Stochastic Fourier Integral Operators for Damage Detection

Lukas Lamplmayr, Michael Oberguggenberger, Martin Schwarz

Unit of Engineering Mathematics,
University of Innsbruck,
{lukas.lamplmayr, michael.oberguggenberger, martin.schwarz}@uibk.ac.at

Abstract. This note reports on progress in a research project which we announced at IPW 2014. The research goal is to model linear wave propagation in random media by means of stochastic Fourier integral operators. We show that the case of plane waves can be completely treated by this method and can be used for damage detection using ultrasonic measurements. The Fourier integral operator method (including stochastic perturbations) is also shown to be applicable in the three dimensional linear elastic, homogeneous, isotropic case and in modelling Rayleigh surface waves. An in-house adaptation of a fast algorithm for computing multidimensional Fourier integral operators is discussed.

1 Introduction

The purpose of this note is to report on progress in a research project which we announced at IPW 2014 [4]. The novel research goal is to model linear wave propagation in random media by means of stochastic Fourier integral operators. The applicative targets are reliability analysis, damage detection, system identification, and calibration of models in the presence of randomly perturbed structures in elasticity and strength of materials. The set-up applies to wave propagation in random media, i.e., materials with stochastically varying properties.

The dynamic response of an elastic medium is described by the equilibrium equations, a system of hyperbolic differential equations. Randomness of the medium causes the coefficients of the equations to become random fields. The direct, bottom-up approach would start by modelling the coefficients as random fields, solving the equations, and inferring stochastic properties of the solution. However, the direct approach faces the difficulty that the realizations of the random fields commonly in use in elastostatics do not have the degree of smoothness required to construct solutions in the propagation case. Thus the stochastic characteristics of the solutions are hardly tractable explicitly. Our approach consists in representing the dynamic response of the structure by means of stochastic Fourier integral operators and in shifting the stochastic modelling from the coefficients to stochastic building blocks of the Fourier integral operator.

A Fourier integral operator consists of a complex exponential term containing the phase function (which describes the propagation geometry) and an amplitude. In the linear, constant coefficient case, the response of the system can be expressed explicitly by means of Fourier integral operators based on so-called half wave equations. In the case of a spatially varying medium, we add parametrically described random terms to the phase function and amplitude of the Fourier integral operator, which carry the burden of modelling the random medium and can still be computed numerically. Since neither the stochastic models of the coefficients nor the ones of the phase function and amplitude are known a priori, but have to be determined by data fitting, our top-down approach appears as justified as the bottom-up approach.

We present the following results: The case of plane waves (corresponding to one dimensional wave propagation) has been completely solved. We apply this to damage detection in a carbon fiber composite plate. The time-dependent response of the structure to an ultrasonic excitation at single locations is modelled by a damped one dimensional wave equation. Calibrating the wave speed and the damping coefficient to the ultrasonic signal allows one to detect damage at a single location. Repeating the measurement on a grid of points admits to locate the damage spatially.

The response of a three dimensional homogeneous and isotropic elastic material can be described by wave potentials which satisfy four uncoupled wave equations. Again, the Fourier integral operator representation based on half waves can be obtained for a time-dependent point excitation inside the medium. A more realistic set-up is the surface excitation of a half space. This leads to Rayleigh waves, which in turn can be modelled by Fourier integral operators, and the stochastic perturbations of the phase functions can be performed.

Finally, it should be noted that the evaluation of a multidimensional Fourier integral operator is computationally expensive. For this reason, we also set up an in-house program implementing the fast butterfly algorithm of [2].

The plan of the paper is as follows. In Section 2, we present the ultrasonic measurement modelled by a one dimensional damped wave equation and demonstrate its capability to detect and locate damage in a carbon fiber composite plate. Section 3 contains a mathematical elaboration of the Fourier integral operator method, generalizing from the one- to the three dimensional case. We describe in more detail how the stochastics are introduced, and we discuss computational issues. Section 4 addresses the three dimensional case. We first show how the full space problem (linear elastic, homogeneous, isotropic) with point source excitation can be handled by Fourier integral operators. In the second subsection, Fourier integral operator modelling of Rayleigh waves is addressed. A stochastic perturbation in the phase function allows one to compute a confidence band around the response.

2 Preliminary Studies

2.1 Ultrasonic Impulse Echo

In this section we address damage detection in a carbon fiber composite plate. The particular plate is damaged by a high speed impact.

A Piezo crystal transducer produces an ultrasonic pulse at the surface of the plate, which then goes through the plate. The pulse is reflected at the bottom and goes back to the top. The transducer measures the amplitude at the top over time.

As a first model we assume plane waves in the medium. This can be justified, since the diameter of the transducer is larger than the thickness of the plate (approximately 2 : 1). Plane waves can be reduced to the one dimensional wave equation with wave speed c. In order to fit the model to measured data, a damping term b is added to the equation. The bottom side of the plate is assumed to be stress free. During the excitation ($t < T_{\text{ex}}$) the solution on the top side is assumed to be known. After that we assume to have a stress free boundary on top.

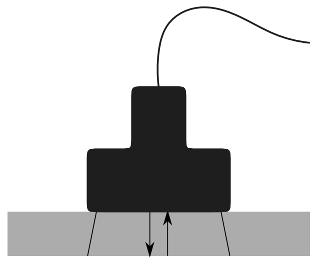

Figure 1: Measurement principle.

Since one cannot obtain the plate thickness and the wave speed simultaneously, we assume that the thickness of the plate is 1, and we do a space dimension free computation.

This results in the following equations:

$$u(z,t) = \begin{cases} v(z,t) & \text{if } t \in [0, T_{\text{ex}}] \\ w(z,t) & \text{if } t \in (T_{\text{ex}}, T_{\text{end}}], \end{cases} \tag{1a}$$

where

$$\begin{cases} \partial_{tt} v(z,t) + b\,\partial_t v(z,t) - c^2\,\partial_{zz} v(z,t) = 0 & & \\ & v(z,0) = 0 & \partial_t v(z,0) = 0 \\ & v(0,t) = f(t) & \partial_z v(1,t) = 0 \end{cases} \tag{1b}$$

and

$$\begin{cases} \partial_{tt} w(z,t) + b\,\partial_t w(z,t) - c^2\,\partial_{zz} w(z,t) = 0 & & \\ & w(z,T_{\text{ex}}) - v(z,T_{\text{ex}}) = 0 & \partial_t w(z,T_{\text{ex}}) - \partial_t v(z,T_{\text{ex}}) = 0 \\ & w(0,t) = 0 & \partial_z w(1,t) = 0. \end{cases} \tag{1c}$$

The signal measured by the transducer is assumed to be $u(0,t)$. We calibrate the parameters b and c, by minimizing the L^2-error between measurement and simulation. In Figure 2 one can see the best fit compared to the measurement. One observes a higher damping term and a lower wave speed in the damaged region. This is to be expected, since damage lowers the elastic properties of a medium (e.g. Young's module, which is proportional to c^2), and delamination leads to a higher viscous behavior.

In order to get a spatial resolution, the top of the plate is measured on a grid with a spatial distance 5×5 mm. Furthermore, the damaged region is measured with grid size 1×1 mm. In Figure 3 one can see the optimized parameters b and c at the corresponding measurement location.

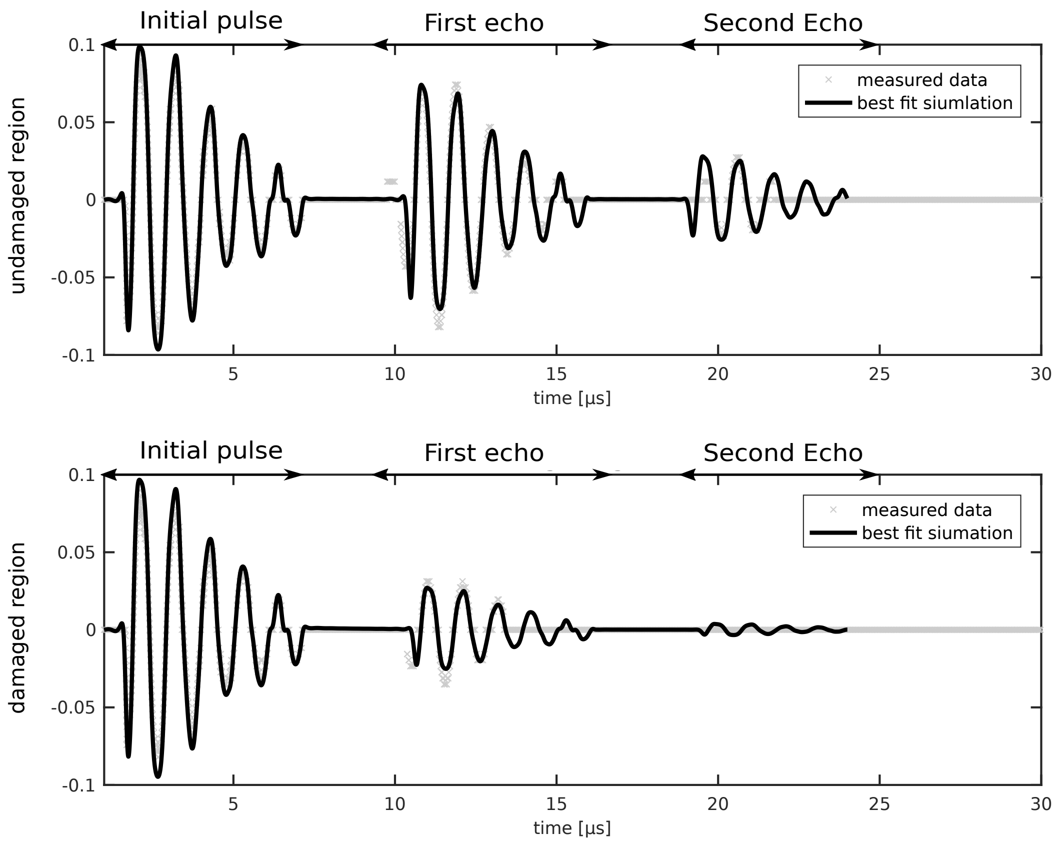

Figure 2: Measured and simulated signal in an undamaged and a damaged region.

3 Fourier Integral Operators

3.1 1D Wave equation

For simplicity just consider equation (1b). By the reflection principle, the space coordinate can be extended to the whole real line. The equation reads

$$\begin{cases} \partial_{tt}v(z,t) + b\,\partial_t v(z,t) - c^2\,\partial_{zz}v(z,t) = \sum_{k\in\mathbb{Z}} 2c^2\partial_z\delta(z-2kL)f(t) \\ v(z,0)=0, \qquad \partial_t v(z,0)=0, \end{cases} \tag{2}$$

where δ is the Dirac delta function. Furthermore we decompose (2) into a *half wave equation* by substituting $v_\pm = (\partial_t + \frac{b}{2})v \pm iA(D_z)v$, where $i=\sqrt{-1}$ and $D_z = -i\partial_z$ and $A(D_z)$ is the pseudodifferential operator with symbol $A(\xi) = \sqrt{c^2\|\xi\|^2 - \frac{b^2}{4}}$. The half wave equation reads:

$$\begin{cases} \left(\partial_t + \frac{b}{2}\right) v_\pm(z,t) \mp iA(D_z)v_\pm(z,t) = \sum_{k\in\mathbb{Z}} 2c^2\partial_z\delta(z-2kL)f(t) \\ v_\pm(z,0)=0. \end{cases} \tag{3}$$

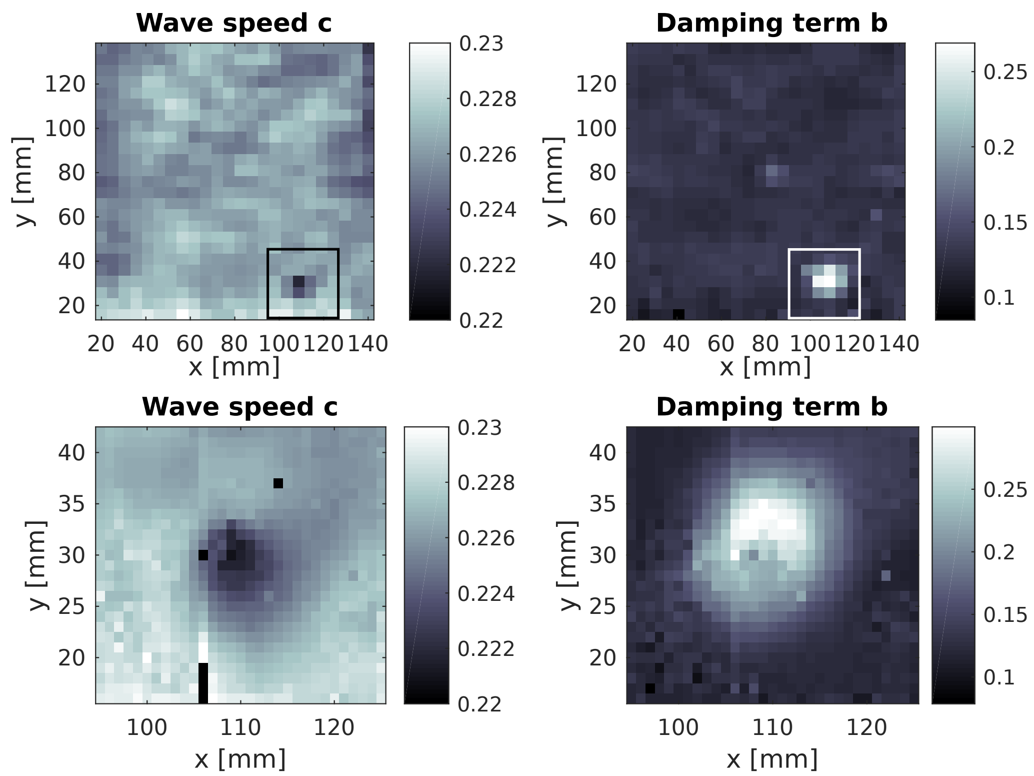

Figure 3: Spatial resolution of the parameter b and c. Top: overall plate. Bottom: damaged region.

Taking the spatial Fourier transform leads to the ordinary differential equation

$$\begin{cases} \left(\partial_t + \frac{b}{2}\right) \widehat{v}_\pm(\xi,t) \mp iA(\xi)\widehat{v}_\pm(\xi,t) = \sum_{k\in\mathbb{Z}} 2ic^2\xi e^{2i\xi kL} f(t) \\ \widehat{v}_\pm(\xi,0) = 0 \end{cases} \tag{4}$$

which is then solved by

$$\widehat{v}_\pm(\xi,t) = 2ic^2\xi \sum_{k\in\mathbb{Z}} e^{2i\xi kL} \int_0^t e^{\pm iA(\xi)(t-s)-\frac{b}{2}(t-s)} f(s)\mathrm{d}s.$$

Taking the inverse spatial Fourier transform and interchanging the order of integration leads to the solution operator

$$v_\pm(z,t) = \sum_{k\in\mathbb{Z}} \int_0^t \int_{\mathbb{R}} \frac{ic^2\xi}{\pi} e^{2i\xi kL} e^{i\xi z \pm iA(\xi)(t-s)-\frac{b}{2}(t-s)} f(s)\,\mathrm{d}\xi\,\mathrm{d}s. \tag{5}$$

By noting that $\frac{1}{2}(v_+(z,t) + v_-(z,t)) = (\partial_t + \frac{b}{2})v(z,t)$ we get v by

$$v(z,t) = \int_0^t \frac{v_+(z,s) + v_-(z,s)}{2} e^{-\frac{b}{2}(t-s)}\mathrm{d}s.$$

So the solution can be expressed by time integration of sums of Fourier integral operators. Note that these can be only interpreted as oscillatory integrals (see e.g. [8]).

3.2 Three dimensional approximative solution

Consider the three dimensional perturbed wave equation, with wave speed $c^2(\boldsymbol{x}) = c_0^2 + \varepsilon R(\boldsymbol{x})$, ε small and R smooth,

$$\begin{cases} \partial_{tt} u(\boldsymbol{x}, t) = (c_0^2 + \varepsilon R(\boldsymbol{x}))\Delta u(\boldsymbol{x}, t) \\ \quad u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \qquad \partial_t u(\boldsymbol{x}, 0) = u_1(\boldsymbol{x}) \end{cases} \tag{6}$$

Then u can be written as a first order hyperbolic system of pseudodifferential equations [9] as follows: Let $\sqrt{1-\Delta}$ be the pseudodifferential operator with symbol $\sqrt{1+\|\boldsymbol{\xi}\|^2}$. Defining $v_1 := \sqrt{1-\Delta}\, u$ and $v_2 := \partial_t u$ leads to

$$\begin{bmatrix} \partial_t v_1(\boldsymbol{x}, t) \\ \partial_t v_2(\boldsymbol{x}, t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \sqrt{1-\Delta} \\ \frac{(c_0^2 + \varepsilon R(\boldsymbol{x}))\Delta}{\sqrt{1-\Delta}} & 0 \end{bmatrix}}_{:= A(\boldsymbol{x}, D_{\boldsymbol{x}})} \begin{bmatrix} v_1(\boldsymbol{x}, t) \\ v_2(\boldsymbol{x}, t) \end{bmatrix}, \tag{7}$$

with initial condition $\boldsymbol{v_0}(\boldsymbol{x}) = [\sqrt{1-\Delta}\, u_0(\boldsymbol{x}), u_1(\boldsymbol{x})]^{\mathrm{T}}$. If $\varepsilon = 0$ the pseudodifferential matrix $A(\boldsymbol{x}, D_{\boldsymbol{x}})$ is easily diagonalized and one arrives at the half wave equation

$$\begin{bmatrix} \partial_t v_+(\boldsymbol{x}, t) \\ \partial_t v_-(\boldsymbol{x}, t) \end{bmatrix} = \begin{bmatrix} i c_0 \sqrt{-\Delta} & 0 \\ 0 & -i c_0 \sqrt{-\Delta} \end{bmatrix} \begin{bmatrix} v_+(\boldsymbol{x}, t) \\ v_-(\boldsymbol{x}, t) \end{bmatrix}.$$

In any case, equation (7) is a symmetrizable system and

$$\boldsymbol{v}(\boldsymbol{x}, t_0) \approx \frac{1}{(2\pi)^3} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{y}} e^{A(\boldsymbol{x}, \boldsymbol{\xi})t} \boldsymbol{v_0}(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \mathrm{d}\boldsymbol{\xi},$$

where $e^{A(\boldsymbol{x}, \boldsymbol{\xi})t}$ is the matrix exponential of $A(\boldsymbol{x}, \boldsymbol{\xi})t$. If $\varepsilon = 0$ this solution is exact. Otherwise one can expect a linear convergence in some Sobolev norm for t_0 small (see [6, 7]). So for small time steps or small ε one has a good approximation.

3.3 Fast Fourier integral operators

For this section we assume that the dimension $n = 2$ or $n = 3$. The Fourier integral operators can be numerically evaluated by spatial discretization. One has

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\phi(\boldsymbol{x}, \boldsymbol{\xi}, t) - i\boldsymbol{y}\cdot\boldsymbol{\xi}} f(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \mathrm{d}\boldsymbol{\xi} \approx \Delta_\xi \Delta_y \sum_{k \in \mathcal{G}_\xi} e^{i\phi(\boldsymbol{x}_l, \boldsymbol{\xi}_k, t)} \sum_{j \in \mathcal{G}_y} e^{i\boldsymbol{y}_j \cdot \boldsymbol{\xi}_k} f(\boldsymbol{y}_j), \tag{8}$$

where $\mathcal{G}_y$ is the spatially discretized and truncated grid in $\mathbb{R}^n$, e.g. $\{\frac{2\pi}{N}(k_1, \ldots, k_n)^{\mathrm{T}}, k_i = 1, \ldots, N\}$ and $\mathcal{G}_\xi$ the corresponding grid on the phase side, e.g. $\{(j_1, \ldots, j_n)^{\mathrm{T}}, j_i = 1, \ldots, N\}$. Furthermore, Δ_y and Δ_ξ is the volume of a single grid cell in the corresponding discretization. The second sum can be computed by the fast Fourier transform algorithm. So a term of form

$$\sum_{k \in \mathcal{G}_\xi} e^{i\phi(\boldsymbol{x}_l, \boldsymbol{\xi}_k, t)} g(\boldsymbol{\xi}_k)$$

is remaining. A direct evaluation of this term is very costly and is of complexity $\mathcal{O}((N^n)^2)$. Following [2] there is a *butterfly algorithm* to compute this term using a divide and conquer strategy. Thereby the complexity reduces to $\mathcal{O}((N^n)\log(N))$. In Figure 4 one can see the computational time for one evaluation of the two dimensional fast Fourier integral operator on a work station (Intel Xeon E5-1650) implemented in MATLAB. For the parallel evaluation we use four cores.

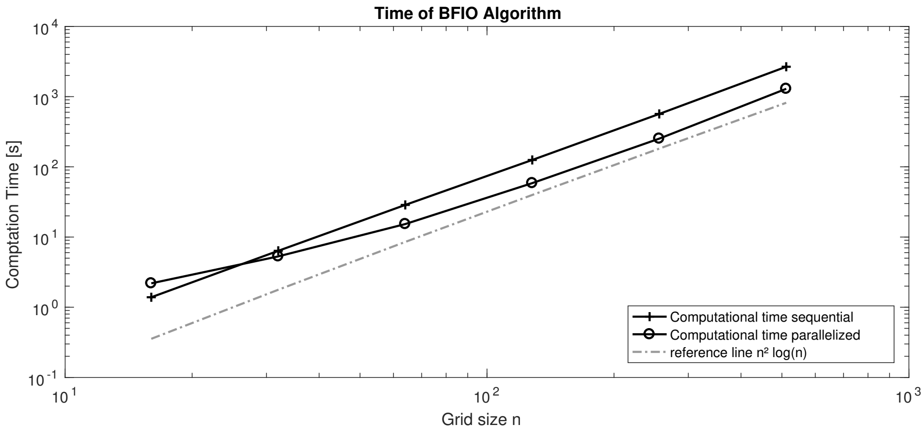

Figure 4: Time for one evaluation of the fast Fourier integral operator.

3.4 Stochastics

In standard engineering approaches, uncertainty is modeled in the coefficients of the model by e.g. random fields. However, it is very difficult to track the stochastic properties from the coefficients to the solutions. So, we use a top-down approach. Instead of putting the randomness into the model, we solve a deterministic model by means of Fourier integral operators and add parametrized randomness into the solution operator. To construct the perturbed Fourier integral operator, we use the following strategy: Let

$$\frac{1}{(2\pi)^3}\iint_{\mathbb{R}^3\times\mathbb{R}^3} e^{i\Phi(\boldsymbol{x},\boldsymbol{\xi},t)-i\boldsymbol{\xi}\cdot\boldsymbol{y}} a(\boldsymbol{\xi},t)u(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{\xi} \tag{9}$$

be a term of the solution operator for the wave equation with constant wave speed c_0. Then add a random perturbation $r_1(\boldsymbol{\xi},\boldsymbol{x}), r_2(\boldsymbol{\xi})$ to the solution operator and

$$\frac{1}{(2\pi)^3}\iint_{\mathbb{R}^3\times\mathbb{R}^3} e^{i(\Phi(\boldsymbol{x},\boldsymbol{\xi},t)+r_1(\boldsymbol{\xi},\boldsymbol{x}))-i\boldsymbol{\xi}\cdot\boldsymbol{y}} (a(\boldsymbol{\xi},t)+r_2(\boldsymbol{\xi}))u(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{\xi} \tag{10}$$

is the perturbed, stochastic solution operator. This representation is very convenient, since one can interpret r_1 as the perturbation geometry of the wave propagation, and r_2 as the perturbation amplitude of the wavenumber.

4 Three dimensional linear elasticity

We consider the equilibrium equation in three dimensional elasticity theory. To avoid technicalities we assume the homogeneous isotropic case:

$$\frac{\lambda+\mu}{\rho}(\nabla\times\nabla\times\boldsymbol{u}(\boldsymbol{x},t))+\frac{\mu}{\rho}\Delta\boldsymbol{u}(\boldsymbol{x},t)-\partial_{tt}\boldsymbol{u}(\boldsymbol{x},t)=-\boldsymbol{f}(\boldsymbol{x},t), \tag{11}$$

where $\boldsymbol{f}$ is the body force, λ and μ the Lamé constants and ρ the density. The lateral resp. the transverse wave speed are $c_l^2=\frac{\lambda+2\mu}{\rho}$ resp. $c_s^2=\frac{\mu}{\rho}$. In this case, there exist [3, 5] potentials Φ and $\boldsymbol{\Psi}=(\Psi_1,\Psi_2,\Psi_3)^{\mathrm{T}}$ with

$$\begin{aligned}\nabla\Phi(\boldsymbol{x},t)+\nabla\times\boldsymbol{\Psi}(\boldsymbol{x},t)&=\boldsymbol{u}(\boldsymbol{x},t)\\ \nabla\cdot\boldsymbol{\Psi}(\boldsymbol{x},t)&=0,\end{aligned} \tag{12}$$

where Φ and $\boldsymbol{\Psi}$ satisfy the wave equations

$$\begin{aligned}\partial_{tt}\Phi(\boldsymbol{x},t)-c_l^2\Delta\Phi(\boldsymbol{x},t)&=-\varphi(\boldsymbol{x},t)\\ \partial_{tt}\Psi_j(\boldsymbol{x},t)-c_s^2\Delta\Psi_j(\boldsymbol{x},t)&=-\psi_j(\boldsymbol{x},t),\quad j=1,2,3,\end{aligned} \tag{13}$$

and φ and $\boldsymbol{\psi}$ have to be determined from

$$\begin{aligned}\nabla\varphi(\boldsymbol{x},t)+\nabla\times\boldsymbol{\psi}(\boldsymbol{x},t)&=-\boldsymbol{f}(\boldsymbol{x},t)\\ \nabla\cdot\boldsymbol{\psi}(\boldsymbol{x},t)&=0.\end{aligned}$$

So it suffices to solve the decoupled problem for the wave potentials. One can compute φ and $\boldsymbol{\psi}$ by inverting the Laplace equation as follows: We make the ansatz

$$\begin{aligned}\varphi(\boldsymbol{x},t)&:=\partial_x g_1(\boldsymbol{x},t)+\partial_y g_2(\boldsymbol{x},t)+\partial_z g_3(\boldsymbol{x},t)\\ \psi_1(\boldsymbol{x},t)&:=\partial_z g_2(\boldsymbol{x},t)-\partial_y g_3(\boldsymbol{x},t)\\ \psi_2(\boldsymbol{x},t)&:=-\partial_z g_1(\boldsymbol{x},t)+\partial_x g_3(\boldsymbol{x},t)\\ \psi_3(\boldsymbol{x},t)&:=\partial_y g_1(\boldsymbol{x},t)-\partial_x g_2(\boldsymbol{x},t),\end{aligned} \tag{14}$$

for a function $g_j, j=1,2,3$ to be determined. Then

$$\nabla\varphi(\boldsymbol{x},t)+\nabla\times\boldsymbol{\psi}(\boldsymbol{x},t)=(\Delta g_1(\boldsymbol{x},t),\Delta g_2(\boldsymbol{x},t),\Delta g_3(\boldsymbol{x},t))^{\mathrm{T}}.$$

By inverting the Laplace operator one gets $g_j(\boldsymbol{x},t)=\Delta^{-1}f_j(\boldsymbol{x},t)$.

4.1 Waves within the medium

To validate the solution given by the Fourier integral operator, a three dimensional finite element model of a cuboid is excited by a force acting along the z-axis with frequency 2π MHz. We assume the material parameters of aluminum (Young's Module $E = 70$ GPa and Poisson ratio $\nu = 0.35$). The model is fixed along the top edges. The model has $512 \times 512 \times 12$ elements. In Figure 6 one can see the comparison of the solution of the Fourier integral operator and the finite element model at time 2μs. One sees a good coherence, considering the variation in z-direction in the finite element model itself.

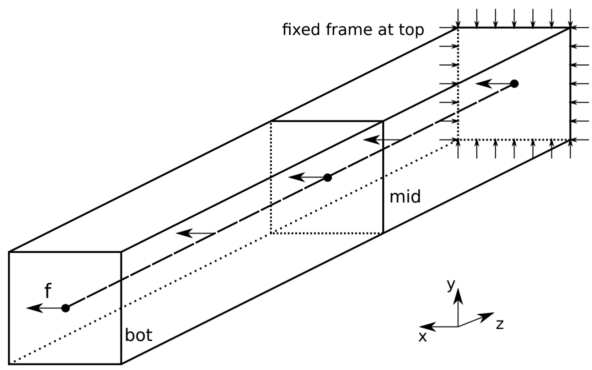

Figure 5: Three dimensional dynamic simulation.

We study the propagation induced by body forces. As initial data we assume that $\boldsymbol{u}(\boldsymbol{x},0) \equiv \partial_t \boldsymbol{u}(\boldsymbol{x},0) \equiv 0$. The force term $\boldsymbol{f}(\boldsymbol{x},t) = [\delta(x)\delta(y)\sin(2\pi t),0,0]^{\mathrm{T}}$ is applied to the problem. By this, $\boldsymbol{u}$ does not depend on z and thereby the displacement is only in (x,y)-direction. This is a two dimensional plane strain problem. By that, $g_1(x,y,t) = E(x,y)\sin(2\pi t)$, where $E(x,y) = -(2\pi)^{-1}\log(\sqrt{x^2+y^2})$ is the fundamental solution of the two dimensional Laplace operator.

4.2 Rayleigh surface waves

Consider the half space $\{(x,y,z) \in \mathbb{R}^3, z \geq 0\}$. We assume a stress-free surface and exponential decay in depth [1], i.e.

$$\Phi \propto e^{-\zeta_l z + i(\xi x + \eta y \pm c\kappa t)} \quad \text{resp.} \quad \Psi_j \propto e^{-\zeta_s z + i(\xi x + \eta y \pm c\kappa t)}, \quad j = 1,2,3, \tag{15}$$

where $\kappa = \sqrt{\xi^2 + \eta^2}$. If no body force $\boldsymbol{f}$ is present, equation (13) is solved if

$$\zeta_l^2 = \kappa^2 \left(1 - \frac{c^2}{c_l^2}\right) \qquad \text{and} \qquad \zeta_s^2 = \kappa^2 \left(1 - \frac{c^2}{c_s^2}\right). \tag{16}$$

Since $z > 0$ we choose the positive root of (16) to satisfy the exponential decay assumption. The stresses in terms of potentials read as follows:

$$\begin{aligned} \tau_{13} &= c_s^2(2\partial_{xz}\Phi - \partial_{xy}\Psi_1 + (\partial_{xx} - \partial_{zz})\Psi_2 + \partial_{yz}\Psi_3) \\ \tau_{23} &= c_s^2(2\partial_{yz}\Phi + (\partial_{zz} - \partial_{yy})\Psi_1 + \partial_{xy}\Psi_2 - \partial_{yz}\Psi_3) \\ \tau_{33} &= (c_l^2 - 2c_s^2)\Delta\Phi + 2c_s^2\partial_{zz}\Phi - 2c_s^2\partial_{yz}\Psi_1 + 2c_s^2\partial_{xz}\Psi_2. \end{aligned} \tag{17}$$

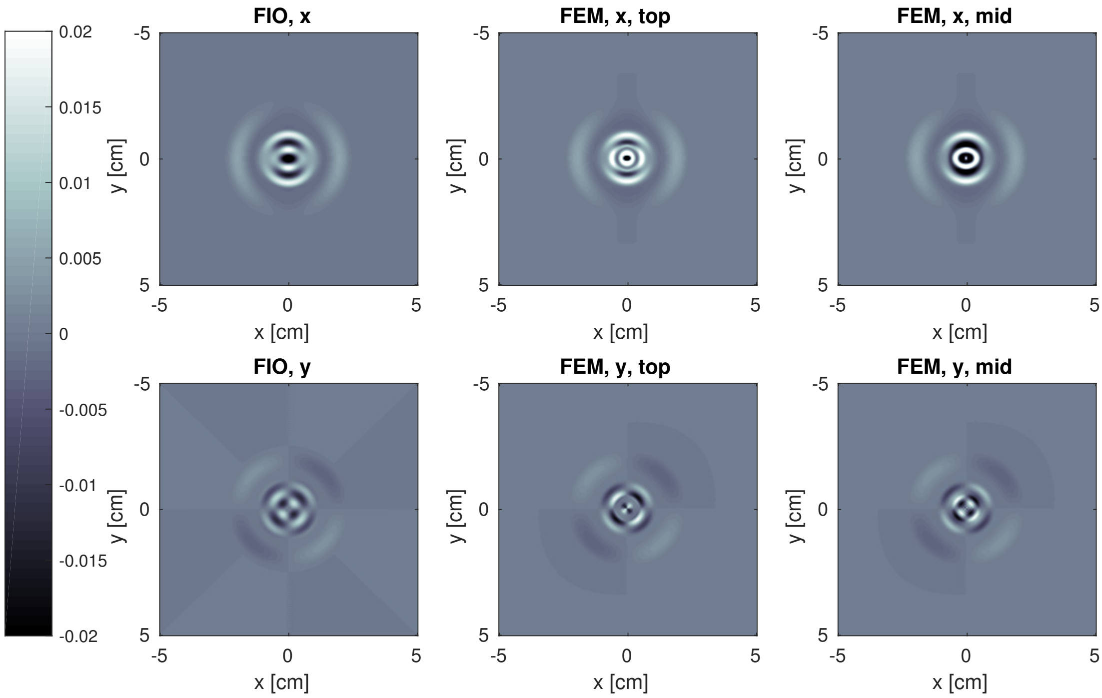

Figure 6: Top: Amplitude in x direction. Bottom: Amplitude in y direction.

We plug (15) into (17) and set $\tau_{13} = \tau_{23} = \tau_{33} = 0$ for $z = 0$. Since in addition one has $\nabla \cdot \mathbf{\Psi} = 0$, the following (4×4)-system of equations arises:

$$\begin{bmatrix} -2i\zeta_l\xi & \xi\eta & -\xi^2 - \zeta_s^2 & -i\zeta_s\eta \\ -2i\zeta_l\eta & \eta^2 + \zeta_s^2 & -\xi\eta & i\zeta_s\xi \\ -(\xi^2+\eta^2)(c_l^2 - 2c_s^2) + c_l^2\zeta_l^2 & 2ic_s^2\zeta_s\eta & -2ic_s^2\zeta_s\xi & 0 \\ 0 & i\xi & i\eta & -\zeta_s \end{bmatrix} \cdot \begin{bmatrix} \Phi \\ \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{bmatrix}_{z=0} = 0.$$

In order to get a nontrivial solution, the determinant of this system has to be zero:

$$\kappa^6 c^2 \left[\left(2 - \frac{c^2}{c_s^2}\right)^2 - 4\sqrt{1 - \frac{c^2}{c_s^2}}\sqrt{1 - \frac{c^2}{c_l^2}} \right] = 0, \tag{18}$$

which is well known as the Rayleigh equation. If c satisfies (18),

$$\begin{bmatrix} \Phi \\ \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{bmatrix} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} (C_1 e^{i(\xi x + \eta y + c\kappa t)} + C_2 e^{i(\xi x + \eta y - c\kappa t)}) \begin{bmatrix} (\zeta_s^2 + \kappa^2)e^{-\zeta_l z} \\ 2i\eta\zeta_l e^{-\zeta_s z} \\ -2i\xi\zeta_l e^{-\zeta_s z} \\ 0 \end{bmatrix} \mathrm{d}(\xi, \eta) \tag{19}$$

describe the Rayleigh surface waves.

Let $\widehat{u}_{30}(\xi,\eta) := \mathcal{F}_{x,y}[u_3\big|_{t=0,z=0}](\xi,\eta)$ and $\widehat{u}_{31} := \mathcal{F}_{x,y}[\partial_t u_3\big|_{t=0,z=0}](\xi,\eta)$, where $\mathcal{F}_{x,y}$ is the spatial Fourier transform in the (x,y)-plane. If

$$C_1 = \frac{c_s^2}{2\kappa\zeta_l}\left(\widehat{u}_{30} + \frac{\widehat{u}_{31}}{ic\kappa}\right) \quad \text{and} \quad C_2 = \frac{c_s^2}{2\kappa\zeta_l}\left(\widehat{u}_{30} - \frac{\widehat{u}_{31}}{ic\kappa}\right), \tag{20}$$

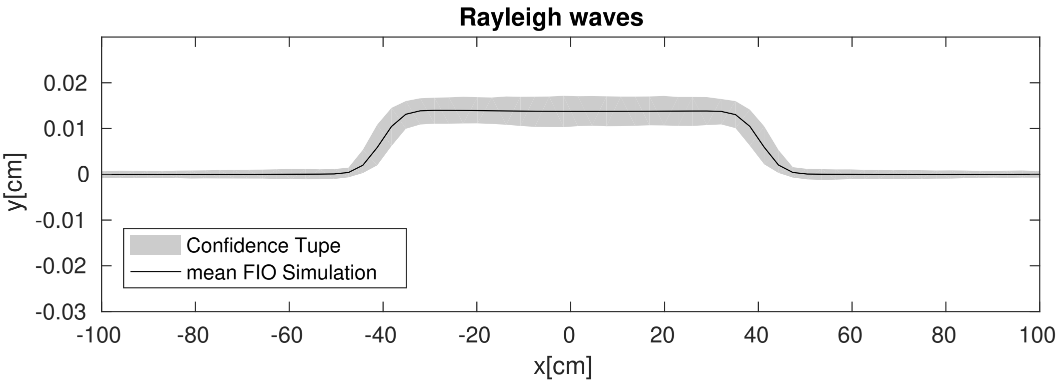

Figure 7: Solution of the Rayleigh waves on the surface, including a 95% confidence tube.

the solution satisfies given initial data at $z = 0$. So we end up by Fourier integral operators (equation (19)) describing Rayleigh surface waves.

For an example we use again the material parameters of aluminum and construct the Fourier integral operator as above. Let $u_{30}(x, y) = 0$ and $u_{31}(x, y) = \exp(-x^2)$. This makes the solution constant in y-direction and we have again a plain strain problem. Adding random noise of the form $r_1(x, y, t) = r_{11}(x)\xi \pm r_{12}(x)\kappa t$ to the phase functions and a confidence tube is produced. In Figure 7, one can see the amplitude at time 50 μs and a symmetric 95%-confidence tube (upper and lower bound in y-direction) for given Gaussian random fields r_1 and r_2. The sample size is 500.

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References

[1] Achenbach, J.D.: *Wave Propagation in Elastic Solids.* North-Holland, Amsterdam, 2010

[2] Candès, E. ; Demanet, L. ; Ying, L.: A Fast Butterfly Algorithm for the Computation of Fourier Integral Operators. In: *Multiscale Modeling & Simulation* (2009)

[3] Graff, K.F.: *Wave Motion in Elastic Solids.* Clarendon Press, Oxford, 1975

[4] Oberguggenberger, M. ; Schwarz, M.: Fourier integral operators in stochastic structural analysis. In: *12th International Probabilistic Workshop* (2014)

[5] Poruchikov, V.B.: *Methods of the Classical Theory of Elastodynamics.* Springer, Berlin Heidelberg, 1993

[6] Rousseau, J. L.: Fourier Integral Operator Approximation of Solutions to First-Order Hyperbolic Pseudodifferential Equations I: Convergence in Sobolev Spaces. In: *Communications in Partial Differential Equations* (2006)

[7] Rousseau, J. L.: Fourier Integral Operator Product Representation of Solutions to First Order Symmetrizable Hyperbolic Systems. In: *Journal d'Analyse Mathématique* (2009)

[8] Shubin, M.A.: *Pseudodifferential Operators and Spectral Theory.* Springer, Berlin Heidelberg, 2001

[9] Taylor, M.E.: *Pseudodifferential Operators.* Princeton University Press, Princeton, 1981