphi(v-w)\, f(w)\, \mathrm{d}w\,, \quad v \in \Omega\,.$$

Approach.

- Application of Fourier spectral method
- Quadrature approximation nearby singularity of kernel
- Identification of basic integrals involving Fourier functions

Implementation.

- Use of fast Fourier techniques (FFT / IFFT)
- Summation along certain directions (*einsum*)
- Observation of reduced computational complexity

Fourier spectral method

Fourier functions. Consider well-known Fourier functions

$$\mathscr{F}_\kappa^{(\alpha)}(\xi) = \frac{1}{\sqrt{\alpha_2-\alpha_1}}\, \mathrm{e}^{\mu_\kappa^{(\alpha)}(\xi-\alpha_1)}, \quad \mu_\kappa^{(\alpha)} = \frac{2\pi\mathrm{i}\kappa}{\alpha_2-\alpha_1},$$

$$\mathscr{F}_\kappa^{(\alpha)}(\alpha_1) = \frac{1}{\sqrt{\alpha_2-\alpha_1}} = \mathscr{F}_\kappa^{(\alpha)}(\alpha_2), \quad \partial_\xi \mathscr{F}_\kappa^{(\alpha)}(\xi) = \mu_\kappa^{(\alpha)}\, \mathscr{F}_\kappa^{(\alpha)}(\xi),$$

$$\xi \in \mathbb{R}, \quad \kappa \in \mathbb{Z}, \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2, \quad \alpha_1 < \alpha_2.$$

In particular, in three dimensions, denote

$$F_m(v) = \mathscr{F}_{m_1}^{(b_{11},b_{12})}(v_1)\, \mathscr{F}_{m_2}^{(b_{21},b_{22})}(v_2)\, \mathscr{F}_{m_3}^{(b_{31},b_{32})}(v_3),$$

$$v = (v_1, v_2, v_3) \in \Omega_{\mathscr{F}}, \quad m = (m_1, m_2, m_3) \in \mathscr{M},$$

$$\Omega_{\mathscr{F}} = [b_{11}, b_{12}] \times [b_{21}, b_{22}] \times [b_{31}, b_{32}],$$

$$\mathscr{M} = \{-\tfrac{1}{2}\, M_1, \dots, \tfrac{1}{2}\, M_1 - 1\} \times \cdots \times \{-\tfrac{1}{2}\, M_3, \dots, \tfrac{1}{2}\, M_3 - 1\} \subset \mathbb{Z}^3.$$

Fourier spectral method

General benefits.

- High accuracy for localised regular functions
- High efficiency (implementation based on FFT / IFFT)

Fourier spectral method

Within our setting. Recall the typical form of the decisive integrals

$$I_{w000\varphi000f000}(v) = \int_{\Omega} \varphi(v-w)\, f(w)\, \mathrm{d}w\,, \quad v \in \Omega\,.$$

Specific advantage. The particular identity

$$F_{\ell}(v-w)\, F_m(w) = \Gamma_{\ell}^{(b)}\, F_{\ell}(v)\, F_{m-\ell}(w)\,,$$

$$\Gamma_{\ell}^{(b)} = \mathrm{e}^{-\mu_{\ell_1}^{(b_{11},b_{12})}\, b_{11}}\, \mathrm{e}^{-\mu_{\ell_2}^{(b_{21},b_{22})}\, b_{21}}\, \mathrm{e}^{-\mu_{\ell_3}^{(b_{31},b_{32})}\, b_{31}}\,,$$

suggests to study Fourier series expansions for both, the density function and the (regularised) integral kernel.

Fourier spectral method

Fourier series expansions. Use approximations based on Fourier series expansions for the integral kernel and its (known) derivatives

$$\varphi(v) \approx \sum_{\ell \in \mathscr{L}} \varphi_\ell \, F_\ell(v) \,,$$

$$\partial_{v_1}^{j_1} \partial_{v_2}^{j_2} \partial_{v_3}^{j_3} \varphi(v) \approx \sum_{\ell \in \mathscr{L}} \varphi_{j_1 j_2 j_3 \ell} \, F_\ell(v) \,.$$

Employ a Fourier series expansion for the density function and the resulting representations for its derivatives

$$f(v) \approx \sum_{m \in \mathscr{M}} f_m \, F_m(v) \,,$$

$$\partial_{v_1}^{k_1} \partial_{v_2}^{k_2} \partial_{v_3}^{k_3} f(v) \approx \sum_{m \in \mathscr{M}} f_m \left(\mu_{m_1}^{(b_{11}, b_{12})}\right)^{k_1} \left(\mu_{m_2}^{(b_{21}, b_{22})}\right)^{k_2} \left(\mu_{m_3}^{(b_{31}, b_{32})}\right)^{k_3} F_m(v) \,.$$

Basic integrals

Calculations. Straightforward calculations yield explicit representations for basic integrals involving monomials and Fourier functions

$$I_j(k,m_j) = \int_{b_{j1}}^{b_{j2}} w_j^k \, \mathscr{F}_{m_j}^{(b_{j1},b_{j2})}(w_j)\,\mathrm{d}w_j = \begin{cases} \sqrt{b_{j2}-b_{j1}}\,, & m_j=0, \quad k=0, \\ \dfrac{b_{j2}^2-b_{j1}^2}{2\sqrt{b_{j2}-b_{j1}}}\,, & m_j=0, \quad k=1, \\ \dfrac{b_{j2}^3-b_{j1}^3}{3\sqrt{b_{j2}-b_{j1}}}\,, & m_j=0, \quad k=2, \\ 0\,, & m_j\neq 0, \quad k=0, \\ \dfrac{\sqrt{b_{j2}-b_{j1}}}{\mu_{m_j}^{(b_{j1},b_{j2})}}\,, & m_j\neq 0, \quad k=1, \\ \dfrac{(b_{j2}^2-b_{j1}^2)\,\mu_{m_j}^{(b_{j1},b_{j2})}-2\,(b_{j2}-b_{j1})}{\sqrt{b_{j2}-b_{j1}}\,(\mu_{m_j}^{(b_{j1},b_{j2})})^2}\,, & m_j\neq 0, \quad k=2, \end{cases} \qquad j\in\{1,\dots,d\}.$$

Reductions. The observed simplification leads to significant reductions in computational complexity for three dimensions.

A few more (surprising) details ...

Implementation and observation. Essential ingredients for the efficient implementation are

- matrix multiplications,
- summations along certain directions (*einsum*),
- inverse fast Fourier transforms.

A few more (surprising) details ...

Reduced computational complexity (Arising inner sums). A crucial observation is that the costs for the computation of inner sums amount to the evaluation of single and double sums, but no triple sums.

Type of decisive integral	# Summations along directions (*einsum*)
w 000	0
w 100, w 010, w 001	1
w 200, w 020, w 002	1
w 110, w 101, w 011	2

A few more (surprising) details ...

Reduced computational complexity (quadrature approximation on small domain). Test problem (regular integral kernel, unbounded domain, known solution) in two dimensions. Numerical evaluation of the Landau operator based on 256×256 uniform grid points covering the truncated velocity domain $[-10, 10] \times [-10, 10]$. Precomputation times observed for a quadrature approximation based on 5×5 grid points versus a quadrature approximation on the whole domain based on 256×256 grid points. In both cases, an overall relative accuracy of about $4 \cdot 10^{-11}$ is obtained.

Quadrature on a small neighbourhood	Precomputation time CT
Quadrature on the whole domain	88×CT

Arising operators

A glance at the operators arising in the final representation of the Landau operator ...

$$
\begin{aligned}
\big(Q_{000}(f)\big)(v) = &-v_1^2\big(I_{w000\,\varphi010\,f010}(v)+I_{w000\,\varphi001\,f001}(v)\big)+v_1\,v_2\big(I_{w000\,\varphi010\,f100}(v)+I_{w000\,\varphi100\,f010}(v)\big)\\
&+v_1\,v_3\big(I_{w000\,\varphi001\,f100}(v)+I_{w000\,\varphi100\,f001}(v)\big)-v_2^2\big(I_{w000\,\varphi100\,f100}(v)+I_{w000\,\varphi001\,f001}(v)\big)\\
&+v_2\,v_3\big(I_{w000\,\varphi001\,f010}(v)+I_{w000\,\varphi010\,f001}(v)\big)-v_3^2\big(I_{w000\,\varphi100\,f100}(v)+I_{w000\,\varphi010\,f010}(v)\big)\\
&+v_1\big(2\,I_{w000\,\varphi000\,f100}(v)-I_{w010\,\varphi010\,f100}(v)-I_{w001\,\varphi001\,f100}(v)-I_{w010\,\varphi100\,f010}(v)+2\,I_{w100\,\varphi010\,f010}(v)\\
&-I_{w001\,\varphi100\,f001}(v)+2\,I_{w100\,\varphi001\,f001}(v)\big)\\
&+v_2\big(2\,I_{w010\,\varphi100\,f100}(v)-I_{w100\,\varphi010\,f100}(v)+2\,I_{w000\,\varphi000\,f010}(v)-I_{w100\,\varphi100\,f010}(v)-I_{w001\,\varphi001\,f010}(v)\\
&-I_{w001\,\varphi010\,f001}(v)+2\,I_{w010\,\varphi001\,f001}(v)\big)\\
&+v_3\big(2\,I_{w001\,\varphi100\,f100}(v)-I_{w100\,\varphi001\,f100}(v)+2\,I_{w001\,\varphi010\,f010}(v)-I_{w010\,\varphi001\,f010}(v)+2\,I_{w000\,\varphi000\,f001}(v)\\
&-I_{w100\,\varphi100\,f001}(v)-I_{w010\,\varphi010\,f001}(v)\big)\\
&-2\,I_{w100\,\varphi000\,f100}(v)-I_{w020\,\varphi100\,f100}(v)-I_{w002\,\varphi100\,f100}(v)+I_{w110\,\varphi010\,f100}(v)+I_{w101\,\varphi001\,f100}(v)\\
&-2\,I_{w010\,\varphi000\,f010}(v)+I_{w110\,\varphi100\,f010}(v)-I_{w200\,\varphi010\,f010}(v)-I_{w002\,\varphi010\,f010}(v)+I_{w011\,\varphi001\,f010}(v)\\
&-2\,I_{w001\,\varphi000\,f001}(v)+I_{w101\,\varphi100\,f001}(v)+I_{w011\,\varphi010\,f001}(v)-I_{w200\,\varphi001\,f001}(v)-I_{w020\,\varphi001\,f001}(v).
\end{aligned}
$$

... The numerical verification of the correctness of all terms on test problems with known solutions seemed to be a good idea ... ;-)

Constant and singular integral kernels

Detailed study of Maxwellian molecules case.
BKW solution on unbounded domain.
Conclusions for generalised Coulomb case.

Test problems

Test problems. Study of the different approaches and validation of the implementation by means of test problems with constant, regular, and singular kernels.

Maxwellian molecules case (BWK solution)

Landau equation. The BKW solution to the Landau equation involving a constant kernel is given by

$$\varphi(v) = C = \tfrac{1}{24}\,, \quad (\alpha_1, \alpha_2, \alpha_3) = \left(\tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{6}\right), \quad K(t) = 1 - \tfrac{1}{2}\,\mathrm{e}^{-\alpha_3 t}\,,$$

$$f(v,t) = \frac{1}{(2\pi K(t))^{3/2}}\,\mathrm{e}^{-\frac{1}{2}\frac{1}{K(t)}|v|^2}\left(\alpha_1 - \alpha_2 \frac{1}{K(t)} + \frac{1}{2}\frac{1-K(t)}{(K(t))^2}|v|^2\right),$$

$$\partial_t f(v,t) = \big(Q(f,f)\big)(v,t)\,, \quad (v,t) \in \Omega \times [t_0, T]\,.$$

Landau operator. For a particular choice of the density function, the associated Landau operator reads as

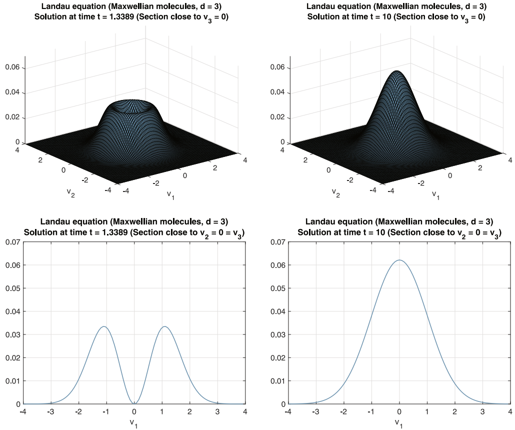
$$f(v) = \frac{1}{2\pi^{3/2}}\,\mathrm{e}^{-|v|^2}\left(2|v|^2 - 1\right),$$

$$\big(Q(f,f)\big)(v) = \frac{1}{6\pi^{3/2}}\,\mathrm{e}^{-|v|^2}\left(|v|^4 - 5|v|^2 + \frac{15}{4}\right), \quad v \in \Omega\,.$$

**An ideal situation to verify our approach
and perform first numerical tests. :-)**

Test problem (Constant kernel)

Maxwellian molecules case. Numerical illustration of the BKW solution to the Landau equation with constant kernel (see also CARRILLO ET AL. (2020)).



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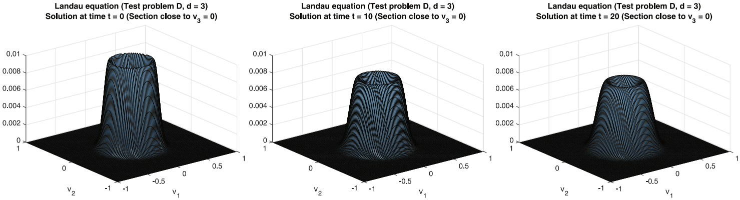
Test problem (General singular kernels)

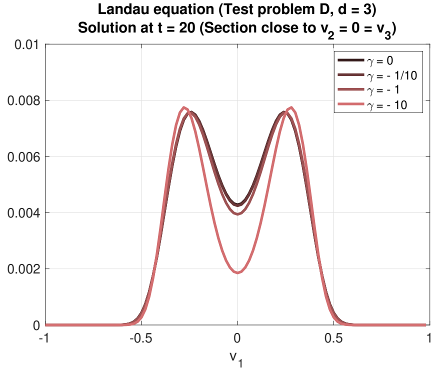
Coulomb case and generalisations. Time evolution of Landau equations with Coulomb interaction ($\gamma = 0$) and related singular kernels

$$\varphi(u) = C\,|u|^{\beta}\,\mathrm{e}^{\gamma\,|u|}\,, \quad \beta = -3\,, \quad \gamma < 0\,, \quad u \in \mathbb{R}^3\,.$$



https://techmath.uibk.ac.at/mecht/MyHomepage/Research/Movie_CoulombPotential_Solution3d.m4v

Test problem (General singular kernels)



Comparison of the solution profiles for different exponents $\gamma \in \{0, -\frac{1}{10}, -1, -10\}$.

Conclusions

Summary. Study of flexible novel approaches for the reliable and efficient evaluation of Landau collision operators with general singular kernels.

- Significant improvement in accuracy (exponential convergence of Fourier spectral method on main part of velocity domain).
- Significant reduction of computational effort (outsourcing to precomputations, time evolution based on FFT / IFFT).

Open tasks and future objectives.

- Specification of the spatial and temporal discretisation errors.
- Generalisation of the spectral method (e.g. Hermite functions).
- Extensions to Vlasov–(Maxwell–)Landau equations by exponential operator splitting methods.

Thank you!