Novel approaches for the reliable and efficient numerical evaluation of Landau-type operators

Mechthild Thalhammer
Leopold–Franzens-Universität Innsbruck, Austria

Hyperbolic Problems: A Comprehensive Approach
Würzburg, Germany, March 2026

Future objectives

Future objectives. The numerical simulation of kinetic problems such as Vlasov–(Maxwell–)Landau-type equations remains a challenge, due to the structural complexity of the underlying partial differential equations and the related high computational effort.

- Our generally applicable and expedient approach relies on the application of operator splitting methods.
- Reliable and efficient solvers for Landau-type equations represent fundamental components of the entire algorithms.

Vlasov–Landau-type equation. Relevant models of plasma physics are given by inhomogeneous Vlasov–Landau-type equations

$$\partial_t f + v \cdot \nabla_x f - F \cdot \nabla_v f = Q(f, f)\,.$$

Here, $(x, v, t) \in \Omega^{(x)} \times \Omega^{(v)} \times [t_0, T] \subseteq \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ describe position, velocity, and time, $f : \Omega^{(x)} \times \Omega^{(v)} \times [t_0, T] \to \mathbb{R}$ the distribution of charged particles, $F : \Omega^{(x)} \times \Omega^{(v)} \times [t_0, T] \to \mathbb{R}$ a (given or self-consistent) force field including electromagnetic effects, and $Q(f, f)$ the Landau-type operator capturing collisions between particles.

Current objective

For simplicity, we refer to Landau-type operators as Landau operators.

Current objective. We focus on the study of the collision integral, that is, we design reliable and efficient methods for

- the numerical evaluation of Landau operators and
- the time evolution of spatially homogeneous Landau equations.

Manuscript (in revision). J. A. CARRILLO, M. TH. *Novel approaches for the reliable and efficient numerical evaluation of Landau-type operators.*

Model problem

Starting point. We consider spatially homogeneous Landau equations

$$\partial_t f = \nabla_v \cdot Q^{(c)}(f,f)\,.$$

The arising integral operators can be cast into the general form

$$Q^{(c)}(f,f)(v) = C\int_\Omega \varphi(v-w)\,P(v-w)\left(f(w)\,\nabla_v f(v) - \nabla_w f(w)\,f(v)\right)\mathrm{d}w\,,$$
$$P(u) = u^T u\,I - u\,u^T\,,\quad u\in\Omega\subseteq\mathbb{R}^d\,.$$

Kernels. We study integral kernels with (a single) strong singularity (at the origin) and allow for a coupling of all velocity directions, e.g.

$$\varphi(u) = C\,|u|^\beta\,\mathrm{e}^{\gamma|u|}\,,\quad \beta,\gamma<0\,,\quad u\in\mathbb{R}^d\,.$$

In the relevant special case of Coulomb interaction ($d=3,\beta=-3,\gamma=0$), simplifications apply to our approach, see Pareschi, Russo, Toscani (2000), Zhang, Gamba (2017).

Our strategy

Our strategy. We consider Landau equations as nonlocal drift-diffusion equations and introduce numerical methods in the spirit of collocation (in view of future applications to kinetic equations)

$$Q(f,f) = \nabla_v \cdot Q^{(c)}(f,f), \quad Q^{(c)}(f,f)(v) = I(f)(v)\,\nabla_v f(v) + J(f)(v)\,f(v),$$

$$Q^{(c)}(f,f)(v) = C \int_\Omega \varphi(v-w)\,P(v-w)\,\big(f(w)\,\nabla_v f(v) - \nabla_w f(w)\,f(v)\big)\,\mathrm{d}w.$$

- We identify fundamental integrals involving the singular integral kernels $\varphi : \mathbb{R}^3 \to \mathbb{R}$, polynomials of degree two $p : \mathbb{R}^3 \to \mathbb{R}$, and regular functions $g : \mathbb{R}^3 \to \mathbb{R}$ reflecting the values of the density function or its derivatives

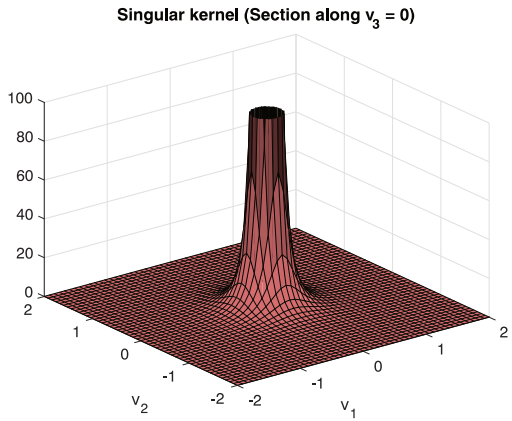
$$\int_{\mathbb{R}^3} \varphi(v-w)\,p(v-w)\,g(w)\,\mathrm{d}w.$$

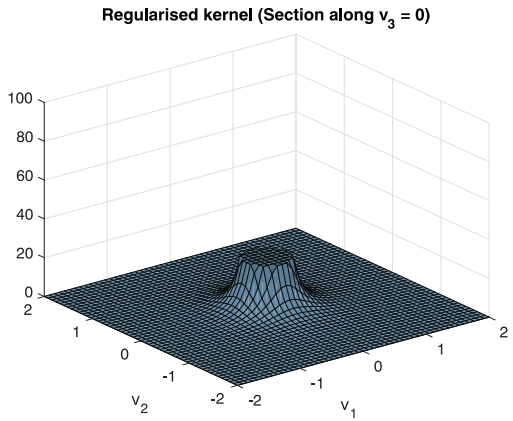
Our strategy

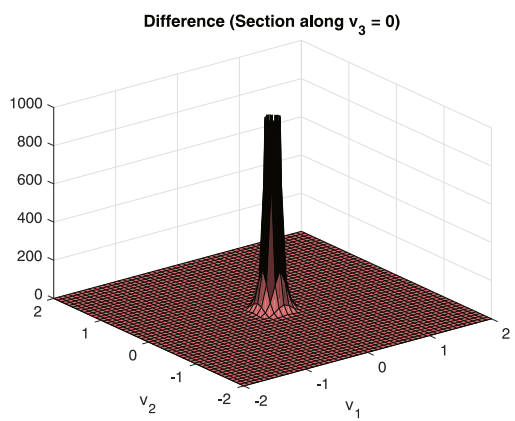
- We use decompositions involving suitable regularisations of the kernels (obtained by interpolation nearby the singularity) such that the remaining difference vanishes on the main part of the velocity domain (see visualisation)

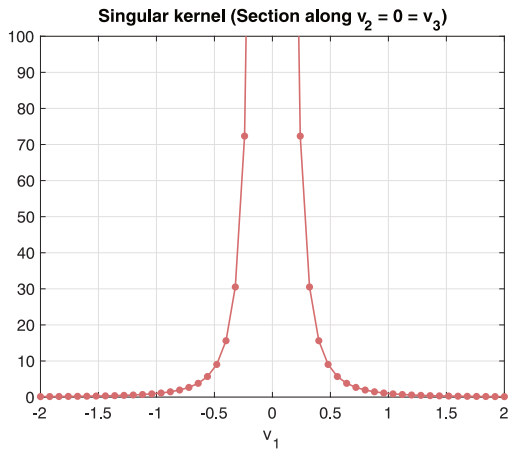
$$\begin{aligned}
&\int_{\mathbb{R}^3} \varphi(v-w)\, p(v-w)\, g(w)\, \mathrm{d}w \\
&\quad = \int_{\mathbb{R}^3} \psi(v-w)\, p(v-w)\, g(w)\, \mathrm{d}w \\
&\qquad + \int_{\mathbb{R}^3} (\varphi-\psi)(v-w)\, p(v-w)\, g(w)\, \mathrm{d}w .
\end{aligned}$$

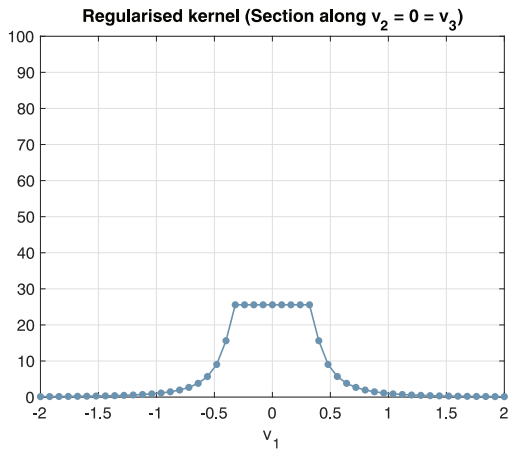
Our strategy

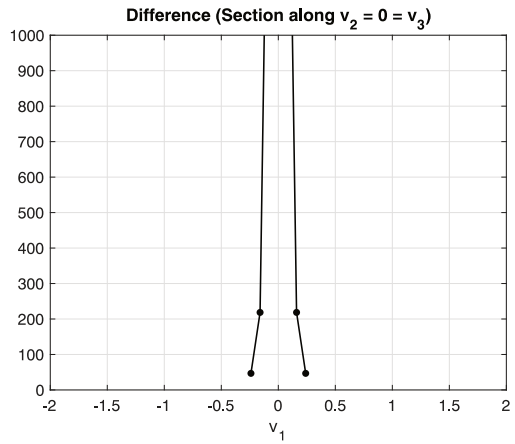












For a kernel with isolated singularity at the origin, a regularised kernel is obtained by interpolation on a small neighbourhood of the origin. The remaining difference vanishes on the main part of the velocity domain.

Our strategy

- For the numerical computation of the integrals, we employ series expansions and quadrature approximations (for few grid points).
- We favour the Fourier spectral method

$$\sum_{m\in\mathcal{M}} g_m \mathscr{F}_m \approx g\,, \qquad \sum_{m\in\mathcal{M}} \psi_m \mathscr{F}_m \approx \psi\,,$$

because of the particular properties of the Fourier functions (complex exponentials, multiplicativity)

$$\mathscr{F}_\kappa(\xi) = \frac{1}{\sqrt{2\,b}}\,\mathrm{e}^{\mu_\kappa(\xi+b)}\,, \quad \xi\in\mathbb{R}, \quad \kappa\in\mathbb{Z}\,,$$
$$\mathscr{F}_m(u) = \mathscr{F}_{m_1}(u_1)\,\mathscr{F}_{m_2}(u_2)\,\mathscr{F}_{m_3}(u_3)\,, \quad u\in\mathbb{R}^3, \quad m\in\mathbb{Z}^3\,,$$

and its highly efficient practical implementation by fast transforms.

- We recall that g represents f and $\nabla_v f(v)$. A novel aspect is the introduction and Fourier expansion of the regularised kernel ψ.

Our strategy

The numerical computation of the fundamental integrals relies on

- integrals with known explicit representations (1d, precomputation)

$$\int_{-b}^{b} \xi^i \, \mathscr{F}_\kappa(\xi) \, \mathrm{d}\xi, \quad i \in \{0,1,2\}, \quad \kappa \in \left\{-\tfrac{M}{2}, \ldots, \tfrac{M}{2}-1\right\},$$

- quadrature approximations of integrals (3d, precomputation)

$$\int_{\text{Small domain}} (\varphi - \psi)(w) \, p(w) \, \mathscr{F}_m(w) \, \mathrm{d}w, \quad m \in \left\{-\tfrac{M}{2}, \ldots, \tfrac{M}{2}-1\right\}^3,$$

- fast Fourier transforms, and summations along certain directions.

Computational effort

Computational effort. The numerical evaluation of the Landau operator in a substep of the time evolution relies on

- (inverse) Fourier transforms (computationally most elaborate components),
- summations, and pointwise multiplications (parallelisation).

Precomputation. Compared to a quadrature approximation on the whole domain, a suitable adjustment of the small neighbourhood of the singularity makes it possible to significantly reduce the precomputation time and effort for the same accuracy.

Numerical comparison. Test problem C (regular integral kernel, unbounded domain, known solution, two velocity dimensions). Numerical evaluation of the Landau operator based on 256×256 uniform grid points covering the truncated velocity domain $[-10, 10]^2$. Precomputation times observed for a quadrature approximation based on 5×5 grid points versus a quadrature approximation on the whole domain based on 256×256 grid points. In both cases, an overall relative accuracy of about $4 \cdot 10^{-11}$ is obtained.

Quadrature on a small neighbourhood	Precomputation time CT
Quadrature on the whole domain	88×CT

Alternative approaches

Approaches for the numerical evaluation of Landau collision operators.

Approach CST2
Based on the conservative formulation of the Landau operator.
Uses numerical differentiation of the integral operator.
Adapted to kernels with an isolated singularity at the origin.
The integral transform is applied to the singular kernel and its regularisation.

Approach CST1
Based on the conservative formulation of the Landau operator.
Uses numerical differentiation of the integral operator.
Adapted to kernels with an isolated singularity at the origin.
The integral transform is applied to the singular kernel.

Approach NST1
Based on the non-conservative formulation of the Landau operator.
Avoids numerical differentiation of the integral operator.
Adapted to kernels with an isolated singularity at the origin.
The integral transform is applied to the singular kernel and its derivatives.

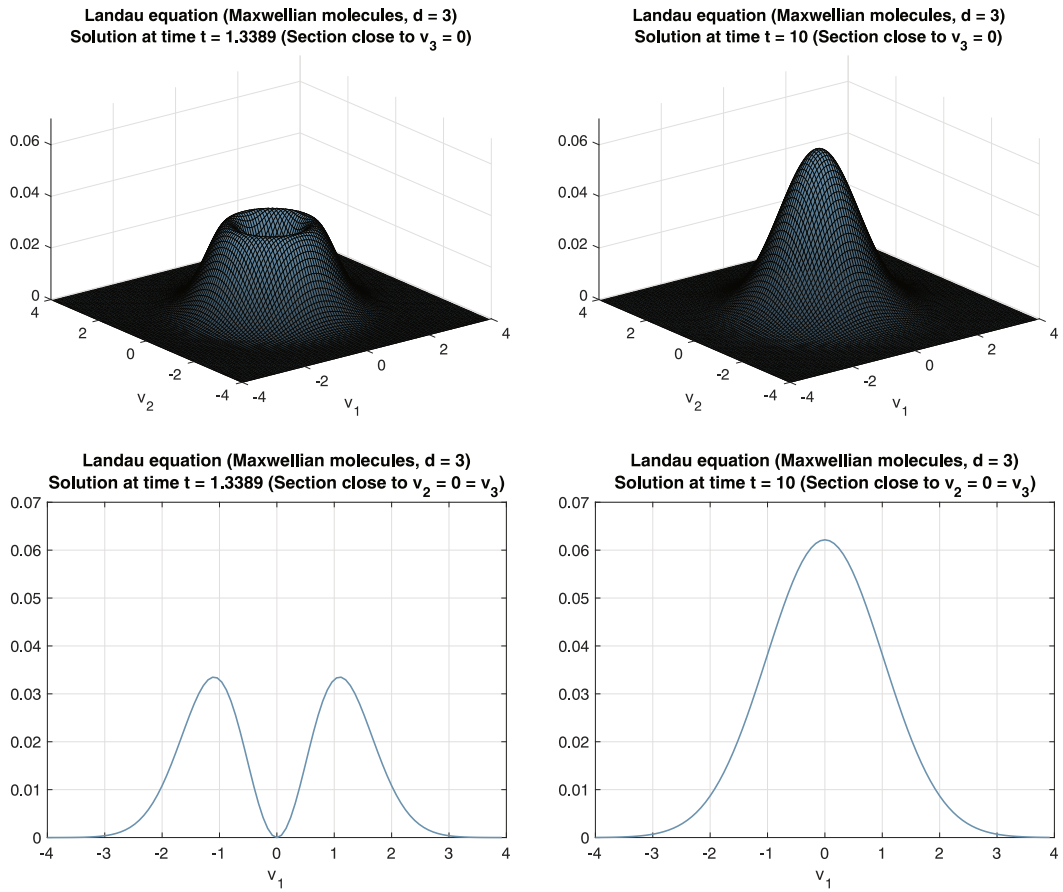
C conservative formulation / N non-conservative formulation. S singular kernel / R simplifications for regular kernels.
T1 integral transform applied to first part / T2 integral transform applied to first and second part.

Test problems

Test problems. Study of the different approaches and validation of the implementation by means of test problems with constant, regular, and singular kernels.

Test problem (Constant kernel)

Maxwellian molecules case. Numerical illustration of the BKW solution to the Landau equation with constant kernel (see also CARRILLO ET AL. (2020)).

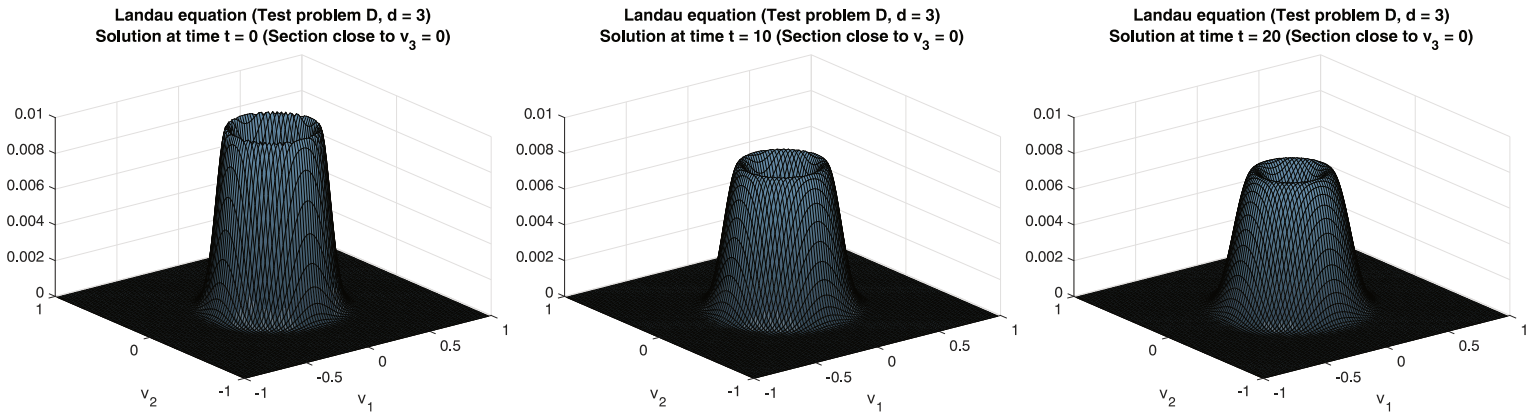


https://techmath.uibk.ac.at/mecht/MyHomepage/Research/Movie_Maxwellian_Solution3d.m4v
https://techmath.uibk.ac.at/mecht/MyHomepage/Research/Movie_Maxwellian_Solution3dSection.m4v

Test problem (General singular kernels)

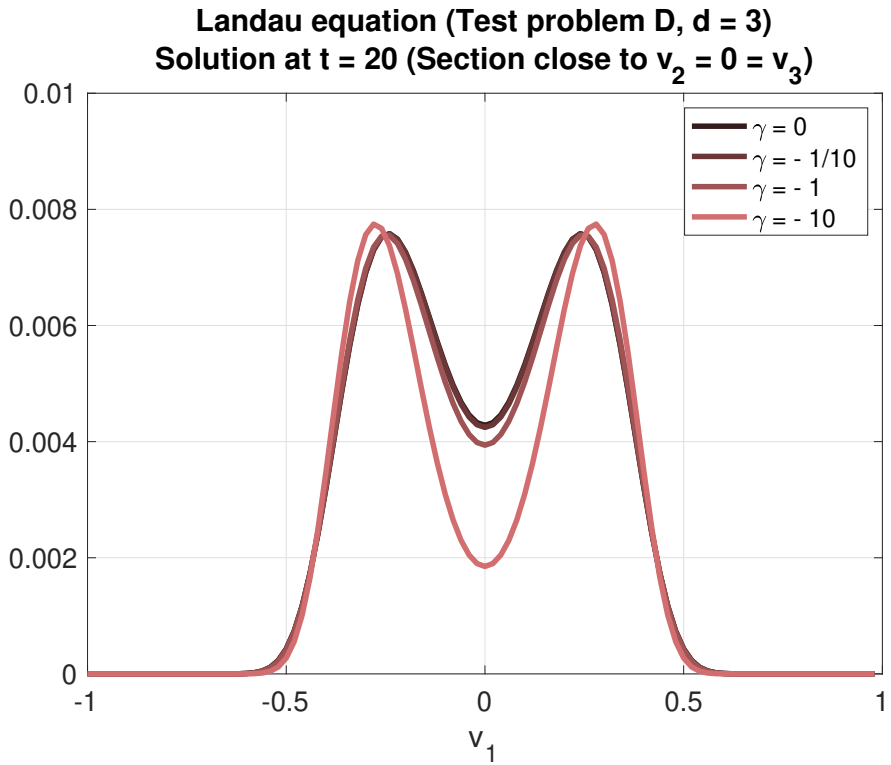
Coulomb case and generalisations. Time evolution of Landau equations with Coulomb interaction ($\gamma = 0$) and related singular kernels

$$\varphi(u) = C\,|u|^{\beta}\,\mathrm{e}^{\gamma|u|}, \quad \beta = -3, \quad \gamma < 0, \quad u \in \mathbb{R}^3.$$



https://techmath.uibk.ac.at/mecht/MyHomepage/Research/Movie_CoulombPotential_Solution3d.m4v

Test problem (General singular kernels)



Comparison of the solution profiles for different exponents $\gamma \in \{0, -\frac{1}{10}, -1, -10\}$.

Conclusions

Summary. Study of flexible novel approaches for the reliable and efficient evaluation of Landau collision operators with general singular kernels.

- Significant improvement in accuracy (exponential convergence of Fourier spectral method on main part of velocity domain).
- Significant reduction of computational effort (outsourcing to precomputations, time evolution based on FFT / IFFT).

Open tasks and future objectives.

- Specification of the spatial and temporal discretisation errors.
- Generalisation of the spectral method (e.g. Hermite functions).
- Extensions to Vlasov–(Maxwell–)Landau equations by exponential operator splitting methods.

Thank you!