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# Probabilistic properties of generalized stochastic processes in algebras of generalized functions

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**Abstract** Stochastic processes are regarded in the framework of Colombeau-type algebras of generalized functions. The notion of point values of Colombeau stochastic processes in compactly supported generalized points is established, which uniquely characterize the process, and relying on this result we prove the measurability of the corresponding random variables with values in the Colombeau algebra of compactly supported generalized constants endowed with the topology generated by sharp open balls. The generalized characteristic function and the generalized correlation function of Colombeau stochastic processes are introduced and their properties are investigated. It is shown that the characteristic function of classical stochastic processes can be embedded into the space of generalized characteristic functions. The generalized expectation and the generalized correlation function can be retrieved from the generalized characteristic function. The structural representation of the correlation function which is supported on the diagonal is given. Examples of generalized characteristic functions related to Gaussian Colombeau stochastic processes are given.

**Keywords** Colombeau stochastic processes · Gaussian processes · Point values of generalized functions · Generalized characteristic functions · Generalized correlation function

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## 1 Introduction

It is often the case that solutions to stochastic partial differential equations (SPDEs) do not exist in the usual sense, rather they are generalized functions. Hence, there has been a great interest in developing a stochastic calculus in algebras of generalized functions (see [8, 10, 12–16]).

Colombeau-type stochastic processes defined as Colombeau functions with values in the space of random variables with finite $p$th moments, $L^p(\mathfrak{O})$, $p \geq 1$, and with values in $\mathcal{L}(\mathfrak{O})$, the space of real random variables (measurable functions) endowed with almost sure convergence, were considered in [8]. Note that in this paper we use the notation $\mathfrak{O}$ for the probability space and $\Omega$ for an open subset of $\mathbb{R}^d$. Further on, we use the conventional notation $\mathcal{C}^k(\Omega)$ for $k$ times continuously differentiable functions and $\mathcal{C}^\infty(\Omega)$ for smooth functions, $\mathcal{D}(\Omega)$ for the smooth test functions with compact support and its dual $\mathcal{D}'(\Omega)$ for the generalized functions. Similarly, $\mathcal{S}(\Omega)$ denotes the Schwartz space of rapidly decreasing functions and $\mathcal{S}'(\Omega)$ the space of tempered distributions.

In [8] the authors have shown that distributional stochastic processes $\xi$, i.e. mappings $\phi \mapsto \langle \xi, \phi \rangle$ which are continuous from $\mathcal{D}(\Omega)$ into $L^p(\mathfrak{O})$ with respect to the strong topology, can be embedded into such algebras. In this paper we continue this approach and study Colombeau-type stochastic processes defined in this sense, but in order to retain measurability properties, we use a sequential approach. We introduce the algebras $\mathcal{G}_{\mathcal{L}}(\mathfrak{O}, \Omega)$, $\mathcal{G}_{\mathcal{M}^\infty}(\mathfrak{O}, \Omega)$ and the vector space $\mathcal{G}_{L^p}(\mathfrak{O}, \Omega)$, where $\mathcal{M}^\infty(\mathfrak{O}) = \cap_{1 \leq p < \infty} L^p(\mathfrak{O})$ is the space of random variables with all seminorms $||| \cdot |||_s = \sup_{1 \leq p \leq s} \| \cdot \|_{L^p}$, $s \in \mathbb{N}$ finite. Elements of these spaces will be called Colombeau stochastic processes (CSPs) with values in $\mathcal{L}(\mathfrak{O})$, $\mathcal{M}^\infty(\mathfrak{O})$ and $L^p(\mathfrak{O})$, respectively. Also, we consider the spaces $\mathcal{G}^k_{\mathcal{L}}(\mathfrak{O}, \Omega)$ and $\mathcal{G}^k_{L^p}(\mathfrak{O}, \Omega)$ in which we perform differentiation only up to order $k$.

We establish the notion of point values $X(\cdot, \tilde{x})$ of CSPs in compactly supported generalized points $\tilde{x}$ and prove measurability of the corresponding random variable with values in a Colombeau algebra of compactly supported generalized constants $\mathcal{R}_c$, endowed with the topology generated by sharp open balls.

We introduce the generalized characteristic function of CSPs with values in $\mathcal{M}^\infty(\mathfrak{O})$ and the generalized characteristic function of CSPs in $\mathcal{G}^k_{L^{kp}}(\mathfrak{O}, \Omega)$. Note that $\mathcal{G}^k_{L^{kp}}(\mathfrak{O}, \Omega)$ is a vector space where the differentiation decreases the order to $\mathcal{G}^{k-1}_{L^{kp}}(\mathfrak{O}, \Omega)$.

Since CSPs are completely characterized by their generalized point values, we use the point values not only to prove measurability of the processes, but also to characterize their characteristic function, expectation, covariance and correlation function. Special attention is put on the study of processes with uncorrelated values whose correlation function is supported by the diagonal. We provide several examples of characteristic functions and correlation functions of Gaussian Colombeau stochastic processes (GCSPs).

The paper is organized as follows: In Section 2 we introduce the basic notions and define several classes of Colombeau stochastic processes. Section 3 is devoted to the characterization of CSPs via their generalized point values. The main result in this section is the proof of the measurability of CSPs via their point values. In Section 4 we define the generalized characteristic function of CSPs. For classical stochastic processes, their characteristic function can be embedded into the space of generalized characteristic functions. We also prove that the moments of a CSP can be retrieved via the generalized characteristic function. Section 5 is devoted to the study of the generalized correlation function of CSPs and its structural characterization. The main result in this section is the structural representation of the correlation function which is supported on the diagonal.

Some further topics and probabilistic properties of CSPs, such as independence, stationarity, stationarity of increments etc. will be considered in our next paper [3].

## 2 Basic notions

### 2.1 Colombeau algebra

In this paper we focus our attention on the sequential approach to Colombeau-type algebras. Instead of considering nets of functions $(u_\varepsilon)_\varepsilon$, $\varepsilon \in (0,1]$, we take sequences of functions $(u_n)_n$ indexed by $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$.

Let $\Omega$ be an open set in $\mathbb{R}^d$ and $\partial^\alpha = \partial_{x_1}^{\alpha_1} ... \partial_{x_d}^{\alpha_d}$, $\alpha \in \mathbb{N}_0^d$. The notation $K \Subset \Omega$ will be used to denote that $K$ is a compact subset of $\Omega$. Recall, [1,4,10], the Colombeau algebra on $\Omega$ is defined as $\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$; see Appendix A.1. $\mathcal{G}(\Omega)$ is an associative, commutative differential algebra. If $(u_n)_n \in \mathcal{E}_M(\Omega)$ is a representative of $u \in \mathcal{G}(\Omega)$, we write $u = [(u_n)_n]$.

**Remark 1** *In the definitions of Appendix A.1 we can consider* $\mathcal{E}^k(\Omega) = (\mathcal{C}^k(\Omega))^{\mathbb{N}}$ *and the corresponding vector spaces* $\mathcal{E}_M^k(\Omega)$ *and* $\mathcal{N}^k(\Omega)$, *where we take* $(u_n)_n \in \mathcal{E}^k(\Omega)$ *and perform differentiation up to order* $k$.

We fix a sequence of mollifiers $\varphi_n \in \mathcal{S}(\mathbb{R}^d), n \in \mathbb{N}$, of the form

$$\varphi_n(x) = n^d \varphi(nx),\ x \in \mathbb{R}^d,\ n \in \mathbb{N}, \tag{1}$$

where $\varphi \in \mathcal{S}(\mathbb{R}^d)$ has the following properties: $\int \varphi(x)\,dx = 1$, $\int x^m \varphi(x)\,dx = 0$, $m \in \mathbb{N}$, and $\varphi$ is positive-definite, i.e. $\hat{\varphi} \geq 0$, where $\hat{\varphi}$ denotes the Fourier transformation of $\varphi$. (For example, one can take $\hat{\varphi} \in \mathcal{D}(\mathbb{R}^d)$, $\hat{\varphi} \geq 0$ and $\hat{\varphi} \equiv 1$ in a neighborhood of zero).

In general, if $u = [(u_n)_n] \in \mathcal{G}(\Omega)$ is associated with an element $f \in \mathcal{D}'(\Omega)$, (see Appendix A.1 for the notion of association), then $\operatorname{supp} u \supseteq \operatorname{supp} f$. It is known that $\operatorname{supp} u$ can be strictly larger than $\operatorname{supp} f$.

### 2.2 Colombeau stochastic processes (CSPs)

Let $(\mathfrak{O}, \mathfrak{U}, P)$ be a probability space.

**Definition 1** Let $k \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{E}_{\mathcal{L}}^k(\mathfrak{O}, \Omega)$ be the set of sequences $(u_n(\omega, x))_n$, $\omega \in \mathfrak{O}$, $x \in \Omega$, $n \in \mathbb{N}$, such that $(u_n(\omega, \cdot))_n \in (\mathcal{C}^k(\Omega))^{\mathbb{N}}$ for almost every (a.e.) $\omega \in \mathfrak{O}$, and for every $x \in \Omega$, $(u_n(\cdot, x))_n$ is a sequence of measurable functions on $\mathfrak{O}$. Define:

$$\mathcal{E}_{M,\mathcal{L}}^k(\mathfrak{O}, \Omega) = \Big\{(u_n)_n \in \mathcal{E}_{\mathcal{L}}^k(\mathfrak{O}, \Omega) :\ (\text{for a.e. } \omega \in \mathfrak{O})(\forall K \Subset \Omega)$$
$$(\forall \alpha \in \mathbb{N}_0^d,\ |\alpha| \leq k)(\exists a \in \mathbb{N}) \left(\sup_{x \in K} |\partial^\alpha u_n(\omega, x)| = \mathcal{O}(n^a)\right)\Big\},$$
$$\mathcal{N}_{\mathcal{L}}^k(\mathfrak{O}, \Omega) = \Big\{(u_n)_n \in \mathcal{E}_{\mathcal{L}}^k(\mathfrak{O}, \Omega) :\ (\text{for a.e. } \omega \in \mathfrak{O})(\forall K \Subset \Omega)$$
$$(\forall \alpha \in \mathbb{N}_0^d,\ |\alpha| \leq k)(\forall b \in \mathbb{N}) \left(\sup_{x \in K} |\partial^\alpha u_n(\omega, x)| = \mathcal{O}(n^{-b})\right)\Big\}.$$

Elements of $\mathcal{E}_{M,\mathcal{L}}^k(\mathfrak{O}, \Omega)$ and $\mathcal{N}_{\mathcal{L}}^k(\mathfrak{O}, \Omega)$ are called moderate and negligible sequences of functions with values in $\mathcal{L}(\mathfrak{O})$, respectively. The elements of the quotient space

$$\mathcal{G}_{\mathcal{L}}^k(\mathfrak{O}, \Omega) = \mathcal{E}_{M,\mathcal{L}}^k(\mathfrak{O}, \Omega)/\mathcal{N}_{\mathcal{L}}^k(\mathfrak{O}, \Omega)$$

are called Colombeau stochastic processes (CSPs) over $\Omega$ with values in $\mathcal{L}(\mathfrak{O})$.

$\mathcal{E}^k_{M,\mathcal{L}}(\mathfrak{O},\Omega)$ is an algebra with respect to multiplication and $\mathcal{N}^k_{\mathcal{L}}(\mathfrak{O},\Omega)$ is an ideal in $\mathcal{E}^k_{M,\mathcal{L}}(\mathfrak{O},\Omega)$, so we have that $\mathcal{G}^k_{\mathcal{L}}(\mathfrak{O},\Omega)$ is an algebra.

**Definition 2** Let $p \in [1,\infty]$ and $k \in \mathbb{N} \cup \{\infty\}$. Let $\mathcal{E}^k_{L^p}(\mathfrak{O},\Omega) = (\mathcal{C}^k_{L^p}(\mathfrak{O},\Omega))^{\mathbb{N}}$ be the set of sequences $(u_n(\omega,x))_n$, $\omega \in \mathfrak{O}$, $x \in \Omega$, $n \in \mathbb{N}$, such that the mapping $x \mapsto u_n(\omega,x)$ is in $\mathcal{C}^k(\Omega)$ for a.e. $\omega \in \mathfrak{O}$, and for every $x \in \Omega$, $u_n(\cdot,x)$ is in $L^p(\mathfrak{O})$. Define:

$$\mathcal{E}^k_{M,L^p}(\mathfrak{O},\Omega) = \Big\{(u_n)_n \in \mathcal{E}^k_{L^p}(\mathfrak{O},\Omega) \,:\, (\forall K \Subset \Omega)(\forall \alpha \in \mathbb{N}^d_0,\ |\alpha| \le k)$$
$$(\exists a \in \mathbb{N}) \left( \sup_{x\in K} \|\partial^\alpha u_n(\cdot,x)\|_{L^p} = \mathcal{O}(n^a) \right) \Big\},$$

$$\mathcal{N}^k_{L^p}(\mathfrak{O},\Omega) = \Big\{(u_n)_n \in \mathcal{E}^k_{L^p}(\mathfrak{O},\Omega) \,:\, (\forall K \Subset \Omega)(\forall \alpha \in \mathbb{N}^d_0,\ |\alpha| \le k)$$
$$(\forall b \in \mathbb{N}) \left( \sup_{x\in K} \|\partial^\alpha u_n(\cdot,x)\|_{L^p} = \mathcal{O}(n^{-b}) \right) \Big\}.$$

Elements of the vector spaces $\mathcal{E}^k_{M,L^p}(\mathfrak{O},\Omega)$ and $\mathcal{N}^k_{L^p}(\mathfrak{O},\Omega)$ are called moderate and negligible sequences of functions with values in $L^p(\mathfrak{O})$, respectively. The elements of the quotient space

$$\mathcal{G}^k_{L^p}(\mathfrak{O},\Omega) = \mathcal{E}^k_{M,L^p}(\mathfrak{O},\Omega)/\mathcal{N}^k_{L^p}(\mathfrak{O},\Omega)$$

are called CSPs over $\Omega$ with values in $L^p(\mathfrak{O})$.

**Remark 2** *Note that pathwise continuity does not imply $L^p$-continuity and neither does $L^p$-continuity imply pathwise continuity. Counterexamples are given in Appendix C. In all definitions above we require pathwise continuity (a.e.) and pathwise differentiability $k$ times, but in general the mappings $x \mapsto u_n(\cdot,x)$ do not have to be continuous or differentiable with respect to the $L^p$-norm.*

*Another fact to note is that pathwise $C^k$-smoothness for a.e. $\omega \in \mathfrak{O}$ can easily be modified to obtain $C^k$-smoothness for every $\omega \in \mathfrak{O}$. Namely, there may exist at most countably many sets $A_{n,\alpha}$, $n \in \mathbb{N}$, $\alpha \le k$, of probability measure zero on which the mapping $x \mapsto u_n(\omega,x)$ is not of class $\mathcal{C}^k$. Then $A = \cup_{n,\alpha} A_{n,\alpha}$ is also a zero-probability set and we can modify the representatives to be of class $\mathcal{C}^k$ by letting $u_n(\omega,x) = 0$ for $\omega \in A$.*

For the case $k = \infty$, in the above definitions we will omit the superscript $\infty$ and use the notation $\mathcal{G}_{\mathcal{L}}(\mathfrak{O},\Omega)$, $\mathcal{G}_{L^p}(\mathfrak{O},\Omega)$ etc.

We note that the operation of multiplication is not closed in the vector space $\mathcal{E}^k_{M,L^p}(\mathfrak{O},\Omega)$.

**Proposition 1** *Let $k \in \mathbb{N} \cup \{\infty\}$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $(u_n)_n \in \mathcal{N}^k_{L^p}(\mathfrak{O},\Omega)$ (resp. $\mathcal{E}^k_{M,L^p}(\mathfrak{O},\Omega)$) and $(v_n)_n \in \mathcal{E}^k_{L^q}(\mathfrak{O},\Omega)$, then $(u_n v_n)_n \in \mathcal{N}^k_{L^r}(\mathfrak{O},\Omega)$ (resp. $\mathcal{E}^k_{L^r}(\mathfrak{O},\Omega)$).*

Note that the operation of differentiation is not closed in the vector space $\mathcal{E}^k_{M,L^p}(\mathfrak{O},\Omega)$, $k < \infty$. Indeed, if $(u_n)_n \in \mathcal{E}^k_{M,L^p}(\mathfrak{O},\Omega)$ and $\alpha \in \mathbb{N}^d_0$, $|\alpha| \le k$, then there exists $a \in \mathbb{N}$ such that $\sup_{x\in K} \|\partial^\beta(\partial^\alpha u_n(\omega,x))\|_{L^p} = \mathcal{O}(n^a)$, for every $\beta \in \mathbb{N}^d_0$, $|\beta| \le k - |\alpha|$. Therefore, $(\partial^\alpha u_n)_n \in \mathcal{E}^{k-|\alpha|}_{M,L^p}(\mathfrak{O},\Omega)$.

**Definition 3** Let $\mathcal{E}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)=(\mathcal{C}^\infty_{\mathcal{M}^\infty}(\mathfrak{O},\Omega))^{\mathbb{N}}$ be the set of sequences $(u_n(\omega,x))_n$, $\omega\in\mathfrak{O}$, $x\in\Omega$, $n\in\mathbb{N}$, such that the mapping $x\mapsto u_n(\omega,x)$ is in $C^\infty(\Omega)$ for a.e. $\omega\in\mathfrak{O}$, and for every $x\in\Omega$, $u_n(\cdot,x)$ is in $\mathcal{M}^\infty(\mathfrak{O})$. Define:

$$\mathcal{E}_{M,\mathcal{M}^\infty}(\mathfrak{O},\Omega)=\Big\{(u_n)_n\in\mathcal{E}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)\,:\,(\forall K\Subset\Omega)(\forall\alpha\in\mathbb{N}_0^d)(\forall s\in\mathbb{N})$$
$$(\exists a\in\mathbb{N})\left(\sup_{x\in K}|||\partial^\alpha u_n(\cdot,x)|||_s=\mathcal{O}(n^a)\right)\Big\},$$
$$\mathcal{N}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)=\Big\{(u_n)_n\in\mathcal{E}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)\,:\,(\forall K\Subset\Omega)(\forall\alpha\in\mathbb{N}_0^d)(\forall s\in\mathbb{N})$$
$$(\forall b\in\mathbb{N})\left(\sup_{x\in K}|||\partial^\alpha u_n(\cdot,x)|||_s=\mathcal{O}(n^{-b})\right)\Big\}.$$

Elements of $\mathcal{E}_{M,\mathcal{M}^\infty}(\mathfrak{O},\Omega)$ and $\mathcal{N}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)$ are called moderate and negligible sequences of functions with values in $\mathcal{M}^\infty(\mathfrak{O})$, respectively. CSPs with values in $\mathcal{M}^\infty(\mathfrak{O})$ are defined as elements of

$$\mathcal{G}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)=\mathcal{E}_{M,\mathcal{M}^\infty}(\mathfrak{O},\Omega)/\mathcal{N}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega).$$

Note that $\mathcal{E}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)$ and $\mathcal{N}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)$ are algebras. $\mathcal{N}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)$ is an ideal in $\mathcal{E}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)$ and $\mathcal{G}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)$ is an algebra.

If elements of a sequence $(u_n)_n$ do not depend on $x\in\Omega$, then the above definitions reduce to the notions of moderate sequences of random variables $(u_n)_n\in\mathcal{E}_{M,\mathcal{L}}(\mathfrak{O})$ or $(u_n)_n\in\mathcal{E}_{M,L^p}(\mathfrak{O})$ or $(u_n)_n\in\mathcal{E}_{M,\mathcal{M}^\infty}(\mathfrak{O})$ and negligible sequences of random variables $(u_n)_n\in\mathcal{N}_{\mathcal{L}}(\mathfrak{O})$ or $(u_n)_n\in\mathcal{N}_{L^p}(\mathfrak{O})$ or $(u_n)_n\in\mathcal{N}_{\mathcal{M}^\infty}(\mathfrak{O})$. Elements of the corresponding quotient spaces $\mathcal{G}_{\mathcal{L}}(\mathfrak{O})$, $\mathcal{G}_{L^p}(\mathfrak{O})$ and $\mathcal{G}_{\mathcal{M}^\infty}(\mathfrak{O})$ are called generalized random variables with values in $\mathcal{L}(\mathfrak{O})$, $L^p(\mathfrak{O})$ and $\mathcal{M}^\infty(\mathfrak{O})$, respectively.

Let $p\geq q$. Then

$$\mathcal{E}_{M,L^\infty}(\mathfrak{O},\Omega)\to\mathcal{E}_{M,\mathcal{M}^\infty}(\mathfrak{O},\Omega)\to\mathcal{E}_{M,L^p}(\mathfrak{O},\Omega)\to\mathcal{E}_{M,L^q}(\mathfrak{O},\Omega)\to\mathcal{E}_{M,L^1}(\mathfrak{O},\Omega),$$

$$\mathcal{N}_{L^\infty}(\mathfrak{O},\Omega)\to\mathcal{N}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)\to\mathcal{N}_{L^p}(\mathfrak{O},\Omega)\to\mathcal{N}_{L^q}(\mathfrak{O},\Omega)\to\mathcal{N}_{L^1}(\mathfrak{O},\Omega),$$

which means that a sequence $(u_n)_n$ determining an element of the left space determines the element of the right space (this is called canonical mapping) although the mapping is not injective. Therefore, for $p\geq q$, there exist canonical mappings

$$\mathcal{G}_{L^\infty}(\mathfrak{O},\Omega)\to\mathcal{G}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)\to\mathcal{G}_{L^p}(\mathfrak{O},\Omega)\to\mathcal{G}_{L^q}(\mathfrak{O},\Omega)\to\mathcal{G}_{L^1}(\mathfrak{O},\Omega).$$

In [4] (Theorem 1.2.3, p.11), it has been shown that in order that an element $(u_n)_n\in\mathcal{E}_M(\Omega)$ is in $\mathcal{N}(\Omega)$ it is enough to prove negligibility of its zeroth derivative. The following result is a generalization of this fact.

**Proposition 2** *$(u_n)_n\in\mathcal{E}_{M,L^p}(\mathfrak{O},\Omega)$ (resp. $\mathcal{E}_{M,\mathcal{L}}(\mathfrak{O},\Omega)$ or $\mathcal{E}_{M,\mathcal{M}^\infty}(\mathfrak{O},\Omega)$) is negligible if and only if the following condition is satisfied:*

$$(\forall K\Subset\Omega)(\forall b\in\mathbb{N})\left(\sup_{x\in K}\|u_n(\cdot,x)\|_{L^p}=\mathcal{O}(n^{-b})\right)$$

$$\left(\textit{resp. } (\textit{for a.e.}\,\omega\in\mathfrak{O})(\forall K\Subset\Omega)(\forall b\in\mathbb{N})\left(\sup_{x\in K}|\partial^\alpha u_n(\cdot,x)|=\mathcal{O}(n^{-b})\right)\ \textit{or}\right.$$

$$\left.(\forall K\Subset\Omega)(\forall s\in\mathbb{N})(\forall b\in\mathbb{N})\left(\sup_{x\in K}|||\partial^\alpha u_n(\cdot,x)|||_s=\mathcal{O}(n^{-b})\right)\right).$$

**Proof.** If $(u_n)_n \in \mathcal{E}_{M,L^p}(\mathfrak{O},\Omega)$ is negligible, then it trivially satisfies negligibility of the zeroth order derivative.

Suppose that $(u_n)_n \in \mathcal{E}_{M,L^p}(\mathfrak{O},\Omega)$ satisfies negligibility of the zeroth order derivative. By induction, it suffices to show that the same is true for $(\partial_{x_i} u_n)_n$ for any $1 \le i \le d$. Let $K \Subset \Omega$ and set $\delta := \min(1, \operatorname{dist}(K, \partial\Omega))$, $K_1 := K + \overline{B}_{\delta/2}(0)$. Since $(u_n)_n \in \mathcal{E}_{M,L^p}(\mathfrak{O},\Omega)$, there exists $a \in \mathbb{N}$ such that $\sup_{x\in K_1} \|\partial^2_{x_i} u_n(\cdot,x)\|_{L^p} = \mathcal{O}(n^a)$ as $n \to \infty$. From the given condition it follows that for any $b \in \mathbb{N}$, $\sup_{x\in K_1} \|u_n(\cdot,x)\|_{L^p} = \mathcal{O}(n^{-(a+2b)})$. Using Taylor's theorem, we obtain

$$u_n(\cdot, x + n^{-(a+b)} e_i) = u_n(\cdot,x) + \partial_{x_i} u_n(\cdot,x) n^{-(a+b)} + \frac{1}{2}\partial^2_{x_i} u_n(\cdot, x_\theta) n^{-2(a+b)},$$

i.e.

$$\partial_{x_i} u_n(\cdot,x) = (u_n(\cdot, x + n^{-(a+b)} e_i) - u_n(\cdot,x)) n^{a+b} - \frac{1}{2}\partial^2_{x_i} u_n(\cdot, x_\theta) n^{-(a+b)},$$

where $x_\theta = x + \theta n^{-(a+b)} e_i \in K_1$ for some $\theta \in (0,1)$. Therefore,

$$\begin{aligned}\sup_{x\in K} \|\partial_{x_i} u_n(\cdot,x)\|_{L^p} \le & \left(\sup_{x\in K_1} \|u_n(\cdot, x + n^{-(a+b)} e_i)\|_{L^p} + \sup_{x\in K_1} \|u_n(\cdot,x)\|_{L^p}\right) n^{a+b} \\ & + \frac{1}{2} \sup_{x\in K_1} \|\partial^2_{x_i} u_n(\cdot,x)\|_{L^p} n^{-(a+b)} \le C n^{-b}.\end{aligned}$$

□

The notions of generalized expectation and generalized correlation function of CSPs $u$ with values in $L^2(\mathfrak{O})$ are recalled in Appendix B.2. In the sequel we will use this phrase if $u \in \mathcal{G}_{L^2}(\mathfrak{O},\Omega)$ or $\mathcal{G}^k_{L^2}(\mathfrak{O},\Omega)$. In the cases where it is important for $k$ to be specified, we will do that.

## 3 Generalized point values of Colombeau stochastic processes

Generalized point values of Colombeau functions are introduced in [1]; see Appendix A.2 for the definition of the set of compactly supported generalized real numbers $\mathcal{R}_c$ and compactly supported generalized points $\tilde{\Omega}_c$. Now we introduce in a similar manner the notion of a generalized point value of a CSP with values in $\mathcal{L}(\mathfrak{O})$, in $L^p(\mathfrak{O})$ or in $\mathcal{M}^\infty(\mathfrak{O})$, respectively.

**Proposition 3** *Let $u = [(u_n)_n]$ belong to $\mathcal{G}_{\mathcal{L}}(\mathfrak{O},\Omega)$ or $\mathcal{G}_{L^p}(\mathfrak{O},\Omega)$ or $\mathcal{G}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)$. Then, for fixed $\tilde{x} = [(x_n)_n] \in \tilde{\Omega}_c$, $u(\omega,\tilde{x}) = [(u_n(\omega,x_n))_n]$ is a generalized random variable in $\mathcal{G}_{\mathcal{L}}(\mathfrak{O})$ or $\mathcal{G}_{L^p}(\mathfrak{O})$ or $\mathcal{G}_{\mathcal{M}^\infty}(\mathfrak{O})$. It is called the point value of $u$ (in $\mathcal{L}(\mathfrak{O})$ or $L^p(\mathfrak{O})$ or $\mathcal{M}^\infty(\mathfrak{O})$) at the generalized point $\tilde{x} \in \tilde{\Omega}_c$.*

**Proof.** Since $\tilde{x} = [(x_n)_n] \in \tilde{\Omega}_c$, there exists some $K \Subset \Omega$ such that $x_n \in K$ for all $n \in \mathbb{N}$. If $u = [(u_n)_n]$ belongs to $\mathcal{G}_{\mathcal{L}}(\mathfrak{O},\Omega)$, then by definition $u_n(\omega) = u_n(\omega, x_n)$, $\omega \in \mathfrak{O}$, is a measurable function on $\mathfrak{O}$ for any $n \in \mathbb{N}$. Also, for a.e. $\omega \in \mathfrak{O}$, it holds that

$$|u_n(\omega)| = |u_n(\omega,x_n)| \le \sup_{x\in K} |u_n(\omega,x)| \le C n^a,$$

for some $a \in \mathbb{N}$ and therefore $(u_n)_n \in \mathcal{E}_{M,\mathcal{L}}(\mathfrak{O})$. If $u = [(u_n)_n]$ belongs to $\mathcal{G}_{L^p}(\mathfrak{O},\Omega)$ or $\mathcal{G}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)$, then in a similar way it can be shown that $(u_n)_n$ belongs to $\mathcal{E}_{M,L^p}(\mathfrak{O})$ or $\mathcal{E}_{M,\mathcal{M}^\infty}(\mathfrak{O})$.

Let $\tilde{y} = [(y_n)_n] \in \tilde{\Omega}_c$ such that $\tilde{x} \sim \tilde{y}$, i.e. $|x_n - y_n| = \mathcal{O}(n^{-m})$ for any $m \in \mathbb{N}$. If $u \in \mathcal{G}_{\mathcal{L}}(\mathfrak{O}, \Omega)$, then for a.e. $\omega \in \mathfrak{O}$ there exists $a \in \mathbb{N}$ such that $\sup_{x\in K} |\nabla u_n(\omega, x)| = \mathcal{O}(n^a)$. Let us show that $u(\omega, \tilde{x}) - u(\omega, \tilde{y}) \in \mathcal{N}_{\mathcal{L}}(\mathfrak{O})$. For arbitrary $m \in \mathbb{N}$, we have

$$\begin{aligned}|u_n(\omega, x_n) - u_n(\omega, y_n)| &\leq |x_n - y_n| \int_0^1 |\nabla u_n(\omega, x_n + \sigma(y_n - x_n))| \, d\sigma \\ &\leq C_1 n^{-(m+a)} C_2 n^a = C n^{-m},\end{aligned}$$

for a.e. $\omega \in \mathfrak{O}$, since the point $x_n + \sigma(y_n - x_n)$ remains within some compact subset of $\Omega$.

If $u \in \mathcal{G}_{L^p}(\mathfrak{O}, \Omega)$ (the proof can be conducted in a similar way if $u \in \mathcal{G}_{\mathcal{M}^\infty}(\mathfrak{O}, \Omega)$), then there exists $a \in \mathbb{N}$ such that $\sup_{x\in K} \|\nabla u_n(\cdot, x)\|_{L^p} = \mathcal{O}(n^a)$. Using the generalized Minkowski inequality, we obtain

$$\begin{aligned}&\|u_n(\cdot, x_n) - u_n(\cdot, y_n)\|_{L^p} \leq \\ &\leq |x_n - y_n| \left[ \int_{\mathfrak{O}} \left| \int_0^1 |\nabla u_n(\omega, x_n + \sigma(y_n - x_n)| \, d\sigma \right|^p dP(\omega) \right]^{\frac{1}{p}} \\ &\leq |x_n - y_n| \int_0^1 \left[ \int_{\mathfrak{O}} |\nabla u_n(\omega, x_n + \sigma(y_n - x_n))|^p \, dP(\omega) \right]^{\frac{1}{p}} d\sigma \\ &= |x_n - y_n| \int_0^1 \|\nabla u_n(\cdot, x_n + \sigma(y_n - x_n)\|_{L^p} \, d\sigma \\ &\leq C_1 n^{-(m+a)} C_2 n^a \int_0^1 d\sigma \\ &= C n^{-m},\end{aligned}$$

for arbitrary $m \in \mathbb{N}$. Therefore, $u(\omega, \tilde{x}) - u(\omega, \tilde{y}) \in \mathcal{N}_{L^p}(\mathfrak{O})$. □

### 3.1 Measurability of Colombeau stochastic processes

In the definition of CSPs we required measurability of each representative $u_n(\cdot, x)$ i.e. $u_n(\omega, x)$ is a classical stochastic process for $n \in \mathbb{N}$. We also required pathwise smoothness or that a.e. path $x \mapsto u_n(\omega, x)$ is of class $\mathcal{C}^k$. This implies also the joint measurability of the mappings $(\omega, x) \mapsto u_n(\omega, x)$ in $\mathfrak{O} \times \Omega$. Note that by Remark 2 we can assume that the representatives $u_n(\omega, x)$ are defined and of class $\mathcal{C}^k$ for all $\omega \in \mathfrak{O}$. In this section we will prove another fact, that the processes obtained by considering generalized point values of CSPs are also measurable in the appropriate sense.

Let the algebra of generalized numbers $\mathcal{R}_c$ be endowed with the topology generated by the sharp open balls; see Appendix A.3 (also [7,9]). Let $\tilde{x} \in \tilde{\Omega}_c$ and $\omega \in \mathfrak{O}$. Similar to Proposition 3 one has that $(u_n(\omega, x_n))_n$ belongs to $\mathbb{R}_M$.

**Proposition 4** *Let $u = [(u_n)_n]$ belong to $\mathcal{G}_{\mathcal{L}}(\mathfrak{O}, \Omega)$ or $\mathcal{G}_{L^p}(\mathfrak{O}, \Omega)$ or $\mathcal{G}_{\mathcal{M}^\infty}(\mathfrak{O}, \Omega)$. For fixed $\tilde{x} \in \tilde{\Omega}_c$ the mapping $(\mathfrak{O}, \mathfrak{U}) \ni \omega \mapsto u(\omega, \tilde{x}) \in (\mathcal{R}_c, \mathcal{B}(\mathcal{R}_c))$ is measurable, where $\mathcal{B}(\mathcal{R}_c)$ denotes the Borel $\sigma-$algebra generated by the sharp open balls of $\mathcal{R}_c$.*

**Proof.** Let $O = L((y_n)_n, k) = \{[(z_n)_n] \in \mathcal{R}_c : \limsup_{n\to\infty} |y_n - z_n|^{(\log n)^{-1}} < k\}$ be an open ball in $\mathcal{R}_c$. Then,

$$\begin{aligned}&u^{-1}(\cdot, \tilde{x})(O) = \{\omega \in \mathfrak{O} : u(\omega, \tilde{x}) \in O\} \\ &= \{\omega \in \mathfrak{O} : \limsup_{n\to\infty} |u_n(\omega, x_n) - y_n|^{(\log n)^{-1}} < k\} \\ &= \{\omega \in \mathfrak{O} : |u_n(\omega, x_n) - y_n| < n^{\log k} \text{ for all } n > n_0 \text{ for some } n_0 \in \mathbb{N}\} \\ &= \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{\omega \in \mathfrak{O} : |u_m(\omega, x_m) - y_m| < m^{\log k}\}.\end{aligned}$$

This is a measurable set. □

**Remark 3** *Note that the Borel $\sigma-$algebra generated by the sharp open balls is smaller than the Borel $\sigma-$algebra generated by the sharp topology in $\mathcal{R}_c$. This is a consequence of the fact that $\mathcal{R}_c$ is not separable. This fact was also observed in [7]. We show this in the next example.*

**Example 1** *Let $\mathfrak{O} = \mathbb{R}$ be endowed with the $\sigma-$algebra $\mathfrak{U}$ of Lebesgue measurable sets. Consider $u : \mathfrak{O} \to \mathcal{R}_c$ represented by $u_n(\omega) = \omega$, $n \in \mathbb{N}$, i.e. the standard embedding of $\mathbb{R}$ into $\mathcal{R}_c$. Then $u$ is not measurable with respect to the Borel $\sigma-$algebra generated by the sharp topology. Let us show this. Take a set $E \subseteq \mathbb{R} = \mathfrak{O}$ which is not Lebesgue measurable. Denote by $L(\tilde{x}, p)$ the sharp open ball with center $\tilde{x} = [(x_n)_n] \in \mathcal{R}_c$, i.e. the equivalence class of elements $\tilde{y} = [(y_n)_n] \in \mathcal{R}_c$ such that $|y_n - x_n| \le n^{-p}$, $n \to \infty$, for $p > 0$. Take $\tilde{x} = u(\omega_0)$ for some $\omega_0 \in \mathfrak{O}$. Then $u^{-1}(L(\tilde{x}, p)) = \{\omega \in \mathfrak{O} : |\omega - \omega_0| \le n^{-p}, n \to \infty\} = \{\omega_0\}$ is Lebesgue-measurable, which is in compliance with Proposition 4.*

*On the other hand, the set $V = \bigcup_{\omega \in E} L(u(\omega), p)$ is open in the sharp topology, but $u^{-1}(V) = E$ is not Lebesgue measurable.* □

## 4 Generalized characteristic functions of CSPs

Every classical stochastic process $u$ is uniquely defined via its finite-dimensional distributions, while those are uniquely defined via their characteristic functions. Thus, the information about $E(e^{it(u(\cdot,x_1),u(\cdot,x_2),\ldots,u(\cdot,x_m))})$, $m \in \mathbb{N}$, provides enough to determine the process itself. In this section we will introduce the concept of characteristic functions for Colombeau stochastic processes. For technical simplicity we will consider only the one-dimensional distributions $E(e^{itu(\cdot,x)})$, but one can easily carry out the proofs also for all finite-dimensional distributions.

### 4.1 Generalized characteristic functions of CSPs in $\mathcal{G}_{\mathcal{M}^\infty}(\mathfrak{O}, \Omega)$

Let $u = [(u_n)_n]$ be a CSP with values in $\mathcal{M}^\infty(\mathfrak{O})$. Let us show that $(E(e^{itu_n(\cdot,x)}))_n \in \mathcal{E}_M(\mathbb{R} \times \Omega)$. Using that the paths are smooth, it can be shown that $(E(e^{itu_n(\cdot,x)}))_n \in \mathcal{E}(\mathbb{R} \times \Omega)$. We prove moderateness. Let $K_1 = [-t_0, t_0] \times K$, where $K \Subset \Omega$ and $t_0 \in \mathbb{R}$. Thus,

$$\sup_{(t,x)\in K_1} |\partial_t^k E(e^{itu_n(\cdot,x)})| \le \sup_{x\in K} \|u_n(\cdot,x)\|_{L^k}^k \le \sup_{x\in K} |||u_n(\cdot,x)|||_k^k \le Cn^{ak},$$

for some $a \in \mathbb{N}$, since $(u_n)_n \in \mathcal{E}_{M,\mathcal{M}^\infty}(\mathfrak{O}, \Omega)$. The $m$th order derivative with respect to the variable $x$ of $E(e^{itu_n(\cdot,x)})$ is a linear combination of members of the form $E((u_n^{(i_1)}(\cdot,x))^{k_1} \cdot \ldots \cdot (u_n^{(i_s)}(\cdot,x))^{k_s} e^{itu_n(\cdot,x)})$, where $i_1k_1 + \ldots + i_sk_s = m$. So the proof of moderateness is the same as for $E((u_n'(\cdot,x))^m e^{itu_n(\cdot,x)})$ :

$$\sup_{(t,x)\in K_1} \left| \int_{\mathfrak{O}} (u_n'(\omega,x))^k e^{itu_n(\omega,x)} dP(\omega) \right| \le \sup_{x\in K} \|u_n\|_{L^k}^k \le \sup_{x\in K} |||u_n|||_k^k \le Cn^{ak},$$

for some $a \in \mathbb{N}$, since $(u_n)_n \in \mathcal{E}_{M,\mathcal{M}^\infty}(\mathfrak{O}, \Omega)$. (Here and henceforth $u_n'$ denotes some first order derivative of $u_n$.) In a similar way we estimate the mixed derivatives.

Let $u = [(u_n)_n]$ be a CSP with values in $\mathcal{M}^\infty(\mathfrak{O})$. Suppose that $(v_n)_n$ is a negligible sequence of function with values in $\mathcal{M}^\infty(\mathfrak{O})$. Let us show that $(E(e^{it(u_n+v_n)}) -$

$E(e^{itu_n}))_n$ is a negligible sequence of functions. Let $b \in \mathbb{N}$ be arbitrary. Using the mean value theorem we obtain

$$\begin{aligned}&\sup_{(t,x)\in K_1}\left|E\left(e^{it(u_n+v_n)(\cdot,x)}\right)-E\left(e^{itu_n(\cdot,x)}\right)\right| \leq \sup_{(t,x)\in K_1}\int_{\mathfrak{O}}|e^{itv_n(\omega,x)}-1|\,dP(\omega)\\ &\leq \sup_{(t,x)\in K_1}|t|\int_{\mathfrak{O}}|v_n(\omega,x)||e^{i\theta_t v_n(\omega,x)}|\,dP(\omega)\leq C\sup_{x\in K}\int_{\mathfrak{O}}|v_n(\omega,x)|\,dP(\omega)\\ &= \mathcal{O}(n^{-b}),\end{aligned}$$

where $\theta_t$ lies on the segment $(0,t)$ (or $(t,0)$). We proved the negligibility of the zeroth derivative and by [4] this is sufficient. Therefore, $[(E(e^{itu_n(\cdot,x)}))_n]$ does not depend on the representative and it is a well-defined element of $\mathcal{G}(\mathbb{R}\times\Omega)$. This enables us to introduce the next definition.

**Definition 4** Let $u = [(u_n)_n] \in \mathcal{G}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)$. Then

$$L_u(t,x) = [(L_{u_n}(t,x))_n] = [(E(e^{itu_n(\cdot,x)}))_n] \in \mathcal{G}(\mathbb{R}\times\Omega),\ \ t\in\mathbb{R},\, x\in\Omega,$$

is called the generalized characteristic function of $u$.

The generalized characteristic function $L_u(t,x)$ of a CSP $u \in \mathcal{G}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)$ is positive-definite in $t$ for every $x \in \Omega$. The proof is the same as the well known one for the classical characteristic function.

Note that if $u = [(u_n)_n]$ is a CSP with values in $\mathcal{M}^\infty(\mathfrak{O})$, then $L_u(\tilde{t},\tilde{x}) = E(e^{i\tilde{t}u(\cdot,\tilde{x})})$, for every $\tilde{x} \in \tilde{\Omega}_c$ and $\tilde{t} \in \mathcal{R}_c$. Hence, we have $L(\tilde{0},\tilde{x}) = \tilde{1}$ (where $\tilde{0} = (0,0,0,\ldots,0,\ldots)$, $\tilde{1} = (1,1,1,\ldots,1,\ldots)$) for every $\tilde{x} = [(x_n)_n] \in \tilde{\Omega}_c$, since $L_{u_n}(0,x_n) = 1$ for every $n \in \mathbb{N}$.

## 4.2 Embedding results

Let $u(\omega,x)$, $\omega \in \mathfrak{O}$, $x \in \Omega$, be a stochastic process such that $u(\omega,\cdot) \in L^1_{loc}(\Omega)$ for a.e. $\omega \in \mathfrak{O}$. Moreover, assume that $u(\cdot,x) \in \mathcal{M}^\infty(\mathfrak{O})$ for every $x \in \Omega$. Let $(\kappa_n)_n$ be a sequence of smooth functions supported by $\Omega_{-1/n} = \left\{x \in \Omega : d(x,\mathbb{R}^d\setminus\Omega) > \frac{1}{n}\right\}$, $n \geq n_0$, such that $\kappa_n \equiv 1$ on $\Omega_{-2/n}$, $n > n_0$. The embedding of $u$ into the Colombeau algebra $\mathcal{G}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)$ is given by $u \mapsto [(u_n)_n]$, where

$$u_n(\omega,x) = (u\kappa_n * \varphi_n)(\omega,x) = (u*\varphi_n)(\omega,x),$$

(cf. (1) for the definition of $\varphi_n$) for sufficiently large $n$; see Appendix B.1. One can prove that the sequence of characteristic functions $(L_{u_n}(t,x))_n = (E(e^{itu_n(\cdot,x)}))_n$ is in $\mathcal{E}_M(\mathbb{R}\times\Omega)$. (It would not hold without the assumption $u(\cdot,x) \in \mathcal{M}^\infty(\mathfrak{O})$.)

**Remark 4** *Let $u(\omega,\cdot) \in L^1_{loc}(\Omega)$ for a.e. $\omega \in \mathfrak{O}$ and assume that $u(\cdot,x)$ is a measurable function for all $x \in \Omega$ and belongs to $\mathcal{M}^\infty(\mathfrak{O})$. Then $\mathcal{C}_0^\infty(\Omega) \ni \phi \mapsto \int u(\cdot,x)\phi(x)\,dx$ is a strongly continuous mapping $\mathcal{C}_0^\infty(\Omega) \to L^p(\mathfrak{O})$, $p \geq 1$; see Appendix B.1.*

**Proposition 5** *Let $\phi \in \mathcal{C}^\infty_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)$ and assume that $\sup_{x\in K}\|\phi^{(\alpha)}(\cdot,x)\|_{L^p} < \infty$ for every $\alpha \in \mathbb{N}_0$ and every $K \Subset \Omega$. Let $\phi_n(\omega,x) = (\phi(\omega,\cdot)*\varphi_n(\cdot))(x)$, $x \in \Omega$, $\omega \in \mathfrak{O}$, for sufficiently large $n$. Then $(\phi_n(\omega,x))_n - (\phi(\omega,x))_n \in \mathcal{N}_{\mathcal{M}^\infty}(\mathfrak{O},\Omega)$ and $(L_{\phi_n}(t,x))_n - (L_\phi(t,x))_n \in \mathcal{N}(\mathbb{R}\times\Omega)$, where $(\phi)_n$ is a constant sequence.*

**Proof.** One can prove easily that $(\phi_n(\omega,x))_n - (\phi(\omega,x))_n \in \mathcal{E}_{M,\mathcal{M}^\infty}(\mathfrak{O},\Omega)$. Hence, for the proof of negligibility, by [4], it is enough to prove the negligibility of the zeroth order derivative. For simplicity of exposition, we work out the case $d = 1$ only, i.e., $\Omega \subset \mathbb{R}$. First,

$$
\begin{aligned}
&|||\phi_n(\omega,x) - \phi(\omega,x)|||_s = \\
&= \sup_{1\le p\le s} \left( \int_{\mathfrak{O}} |(\phi(\omega,\cdot) * \varphi_n(\cdot))(x) - \phi(\omega,x)|^p \, dP(\omega) \right)^{1/p} \\
&= \sup_{1\le p\le s} \left( \int_{\mathfrak{O}} \left| \int_{\Omega} \left( \phi\left(\omega, x - \frac{z}{n}\right) - \phi(\omega,x) \right) \varphi(z)\, dz \right|^p \, dP(\omega) \right)^{1/p}
\end{aligned}
$$

By Taylor's formula and the fact that $\int z^k \varphi(z)\, dz = 0$ for all $k \in \mathbb{N}$, the inner integral equals

$$
\begin{aligned}
&\int_{\Omega} \left( \sum_{k=1}^{b-1} \frac{\phi^{(k)}(\omega,x)}{k!} \left(-\frac{z}{n}\right)^k + \frac{\phi^{(b)}\left(\omega, x - \theta_n \frac{z}{n}\right)}{b!} \left(-\frac{z}{n}\right)^b \right) \varphi(z)\, dz \\
&\qquad\qquad = \int_{\Omega} \frac{\phi^{(b)}\left(\omega, x - \theta_n \frac{z}{n}\right)}{b!} \left(-\frac{z}{n}\right)^b \varphi(z)\, dz
\end{aligned}
$$

where $0 < \theta_n < 1$. By Minkowski's inequality,

$$
\begin{aligned}
&|||\phi_n(\omega,x) - \phi(\omega,x)|||_s \\
&\le \sup_{1\le p\le s} \frac{1}{b! n^b} \int_{\Omega} \left( \int_{\mathfrak{O}} \left| \phi^{(b)}\left(\omega, x - \theta_n \frac{z}{n}\right) \right|^p dP(\omega) \right)^{1/p} |z|^b |\varphi(z)|\, dz\,.
\end{aligned}
$$

The points $x - \theta_n \frac{z}{n}$, $n \in \mathbb{N}$, remain within some compact set. Since by assumption $\phi^{(b)}$ is uniformly bounded on compact sets with respect to the $L^p$-norm, it follows that the above expression is uniformly bounded by $Cn^{-b}$ on every compact set for every $s \in \mathbb{N}$ and for every $b \in \mathbb{N}$ ($C$ depends on $b$ and $s$).

The fact that $(L_{\phi_n}(t,x))_n - (L_{\phi}(t,x))_n \in \mathcal{N}(\mathbb{R} \times \Omega)$ is proven along the same lines. ☐

**Proposition 6** *If $f \in \mathcal{C}_{\mathcal{M}^\infty}(\mathfrak{O}, \Omega)$, then $[(L_{f*\varphi_n}(t,x))_n]$ is associated to the Colombeau generalized function with representative $(L_f(t,\cdot) * \varphi_n(\cdot))(x)$, $t \in \mathbb{R}$, $x \in \Omega$.*

Note that if $f \in \mathcal{C}_{\mathcal{M}^\infty}(\mathfrak{O}, \Omega)$, then the mapping $x \mapsto f(\omega,x)$ is in $C(\Omega)$ for a.e. $\omega \in \mathfrak{O}$, and for every $x \in \Omega$, $f(\cdot,x)$ is in $\mathcal{M}^\infty(\mathfrak{O})$.

**Proof.** Let $\theta(t,x) \in \mathcal{D}(\mathbb{R} \times \Omega)$. Then

$$
\begin{aligned}
&\int_{\mathbb{R}\times\Omega} (L_{f*\varphi_n}(t,x) - (L_f(t,\cdot) * \varphi_n(\cdot))(x))\, \theta(t,x)\, dtdx = \\
&= \int_{\mathbb{R}\times\Omega} \left( \int_{\mathfrak{O}} e^{it \int_{\mathbb{R}} f(\omega,x-y)\varphi_n(y)\, dy}\, dP(\omega) \right. \\
&\quad \left. - \int_{\mathfrak{O}} \int_{\mathbb{R}} e^{itf(\omega,x-y)} \varphi_n(y)\, dy\, dP(\omega) \right) \theta(t,x)\, dtdx \\
&= \int_{\mathbb{R}\times\Omega} \int_{\mathfrak{O}} \left( e^{it \int_{\mathbb{R}} f(\omega,x-\frac{z}{n})\varphi(z)\, dz} - \int_{\mathbb{R}} e^{itf(\omega,x-\frac{z}{n})}\, \varphi(z)\, dz \right) \theta(t,x)\, dP(\omega)\, dt\, dx,
\end{aligned}
$$

where we used Fubini's theorem and a change of variable $ny = z$. Now, letting $n \to \infty$ we obtain by the Lebesgue dominated convergence theorem that

$$
\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}\times\Omega} (L_{f*\varphi_n}(t,x) - (L_f(t,\cdot) * \varphi_n(\cdot))(x))\, \theta(t,x)\, dtdx \\
&\to \int_{\mathbb{R}\times\Omega} \int_{\mathfrak{O}} \left( e^{it \int_{\mathbb{R}} f(\omega,x)\varphi(z)\, dz} - \int_{\mathbb{R}} e^{itf(\omega,x)}\, \varphi(z)\, dz \right) \theta(t,x)\, dP(\omega)\, dt\, dx \\
&= \int_{\mathbb{R}\times\Omega} \int_{\mathfrak{O}} \left( e^{itf(\omega,x) \int_{\mathbb{R}} \varphi(z)\, dz} - e^{itf(\omega,x)} \int_{\mathbb{R}} \varphi(z)\, dz \right) \theta(t,x)\, dP(\omega)\, dt\, dx \\
&= 0.
\end{aligned}
$$

□

4.3 Generalized characteristic functions of CSPs in $\mathcal{G}^k_{L^{kp}}(\mathfrak{O},\Omega)$

Suppose that $k \leq p$. We will consider elements of $\mathcal{G}^k_{L^{kp}}(\mathfrak{O},\Omega)$.

**Proposition 7** *If* $(u_n)_n \in \mathcal{E}^k_{M,L^{kp}}(\mathfrak{O},\Omega)$, *then* $(e^{itu_n(\omega,x)})_n \in \mathcal{E}^k_{M,L^p}(\mathfrak{O},\mathbb{R}\times\Omega)$ *and* $(E(e^{itu_n(\cdot,x)}))_n \in \mathcal{E}^k_M(\mathbb{R}\times\Omega)$.

**Proof.** Denote $K_1 = [-t_0, t_0] \times K$, where $K \Subset \Omega$ and $t_0 \in \mathbb{R}$. Note that the derivative of order $m \leq k$ of $e^{itu_n(\omega,x)}$ is a linear combination of members of the form $(u_n^{(i_1)}(\omega,x))^{k_1}\cdots(u_n^{(i_s)}(\omega,x))^{k_s}e^{itu_n(\omega,x)}$, where $i_1k_1+\ldots+i_sk_s = m$. For example, for the member $(u_n'(\omega,x))^k e^{itu_n(\omega,x)}$, we have

$$\left(\int_{\mathfrak{O}} |(u_n'(\omega,x))^k e^{itu_n(\omega,x)}|^p\,dP\right)^{\frac{1}{p}} = \left(\int_{\mathfrak{O}} \underbrace{|u_n'(\omega,x)|^p\cdots|u_n'(\omega,x)|^p}_{k}\,dP\right)^{\frac{1}{p}}$$

$$\leq \underbrace{\left(\int_{\mathfrak{O}} |u_n'(\omega,x)|^{pk}\,dP\right)^{\frac{1}{kp}}\cdots\left(\int_{\mathfrak{O}} |u_n'(\omega,x)|^{pk}\,dP\right)^{\frac{1}{kp}}}_{k} = \|u_n'(\cdot,x)\|^k_{L^{kp}}.$$

Now, using $(u_n)_n \in \mathcal{E}^k_{M,L^{kp}}(\mathfrak{O},\Omega)$, we obtain that

$$\sup_{(t,x)\in K_1} \|(u_n'(\cdot,x))^k e^{itu_n(\cdot,x)}\|_{L^p} = \mathcal{O}(n^a),$$

for some $a \in \mathbb{N}$. In a similar way we estimate higher order derivatives. Hence, $(e^{itu_n(\omega,x)})_n \in \mathcal{E}^k_{M,L^p}(\mathfrak{O},\mathbb{R}\times\Omega)$.

Using that the paths are of class $\mathcal{C}^k$, it can be shown that $(E(e^{itu_n(\cdot,x)}))_n \in \mathcal{E}^k(\mathbb{R}\times\Omega)$. Let $j,k \in \mathbb{N}$ such that $j+l \leq k$. Using the previously proven estimates, we obtain

$$\sup_{(t,x)\in K_1} |\partial_t^j \partial_x^l E(e^{itu_n(\cdot,x)})| \leq \sup_{(t,x)\in K_1} \|\partial_t^j \partial_x^l e^{itu_n(\cdot,x)}\|_{L^p} \leq Cn^a,$$

for some $a \in \mathbb{N}$. Therefore, $(E(e^{itu_n(\cdot,x)}))_n \in \mathcal{E}^k_M(\mathbb{R}\times\Omega)$. □

Proposition 7 enables us to introduce the next definition.

**Definition 5** Let $u \in \mathcal{G}^k_{L^{kp}}(\mathfrak{O},\Omega)$. Then

$$L_u(t,x) = [(L_{u_n}(t,x))_n] = [(E(e^{itu_n(\cdot,x)})_n] \in \mathcal{G}^k(\mathbb{R}\times\Omega), \quad t\in\mathbb{R},\, x\in\Omega,$$

is called the generalized characteristic function of $u$.

4.4 Calculating the expectation and correlation function.

The generalized characteristic function determines the process, thus both the expectation and the correlation function can be retrieved from it, as well as all the higher order moments that exist. For the purpose of the second moments (i.e. the correlation function) we will denote by

$$L_u(t,s;x,y) = E(e^{i(t,s)\centerdot(u(\cdot,x),u(\cdot,y))}) = E(e^{itu(\cdot,x)}e^{isu(\cdot,y)}), \quad t,s\in\mathbb{R},\ x,y\in\Omega,$$

the characteristic function of the joint distribution of the random field $(u(\omega,x),u(\omega,y))$. Here $\centerdot$ denotes the scalar product in $\mathbb{R}^2$.

**Theorem 1** *Let $u = [(u_n)_n] \in \mathcal{G}_{\mathcal{M}^\infty}(\mathfrak{O}, \Omega)$, resp. $u \in \mathcal{G}^k_{L^{kp}}(\mathfrak{O}, \Omega)$, for $k \geq 1$, $p \geq 2$, and let $L_u(t,x) \in \mathcal{G}(\mathbb{R}\times\Omega)$, resp. $L_u(t,x) \in \mathcal{G}^k(\mathbb{R}\times\Omega)$ be its generalized characteristic function. Furthermore, let $L_u(t,s;x,y) \in \mathcal{G}(\mathbb{R}^2 \times \Omega^2)$, resp. $L_u(t,s;x,y) \in \mathcal{G}^k(\mathbb{R}^2 \times \Omega^2)$, be the characteristic function of the joint distributions.*

*Then, the generalized expectation $m \in \mathcal{G}(\Omega)$, resp. $m \in \mathcal{G}^k(\Omega)$ satisfies*

$$m(x) = i^{-1}\frac{d}{dt}L_u(t,x)|_{t=0}.$$

*The generalized correlation function $B \in \mathcal{G}(\Omega\times\Omega)$, resp. $m \in \mathcal{G}^k(\Omega\times\Omega)$ satisfies*

$$B(x,y) = -\frac{d}{dt}\frac{d}{ds}L_u(t,s;x,y)|_{(t,s)=(0,0)}.$$

**Proof.** Let $u$ be given by the representative $[(u_n(\omega,x))_n]$. Then, $\frac{d}{dt}L_{u_n}(t,x) = \frac{d}{dt}E(e^{itu_n(\cdot,x)}) = E(iu_n(\cdot,x)e^{itu_n(\cdot,x)})$ and thus $\frac{d}{dt}L_{u_n}(0,x) = iE(u_n(\cdot,x)) = im_n(x)$. Similarly,

$$\begin{aligned}\frac{d}{dt}\frac{d}{ds}L_{u_n}(t,s;x,y) &= \frac{d}{dt}\frac{d}{ds}E(e^{itu_n(\cdot,x)}e^{isu_n(\cdot,y)})\\ &= -E(u_n(\cdot,x)u_n(\cdot,y)e^{i(tu_n(\cdot,x)+su_n(\cdot,y))})\end{aligned}$$

and by smoothness of the representatives $\frac{d}{dt}\frac{d}{ds}L_{u_n}(t,s;x,y) = \frac{d}{ds}\frac{d}{dt}L_{u_n}(t,s;x,y)$. Now, $\frac{d}{dt}\frac{d}{ds}L_{u_n}(0,0;x,y) = -E(u_n(\cdot,x)u_n(\cdot,y)) = -B_n(x,y)$. □

## 4.5 Examples of the generalized characteristic function

The definition and the basic assertions related to Gaussian CSPs, abbreviated as GCSPs, are given in Appendix B.3. We provide in this section examples of the generalized characteristic function related to GCSPs.

**Example 2** *Let us compute the generalized characteristic function of a GCSP $u = [(u_n)_n] \in \mathcal{G}^1_{L^2}(\mathfrak{O}, \mathbb{R})$. Let $B = [(B_{u_n})_n] \in \mathcal{G}^1(\mathbb{R}^2)$ be the generalized correlation function of $u$ determined by a Gaussian representative $(u_n)_n$. According to Appendix B.3, the distribution function of $u_n$ is*

$$P(u_n(\omega,x) \in (-\infty,b)) = \frac{1}{\sqrt{2\pi B_{u_n}(x,x)}}\int_{-\infty}^{b}\exp\left(-\frac{s^2}{2B_{u_n}(x,x)}\right)ds,$$

*so we obtain*

$$L_{u_n}(t,x) = \int_{\mathbb{R}} e^{itu_n(\omega,x)}\,dP(\omega) = \exp\left(-\frac{1}{2}B_{u_n}(x,x)t^2\right).$$

*Therefore, the generalized characteristic function of a GCSP $u$ is*

$$L_u(t,x) = \exp\left(-\frac{1}{2}B(x,x)t^2\right) \in \mathcal{G}^1(\mathbb{R}^2).$$

□

**Example 3** *White noise $w = [(w_n)_n] \in \mathcal{G}^1_{L^2}(\mathfrak{O}, \mathbb{R})$ is a GCSP with zero generalized expectation and with a representative of the generalized correlation function $B_{w_n}(x,y) = \varphi_n(x-y)$, $x,y \in \mathbb{R}$, $n \in \mathbb{N}$; for details see [8]. Since $B_{w_n}(x,x) = \varphi_n(x-x) = \varphi_n(0)$, we obtain $L_{w_n}(t,x) = \exp\left(-\frac{1}{2}\varphi_n(0)t^2\right)$, $n \in \mathbb{N}$.* □

## 5 Characterizations of the generalized correlation function

It is known from [5], that all Schwartz distributions are in fact of the form

$$\phi \mapsto \sum \int f_\alpha \partial^\alpha \phi \, dx, \quad \phi \in \mathcal{C}_0^\infty(\Omega),$$

where the sum is locally finite, i.e. on every compact set there are only a finite number of continuous functions $f_\alpha$ which do not vanish identically.

In this section we will assume that the generalized expectation of a given process equals zero.

Let $\xi$ be a distributional stochastic process on $\Omega \subseteq \mathbb{R}^d$. We refer to Appendix B.1 for the definition of the associated CSP $Cd(\xi) = [(u_n)_n]$, where $u_n = \kappa_n \xi * \varphi_n$, $n \in \mathbb{N}$.

### 5.1 Structural theorems

Denote $Q_\Omega = \{(x,y) \in \Omega \times \Omega : \ x \neq y\}$ and $D_\Omega = \Omega \times \Omega \setminus Q = \{(x,y) \in \Omega \times \Omega : \ x = y\}$. In the case $\Omega = \mathbb{R}^d$, we use notation $Q = \{(x,y) \in \mathbb{R}^{2d} : x \neq y\}$ and $D = \mathbb{R}^{2d} \setminus Q$.

**Theorem 2** *Let $B = [(B_n)_n] \in \mathcal{G}(\Omega \times \Omega)$ be a generalized correlation function which corresponds to a CSP $u = [(u_n)_n]$ over $\Omega$ with values in $L^2(\mathfrak{O})$ such that $u = Cd(\xi)$, where $\xi$ is a distributional stochastic process on $\Omega$.*

*a) Let $F \in \mathcal{D}'(\Omega \times \Omega)$ be the correlation functional of $\xi$. Then $B$ is given by $B = Cd(F)$.*

*b) $B(\tilde{x}, \tilde{y}) = 0$ for all $(\tilde{x}, \tilde{y}) \in (\tilde{Q}_\Omega)_c$, if and only if $F$ is concentrated on the diagonal $x = y$, i.e. $\operatorname{supp} F \subseteq D_\Omega$.*

*c) If $B(\tilde{x}, \tilde{y}) = 0$ for all $(\tilde{x}, \tilde{y}) \in (\tilde{Q}_\Omega)_c$, then $B$ is associated to a generalized function which has a representative of the form*

$$B_n^*(x,y) = \int_\Omega \sum_{j,k \in \mathbb{N}_0} R_{j,k}(s) \varphi_n^{(j)}(x-s) \varphi_n^{(k)}(y-s) ds, \quad x,y \in \Omega, \tag{2}$$

*where for every $n \in \mathbb{N}$ only a finite number of continuous functions $R_{j,k}$ are different from zero on any compact subset of $\Omega$.*

**Proof.** a) Let us show that $B = Cd(F)$. We have

$$\begin{aligned} B_n(x,y) &= \int_{\mathfrak{O}} \left( \int_\Omega \kappa_n(s) \xi(\omega,s) \varphi_n(x-s) \, ds \right) \left( \int_\Omega \kappa_n(s) \xi(\omega,t) \varphi_n(y-t) \, dt \right) dP \\ &= \iint_{\Omega \times \Omega} \kappa_n(s) \kappa_n(t) \varphi_n(x-s) \varphi_n(y-t) \left( \int_{\mathfrak{O}} \xi(\omega,s) \xi(\omega,t) \, dP(\omega) \right) dsdt \\ &= \iint_{\Omega \times \Omega} \kappa_n(s) \kappa_n(t) F(s,t) \varphi_n(x-s) \varphi_n(y-t) \, dsdt \\ &= \big( \kappa_n(x) \kappa_n(y) F(x,y) \big) * \varphi_n(x) \varphi_n(y). \end{aligned}$$

b) We know that $\operatorname{supp} B = \operatorname{supp} F$ for embedded distributions. Now, $F \equiv 0$ in $Q_\Omega$ is equivalent to $B \equiv 0$ in $Q_\Omega$ and Theorem 3 implies the statement.

c) From a) and b) we have $B = Cd(F)$ and $F$ is concentrated on $D_\Omega$. The generalized function $F$ has the form

$$\langle F, \theta \rangle = \int_{\Omega \times \Omega} \sum_{j,k \in \mathbb{N}_0} Q_{j,k}(x,y) \frac{\partial^{j+k}}{\partial x^j \partial y^k} \theta(x,y) dxdy, \ \ \theta \in \mathcal{D}(\Omega \times \Omega),$$

where the $Q_{j,k}(x,y)$ are continuous functions, only a finite number of which are different from zero on any compact set of $\Omega$. Since $F$ is concentrated on the diagonal $D_\Omega$, we obtain that

$$\langle F,\theta\rangle = \int_\Omega \sum_{j,k\in\mathbb{N}_0} R_{j,k}(x)\left(\frac{\partial^{j+k}}{\partial x^j \partial y^k}\theta(x,y)\right)\Big|_{x=y} dx,$$

where we have put $R_{j,k}(x) = Q_{j,k}(x,x)$. The form of $F$ over $\mathbb{R}^d$ was mentioned in [2], page 287. A version of this theorem is also given in Theorem 2.3.5 in [5] for compactly supported distributions. We apply the quoted result of [5] but rewritten in the form of [2].

Put

$$B_n^*(x,y) = \int_\Omega \sum_{j,k\in\mathbb{N}_0} R_{j,k}(s)\varphi_n^{(j)}(x-s)\varphi_n^{(k)}(y-s)ds, \;\; x,y\in\Omega,\; n\in\mathbb{N}.$$

We will show that $[(B_n^*)_n] \approx F$, from which it will follow that $[(B_n^*)_n] \approx B$. For any $\phi \in \mathcal{D}(\Omega\times\Omega)$ we have

$$\begin{aligned}
&\iint_{\Omega\times\Omega} B_n^*(x,y)\phi(x,y)dxdy = \\
&= \sum_{j,k\in\mathbb{N}_0} n^2 \iiint_{\Omega\times\Omega\times\Omega} R_{j,k}(s)\frac{\partial^j}{\partial x^j}\varphi\left(n(x-s)\right)\frac{\partial^k}{\partial y^k}\varphi\left(n(y-s)\right)\phi(x,y)dxdyds \\
&= \sum_{j,k\in\mathbb{N}_0} n^2 \iiint_{\Omega\times\Omega\times\Omega} R_{j,k}(s)\varphi\left(n(x-s)\right)\varphi\left(n(y-s)\right)\frac{\partial^{j+k}}{\partial x^j\partial y^k}\phi(x,y)dxdyds
\end{aligned}$$

where we applied partial integration in the last step. Now, with $t = n(x-s)$, $z = n(y-s)$, we obtain

$$\begin{aligned}
&\iint_{\Omega\times\Omega} B_n^*(x,y)\phi(x,y)dxdy = \\
&= \sum_{j,k\in\mathbb{N}_0} \iiint_{\Omega\times\Omega\times\Omega} R_{j,k}(s)\varphi(t)\varphi(z)A\left(s+\frac{t}{n}, s+\frac{z}{n}\right)dtdzds,
\end{aligned}$$

where $A(s+t/n, s+z/n) = \frac{\partial^{j+k}}{\partial x^j\partial y^k}\phi(x,y)\mid_{x=s+t/n, y=s+z/n}$.

Letting $n\to\infty$ we obtain by the Lebesgue dominated convergence theorem that

$$\begin{aligned}
\iint_{\Omega\times\Omega} B_n^*(x,y)\phi(x,y)dxdy \;&\to \int_\Omega \varphi(t)dt \int_\Omega \varphi(z)dz \int_\Omega \textstyle\sum_{j,k\in\mathbb{N}_0} R_{j,k}(s)A(s,s)ds \\
&= \int_\Omega \textstyle\sum_{j,k\in\mathbb{N}_0} R_{j,k}(s)\frac{\partial^{j+k}}{\partial x^j\partial y^k}\phi(x,y)\mid_{x=y=s} ds = \langle F,\phi\rangle.
\end{aligned}$$

Thus, $[(B_n^*)_n] \approx F$. Since $B = Cd(F) \approx F$, it follows $[(B_n^*)_n] \approx B$. □

**Proposition 8** *Let $B = [(B_n)_n] \in \mathcal{G}(\Omega\times\Omega)$ be a generalized correlation function of a CSP $u = [(u_n)_n]$ over $\Omega$ with values in $L^2(\mathfrak{O})$. Suppose that $B$ is associated to $F \in \mathcal{D}'(\Omega\times\Omega)$. If $B(\tilde{x},\tilde{y}) = 0$ for all $(\tilde{x},\tilde{y}) \in (\tilde{Q}_\Omega)_c$, then*

*a) $F$ is concentrated on $D_\Omega$,*

*b) $B$ is associated to a generalized function which has a representative of the form (2).*

**Proof.** a) Let $B \approx F$, $F \in \mathcal{D}'(\Omega \times \Omega)$. Let $B(\tilde{x}, \tilde{y}) = 0$ for all $(\tilde{x}, \tilde{y}) \in (\tilde{Q}_\Omega)_c$. By Theorem 3, $B\big|_{Q_\Omega} = 0$ in $\mathcal{G}(Q_\Omega)$. In particular, $\sup_{(x,y)\in K} |B_n(x,y)| \to 0$ as $n \to \infty$, for every compact subset $K$ of $Q_\Omega$. We prove that $F$ is concentrated on the diagonal, i.e. $\langle F, \theta\rangle = 0$ for all $\theta \in \mathcal{D}(\Omega \times \Omega)$ with $\operatorname{supp}\theta \subset Q_\Omega$. Indeed, for such $\theta$,

$$\langle F, \theta\rangle = \lim_{n\to\infty} \iint_{\Omega\times\Omega} B_n(x,y)\theta(x,y)dxdy = 0.$$

b) From a) it follows that $F$ is concentrated on $D_\Omega$. The rest of proof is analogous to the proof of Theorem 2 c). □

5.2 Examples of the generalized correlation function of GCSPs.

The following example shows that if $B = [(B_n)_n] \in \mathcal{G}(\Omega \times \Omega)$ is a generalized correlation function of some GCSP and $B$ is associated to $F \in \mathcal{D}'(\Omega \times \Omega)$ which is concentrated on the diagonal $D$, then it does not necessarily follow that $\operatorname{supp} B \subseteq \tilde{D}_c$.

**Example 4** *Let $\varphi \in \mathcal{S}(\mathbb{R})$ be a positive-definite function such that $\int \varphi(x)\,dx = 1$. Define $B_n(x,y) = n\varphi(n(x-y)) + \frac{1}{n}$, $x, y \in \mathbb{R}$, $n \in \mathbb{N}$. Since the sum of two positive-definite functions is positive-definite, it follows that $B = [(B_n(x,y))_n]$ is positive-definite; see Appendix A.4. It is known from [8] (Theorem 4.3, p. 267), that $B$ is a generalized correlation function of some GCSP with zero generalized expectation.*

*Let us show that $B$ is associated in $\mathcal{G}(\mathbb{R}^2)$ to the Dirac delta distribution $\delta(x-y)$. Using the change of variables, $t = n(x-y)$, $y = y$, we obtain*

$$\iint_{\mathbb{R}^2} B_n(x,y)\phi(x,y)\,dxdy = \iint_{\mathbb{R}^2} \left(\varphi(t) + \frac{1}{n}\right) \phi\left(\frac{t}{n} + y, y\right) dtdy,$$

*for any $\phi \in \mathcal{D}(\mathbb{R}^2)$. Now, letting $n \to \infty$ we obtain by the Lebesgue dominated convergence theorem, Fubini's theorem and properties of the mollifier function that the latter expression converges to $\int_{\mathbb{R}} \phi(y,y)\,dy$.*

*We have $\operatorname{supp} B = \mathbb{R}^2$ and $\operatorname{supp}\delta(x-y) = D$.* □

The following example shows that there exists a GCSP which does not have a distributional shadow and which has a generalized correlation function that does not have the form (2).

**Example 5** *Let $B = \delta^2(x-y) \in \mathcal{G}(\mathbb{R}^2)$ be the Colombeau generalized function with the representative $B_n(x,y) = \varphi_n^2(x-y) = n^2\varphi^2(n(x-y))$, $n \in \mathbb{N}$, where $\varphi \in \mathcal{S}(\mathbb{R})$ is a positive-definite function such that $\int_{\mathbb{R}} \varphi(x)\,dx = 1$. Since $\delta^2(x-y)$ is positive-definite, it follows that there exists a GCSP $u = [(u_n)_n]$ whose generalized expectation is zero and generalized correlation function is $B$; see Appendix B.3. This is an example of a GCSP which is not associated with any element of $L(\mathcal{D}(\mathbb{R}), L^2(\mathfrak{O}))$, i.e. it does not have a distributional shadow. Clearly, $B_n$ is supported by the diagonal $D$, thus $B(\tilde{x}, \tilde{y}) = 0$ for all $(\tilde{x}, \tilde{y}) \in \tilde{Q}_c$. We will show that $B$ does not have the form (2), i.e.*

$$\langle B_n(x,y), \phi(x)\psi(y)\rangle \neq$$
$$\neq \left\langle \int_{\mathbb{R}} \sum_{j,k\in\mathbb{N}_0} R_{j,k}(s)\varphi_n^{(j)}(x-s)\varphi_n^{(k)}(y-s)\,ds, \phi(x)\psi(y) \right\rangle, \quad (3)$$

*for $\phi, \psi \in \mathcal{D}(\mathbb{R})$. We have*

$$\langle B_n(x,y), \phi(x)\psi(y)\rangle = n \iint_{\mathbb{R}^2} \varphi^2(t)\phi\left(\frac{t}{n} + y\right)\psi(y)\,dtdy,$$

*where we used the change of variables $t = n(x - y)$, $y = y$. Now, by letting $n \to \infty$ we obtain that the latter expression converges to infinity. Only a finite number of the functions $R_{j,k}$ are different from zero, so the sum on the right hand side of (3) is finite, and it converges to the finite value*

$$\int_\Omega \sum_{j,k\in\mathbb{N}_0} R_{j,k}(s) \frac{\partial^{j+k}}{\partial x^j \partial y^k} \phi(x,y) \mid_{x=y=s} ds$$

*as $n \to \infty$ (see the proof of Theorem 2).* □

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## A Notes on the Colombeau algebra

### A.1 Colombeau algebra

In this Appendix we recall the basic notions from the theory of Colombeau generalized functions. More about the Colombeau theory can be found in [1,4,10]. The notation $a_n = \mathcal{O}(b_n)$ means that $a_n \le Cb_n$, $n > n_0$, for some constant $C > 0$ and $n_0 \in \mathbb{N}$. Elements of $\mathcal{E}(\Omega) = (\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$ are called sequences of smooth functions and denoted by $(u_n)_n$. The space $\mathcal{E}(\Omega)$ endowed with componentwise operations is a differential algebra. Denote by $\mathcal{E}_M(\Omega)$ the vector space of sequences $(u_n)_n$ in $\mathcal{E}(\Omega)$ with the following property: for every compact set $K \Subset \Omega$ and for every $\alpha \in \mathbb{N}_0^d$ there exists $a \in \mathbb{N}$ such that $\sup_{x\in K} |\partial^\alpha u_n(x)| = \mathcal{O}(n^a)$. Elements of $\mathcal{E}_M(\Omega)$ are called moderate sequences of functions. Let $\mathcal{N}(\Omega)$ be the vector space of sequences $(u_n)_n$ in $\mathcal{E}(\Omega)$ with the following property: for every compact set $K \Subset \Omega$, for every $\alpha \in \mathbb{N}_0^d$ and for every $b \in \mathbb{N}$ it holds that $\sup_{x\in K} |\partial^\alpha u_n(x)| = \mathcal{O}(n^{-b})$. Elements of $\mathcal{N}(\Omega)$ are called negligible sequences of functions. Note that $\mathcal{E}_M(\Omega)$ is a differential algebra with pointwise operations. It is the largest differential subalgebra of $\mathcal{E}(\Omega)$ in which $\mathcal{N}(\Omega)$ is a differential ideal.

**Remark 5** *In the definitions above we can consider also functions with continuous derivatives up to kth order instead of smooth functions and thus obtain the spaces $\mathcal{E}^k(\Omega)$, $\mathcal{E}_M^k(\Omega)$, $\mathcal{N}^k(\Omega)$ and $\mathcal{G}^k(\Omega)$.*

Let $T \in \mathcal{E}'(\Omega)$ be a compactly supported distribution. Then $T \hookrightarrow Cd(T) = [((T * \varphi_n)|_\Omega)_n] = ((T*\varphi_n)|_\Omega)_n + \mathcal{N}(\Omega)$ defines a linear embedding of $\mathcal{E}'(\Omega)$ into $\mathcal{G}(\Omega)$; see [9,10]. Since the presheaf $\Omega \mapsto \mathcal{G}(\Omega)$ is a sheaf, it follows that the above embedding can be extended to an embedding of $\mathcal{D}'(\Omega)$ and $\mathcal{C}^\infty(\Omega)$ into $\mathcal{G}(\Omega)$ for any open set $\Omega \subset \mathbb{R}^d$.

Apart from equality, there are other useful (weaker) equivalence relations in the Colombeau algebra $\mathcal{G}(\Omega)$. Recall that $u = [(u_n)_n]$ and $v = [(v_n)_n]$ are associated, denoted by $u \approx v$, if $\lim_{n\to\infty} \int_\Omega (u_n(x) - v_n(x))\phi(x)\, dx = 0$, $\phi \in \mathcal{D}(\Omega)$. Similarly, $u = [(u_n)_n] \in \mathcal{G}(\Omega)$ is associated with an element $f \in \mathcal{D}'(\Omega)$, if $\lim_{n\to\infty} \int_\Omega u_n(x)\phi(x)\, dx = \langle f, \phi\rangle$, $\phi \in \mathcal{D}(\Omega)$ (in this case $f$ is called the distributional shadow of $u$). Not all elements of $\mathcal{G}(\Omega)$ have a distributional shadow. Genuine Colombeau generalized functions, as $\delta^2 = [(\varphi_n^2)_n]$, are not associated with any distribution.

An equivalence relation between association and equality is equality in the sense of distributions [1]. Two elements $u = [(u_n)_n]$ and $v = [(v_n)_n]$ of $\mathcal{G}(\Omega)$ are called equal in the sense of distributions, if $\int_\Omega (u_n(x) - v_n(x))\phi(x)\, dx \in \mathcal{N}(\Omega)$ for every $\phi \in \mathcal{D}(\Omega)$.

The restriction $u|_{\Omega'} \in \mathcal{G}(\Omega')$ is defined as $[(u_n|_{\Omega'})_n]$. The support of $u = [(u_n)_n] \in \mathcal{G}(\Omega)$ is defined as $\operatorname{supp} u = (\cup\{\Omega' \subseteq \Omega : \Omega' \text{ open}, u|_{\Omega'} = 0\})^c$.

### A.2 Generalized constants and generalized point values of Colombeau functions

Let $\mathcal{E}_M = \{(r_n)_n \in \mathbb{R}^{\mathbb{N}} : (\exists a \in \mathbb{N})(|r_n| = \mathcal{O}(n^a))\}$ and $\mathcal{N} = \{(r_n)_n \in \mathcal{E}_M : (\forall b \in \mathbb{N})(|r_n| = \mathcal{O}(n^{-b}))\}$. Then $\mathcal{R} = \mathcal{E}_M/\mathcal{N}$ is called the algebra of generalized constants; see [1,4]. Note that $\mathbb{R}$

can be embedded into $\mathcal{R}$ by $r \mapsto (r)_n + \mathcal{N}$. Let $u = [(u_n)_n] \in \mathcal{G}(\Omega)$ and $x_0 \in \Omega$. Then, by [1], $[(u_n(x_0))_n] \in \mathcal{R}$ is called the point value of the generalized function $u$ at $x_0 \in \Omega$.

Let $\Omega_M = \{(x_n)_n \in \Omega^{\mathbb{N}} : (\exists a \in \mathbb{N})(|x_n| = \mathcal{O}(n^a))\}$, and define the equivalence relation $\sim$ by $(x_n)_n \sim (y_n)_n \Leftrightarrow (\forall b \in \mathbb{N})(|x_n - y_n| = \mathcal{O}(n^{-b}))$. The quotient space $\tilde{\Omega} = \Omega_M / \sim$ is called the set of generalized points. Especially, for $\Omega = \mathbb{R}$ we obtain the set of generalized constants $\tilde{\mathbb{R}} = \mathbb{R}_M / \sim = \mathcal{R}$. The set $\tilde{\Omega}_c = \{\tilde{x} = [(x_n)_n] \in \tilde{\Omega} : (\exists K \Subset \Omega)(x_n \in K)\}$ is called the set of compactly supported generalized points; see [4]. For $\Omega = \mathbb{R}$ we write $\tilde{\mathbb{R}}_c = \mathcal{R}_c$. If $u = [(u_n)_n] \in \mathcal{G}(\Omega)$ and $\tilde{x} \in \tilde{\Omega}_c$, then $u(\tilde{x}) = [(u_n(x_n))_n] \in \mathcal{R}_c$ and it is called the point value of $u$ at the compactly supported generalized point $\tilde{x} \in \tilde{\Omega}_c$; see [4].

**Theorem 3** *[4] Let $u = [(u_n)_n] \in \mathcal{G}(\Omega)$. Then $u = 0$ in $\mathcal{G}(\Omega)$ if and only if $u(\tilde{x}) = 0$ in $\mathcal{R}_c$ for all $\tilde{x} \in \tilde{\Omega}_c$.*

## A.3 The sharp topology

Omitting the general construction of the sharp topology on $\mathcal{G}(\Omega)$, we give its construction on $\mathcal{R}_c$. Let $(f_n)_n \in \mathbb{R}^{\mathbb{N}}$ and $\|(f_n)_n\| = \limsup_{n\to\infty} |f_n|^{(\log n)^{-1}}$. Denote $\mathcal{E}_M = \{(f_n)_n \in \mathbb{R}^{\mathbb{N}} : \|(f_n)_n\| < \infty\}$, $\mathcal{N} = \{(f_n)_n \in \mathcal{E}_M : \|(f_n)_n\| = 0\}$ and $\mathcal{E}_M^c = \{(f_n)_n \in \mathcal{E}_M : (\exists K \Subset \mathbb{R})\,(f_n \in K)\}$. Then $\mathcal{R} = \mathcal{E}_M/\mathcal{N}$ is the algebra of generalized constants and $\mathcal{R}_c = \mathcal{E}_M^c/\mathcal{N}$ is the algebra of compactly supported generalized constants. The mapping $d_c : \mathcal{E}_M^c \times \mathcal{E}_M^c \to \mathbb{R}$, $d_c((f_n)_n, (g_n)_n) \mapsto \|(f_n - g_n)_n\|$ is an ultra-pseudometric on $\mathcal{E}_M^c$, and the mapping $\tilde{d}_c : \mathcal{R}_c \times \mathcal{R}_c \to \mathbb{R}$, $\tilde{d}_c([(f_n)_n], [(g_n)_n]) \mapsto d_c((f_n)_n, (g_n)_n)$ is an ultrametric on $\mathcal{R}_c$. The topology defined by $\tilde{d}_c$ is called the sharp topology on $\mathcal{R}_c$. The sharp open balls of $\mathcal{R}_c$ are of the form

$$L((f_n)_n, k)) = \{[(g_n)_n] \in \mathcal{R}_c : \limsup_{n\to\infty} |f_n - g_n|^{(\log n)^{-1}} < k\}.$$

## A.4 Positive and positive-definite Colombeau generalized functions

We recall the notions of positive and positive-definite generalized functions from $\mathcal{D}'(\Omega)$; see [2]. An $f \in \mathcal{D}'(\Omega)$ is positive, $f \geq 0$, if $\langle f, \phi \rangle \geq 0$ for every $\phi \in \mathcal{D}(\Omega)$, $\phi \geq 0$. An $f \in \mathcal{D}'(\mathbb{R}^d)$ is positive-definite if $\langle f, \phi * \phi^* \rangle \geq 0$, where $\phi^*(x) = \overline{\phi(-x)}$, $\phi \in \mathcal{D}(\mathbb{R}^d)$.

Recall, [11], a Colombeau generalized function $f \in \mathcal{G}(\Omega)$ is positive, $f \geq 0$ in $\Omega$, if and only if there exists a representative $(f_n)_n$ of $f$ such that for every $a > 0$ and $K \Subset \Omega$ there exists $n_0 \in \mathbb{N}$ such that $\inf_{x \in K} f_n(x) + n^{-a} \geq 0$, $n > n_0$. If this inequality holds for some representative, it holds for every real representative of $f$. In [4] it is proved that $f \geq 0$ if and only if there exists a representative $(f_n)_n$ such that $f_n(x) \geq 0$, $x \in \Omega$, $n \in \mathbb{N}$. Moreover, by [11] we have $f = [(f_n)_n] \in \mathcal{G}(\Omega)$ is $\mathcal{D}'-$weakly positive if for every $\phi \in \mathcal{D}(\Omega)$, $\phi \geq 0$, $z_\phi = \left[\left(\int_{\mathbb{R}^d} f_n(t)\phi(t)\,dt\right)_n\right] \geq 0$. If $f = [(f_n)_n] \in \mathcal{G}(\Omega)$ satisfies $f \geq 0$ and $f \leq 0$, then $f = 0$. Also, if $f = [(f_n)_n] \in \mathcal{G}(\Omega)$ satisfies $f \geq 0$ and $f \leq 0$ $\mathcal{D}'-$weakly, then $f = 0$ in the sense of distributions. Let $f \in \mathcal{D}'(\Omega)$. Then, by [11], we have that $Cd(f)$ is $\mathcal{D}'-$weakly positive if and only if $f$ is a positive distribution.

Following [11], we say that $f \in \mathcal{G}(\mathbb{R}^d)$ is positive-definite on $\mathbb{R}^d$ if it has a representative $(f_n)_n$ such that

$$(\forall K \Subset \mathbb{R}^d)(\forall a > 0)(\exists n_0 \in \mathbb{N})(\forall \zeta_1, \ldots, \zeta_m \in \mathbb{C})$$

$$\inf_{x_k, x_j \in K} \sum_{k,j=1}^{m} (f_n(x_k - x_j) + n^{-a})\zeta_k \overline{\zeta_j} \geq 0, \quad n \geq n_0.$$

Recall from [11], $f = [(f_n)_n] \in \mathcal{G}(\mathbb{R}^d)$ is $\mathcal{D}'-$weakly positive-definite if for every $\phi \in \mathcal{D}(\mathbb{R}^d)$, $z_\phi = \left[\left(\int_{\mathbb{R}^d} f_n(t)\phi * \phi^*(t)\,dt\right)_n\right] \geq 0$. Let $f \in \mathcal{D}'(\mathbb{R}^d)$. Then, by [11], $Cd(f)$ is $\mathcal{D}'-$weakly positive-definite if and only if $f$ is a positive-definite distribution.

Positive-definite generalized functions are in connection with translation-invariant positive-definite bilinear functionals; see [2]. A real bilinear functional $B(\phi, \psi)$ on $\mathcal{D}(\mathbb{R}^d)$ is positive-definite if $B(\phi, \phi) \geq 0$ for all $\phi \in \mathcal{D}(\mathbb{R}^d)$. Let $B(\phi, \psi)$ be a positive-definite bilinear functional on $\mathcal{D}(\mathbb{R}^d)$. Recall, [2], if $\phi_1, \ldots, \phi_m \in \mathcal{D}(\mathbb{R}^d)$ are linearly independent functions and $\zeta_1, \ldots, \zeta_m \in \mathbb{R}$, then with $\psi = \sum_{i=1}^m \zeta_i \phi_i$, we obtain $\sum_{i,j=1}^n B(\phi_i, \phi_j)\zeta_i \zeta_j = B(\psi, \psi) \geq 0$. Every real translation-invariant bilinear functional on $\mathcal{D}(\mathbb{R}^d)$ can be written in the form

$$B(\phi, \psi) = \langle F(x - y), \phi(x)\psi(y) \rangle, \tag{4}$$

where $F$ is a generalized function in $\mathcal{D}'(\mathbb{R}^d)$. Moreover, $B(\phi,\psi)$ is positive-definite if and only if the generalized function $F$, which corresponds to $B$ via (4), satisfies $\langle F, \phi * \phi^* \rangle \geq 0$, i.e. $F$ is also positive-definite.

Recall from [11], an element $B \in \mathcal{G}(\Omega \times \Omega)$ is $\mathcal{D}'-$weakly positive-definite if it has a representative $(B_n)_n$ such that for every $\phi \in \mathcal{D}(\Omega)$,

$$\left[\left(\int_{\mathbb{R}^d} B_n(x,y)\phi(x)\phi(y)\,dxdy \geq 0\right)_n\right] \geq 0.$$

## B Notes on Colombeau stochastic processes

### B.1 Embeddings of distributional stochastic processes in CSPs

Recall from [8], $\xi : \mathcal{C}_0^\infty(\Omega) \times L^p(\mathfrak{O}) \to \mathbb{C}$ is a generalized functional stochastic process (or distributional stochastic process) if the mapping $\phi \mapsto \xi(\phi, \cdot)$ is a strongly continuous mapping of $C_0^\infty(\Omega)$ into $L^p(\mathfrak{O})$.

Let $\xi$ be a distributional stochastic process on $\Omega$. Denote by $(\kappa_n)_n$ a sequence of smooth functions supported by $\Omega_{-1/n} = \left\{x \in \Omega : d(x, \mathbb{R}^d \setminus \Omega) > \frac{1}{n}\right\}$, $n > n_0$, such that $\kappa_n \equiv 1$ on $\Omega_{-2/n}$, $n > n_0$. Then the assignment $\xi \hookrightarrow u = [((\kappa_n \xi)(\omega, \varphi_n(x - \cdot)))_n]$, $\omega \in \mathfrak{O}$, $x \in \Omega$, $n \in \mathbb{N}$, defines an embedding of the space of distributional stochastic processes $\xi$ into the space of Colombeau stochastic $L^p(\mathfrak{O})-$valued processes $\mathcal{G}_{L^p}(\mathfrak{O}, \Omega)$; see [8]. We use the notation $u = Cd(\xi)$.

If $\xi$ is compactly supported, then we may take $\xi_n(\omega, x) = \xi(\omega, \varphi_n(x - \cdot))$, $\omega \in \mathfrak{O}$, $x \in \Omega$, $n \in \mathbb{N}$.

### B.2 Generalized expectation and generalized correlation function of CSPs with values in $L^2(\mathfrak{O})$

We now recall results from [8]. Let $u = [(u_n)_n] \in \mathcal{G}_{L^2}(\mathfrak{O}, \Omega)$. The generalized expectation of $u$ is an element $m$ of $\mathcal{G}(\Omega)$ with representative $m_{u_n}(x) = E(u_n(\cdot, x))$, $x \in \Omega$, $n \in \mathbb{N}$. The generalized correlation function of $u$ is an element $B$ of $\mathcal{G}(\Omega \times \Omega)$ with representative $B_{u_n}(x,y) = E(u_n(\cdot,x)u_n(\cdot,y))$, $x, y \in \Omega$, $n \in \mathbb{N}$.

Let $\xi$ be a distributional stochastic process. Suppose that $u = [(u_n)_n]$ is the corresponding element of $\mathcal{G}_{L^2}(\mathfrak{O}, \Omega)$ which is defined in Appendix B.1. Then the representatives of the generalized expectation $m = [(m_{u_n})_n]$ and the generalized correlation function $B = [(B_{u_n})_n]$, as well as the process $u = [(u_n)_n]$ itself, depend on the choice of the mollifier function. However, they define elements of the Colombeau algebra which are equal in the sense of distributions.

It is shown in [8] that $\partial^\alpha m_{u_n}(x) = m_{\partial^\alpha u_n}(x)$, $x \in \Omega$, $n \in \mathbb{N}$ and $\partial_x^\alpha \partial_y^\alpha B_{u_n}(x,y) = B_{\partial^\alpha u_n}(x,y)$, $x, y \in \Omega$, $n \in \mathbb{N}$.

The generalized correlation function $B = [(B_{u_n}(x,y))_n]$ is weakly positive-definite, i.e. it has a representative with bilinear positive-definite functionals. Furthermore, its generalized covariance function $C \in \mathcal{G}(\Omega \times \Omega)$ with the representative $C_{u_n}(x,y) = B_{u_n}(x,y) - m_{u_n}(x)m_{u_n}(y)$, $x, y \in \Omega$, $n \in \mathbb{N}$, is weakly positive-definite.

**Remark 6** *Considering only derivatives up to order $k$ instead of smooth representatives, we obtain elements $[(m_{u_n})_n] \in \mathcal{G}^k(\Omega)$ and $[(B_{u_n}(x,y))_n] \in \mathcal{G}^k(\Omega \times \Omega)$.*

**Remark 7** *A word on terminology. In the spirit of [2], we use the term* correlation function *for the noncentered expression $B_{u_n}(x,y) = E(u_n(\cdot,x)u_n(\cdot,y))$ and* covariance function *for its centered counterpart $C_{u_n}(x,y) = B_{u_n}(x,y) - m_{u_n}(x)m_{u_n}(y)$ throughout the paper.*

### B.3 Gaussian Colombeau stochastic $L^2(\mathfrak{O})$-valued processes

We recall from [8] the basic assertions related to GCSPs with values in $L^2(\mathfrak{O})$. This concept originates from the corresponding one in distribution theory; see [2].

Let $u \in \mathcal{G}_{L^2}(\mathfrak{O}, \Omega)$. It is said that $u$ is a GCSP, if there exists a representative $(u_n)_n$ and $n_0 \in \mathbb{N}$ such that for every $n > n_0$ and arbitrary $x_1, \ldots, x_r \in \Omega \subset \mathbb{R}^d$, the probability that $X_n = (u_n(x_1,\omega), \ldots, u_n(x_r,\omega)) \in B$, where $B$ is a Borel set in $\mathbb{R}^r$, is

$$P(X_n \in B) = \left(\frac{\det A_n}{(2\pi)^d}\right)^{1/2} \int_B \exp\left(-\frac{1}{2}\, s^T A_n s\right) ds, \;\; n > n_0,$$

where $A_n, n \in \mathbb{N}$, is a sequence of non-degenerate positive-definite matrices, and $s^T A_n s = \sum_{i=1}^r \sum_{j=1}^r a_{ijn} s_i s_j$, $n > n_0$. We will call $(u_n)_n$ a Gaussian representative of $u$. Also, instead of $n > n_0$ we will write $n \in \mathbb{N}$.

**Example 6** *Let $(c_n)_n \in \mathcal{N}$ be negligible sequence and $s(\omega, x)$ any non-Gaussian stochastic process. Then $c_n s(\omega, x)$ is a non-Gaussian negligible sequence. If $(u_n)_n$ is a Gaussian representative of a GCSP, then $(u_n + c_n s)_n$ is a non-Gaussian representative of the same GCSP. Therefore not all representatives of a GCSP are Gaussian.* □

Let $u$ be a GCSP with Gaussian representative $(u_n)_n$ and let $(B_{u_n})_n$ be a representative of its generalized correlation function. Then, by [8], $A_n = (B_{u_n}(x_i, x_j))^{-1}$, $n \in \mathbb{N}$, for all $x_1, \ldots, x_d \in \mathbb{R}$.

Also we know by [8] that partial derivatives of a GCSP are again GCSPs.

Let $m = [(m_n(x))_n] \in \mathcal{G}(\Omega)$ and $B = [(B_n(x, y))_n] \in \mathcal{G}(\Omega \times \Omega)$ such that the generalized covariance function $C = [(C_n(x, y))_n] = [(B_n(x, y))_n] - [(m_n(x) m_n(y))_n] \in \mathcal{G}(\Omega \times \Omega)$ has a representative $(C_n)_n$ such that each $C_n$, $n \in \mathbb{N}$, is a positive definite bilinear functional. Then by [8] there exists a GCSP $u \in \mathcal{G}_{L^2}(\mathfrak{O}, \Omega)$ with a Gaussian representative $(u_n)_n$, whose generalized expectation and generalized covariance function are $m$ and $C$. This implies the following assertion.

**Theorem 4** *[8] Let $u = [(u_n)_n] \in \mathcal{G}_{L^2}(\mathfrak{O}, \Omega)$ be a Colombeau stochastic process with generalized expectation $m = [(m_{u_n}(x))_n] \in \mathcal{G}(\Omega)$ and generalized correlation function $B = [(B_{u_n}(x, y))_n] \in \mathcal{G}(\Omega \times \Omega)$. There exists a GCSP with the given generalized expectation and generalized correlation function.*

**Remark 8** *As before, we can consider derivatives up to order $k$ instead of smooth functions.*

**Example 7** *White noise $w$ as a GCSP is defined by a zero generalized expectation and a correlation function that is associated to the Dirac delta $\delta(x - y)$ supported on the diagonal. There are several ways to achieve this: one possibility is to define $B_w(x, y) = [(\varphi_n(x - y))_n]$, another to let $B_w(x, y) = [(\int \varphi_n(s - x)\varphi_n(s - y) ds)_n]$. The two processes are associated in Colombeau sense, but not equal. The variance of white noise is given by $B_w(x, x) = [(\varphi_n(0))_n]$ in the first case and $B_w(x, x) = [(\|\varphi_n(\cdot)\|_{L^2}^2)_n]$ in the second case. The two are equal provided that the mollifier satisfies $\|\varphi(\cdot)\|_{L^2}^2 = \varphi(0)$.* □

## C $L^p$-continuity and pathwise continuity

Let $X(t, \omega)$ be a stochastic process. We say that a process is continuous, if almost all of its paths have this property. Recall Kolmogorov's continuity criterion.

**Theorem 5** *[6] Let $X$ be a process on $\mathbb{R}^d$ and assume that there exist constants $a, b, C > 0$ such that*

$$E|X_s - X_t|^a \leq C|s - t|^{d+b}, \quad s, t \in \mathbb{R}^d.$$

*Then $X$ has a pathwise continuous version.*

The following example shows that pathwise continuity does not imply $L^2$-continuity.

**Example 8** *Take $\varphi \in \mathcal{D}(\mathbb{R})$ such that $supp\, \varphi = [-1, 1]$, $\int \varphi(x)\, dx = 0$ and $\int \varphi^2(x)\, dx \neq 0$. Take $\mathfrak{O} = [-1, 1]$ with the uniform probability distribution. Let*

$$X(t, \omega) = \begin{cases} \frac{1}{\sqrt{t}} \varphi\left(\frac{\omega}{t}\right), & |\omega| < t, \\ 0, & |\omega| \geq t \text{ or } t \leq 0, \end{cases}$$

*Then $E(X(t)) = 0$. All trajectories (except for $\omega = 0$) are smooth. But*

$$E((X(t))^2) = \begin{cases} \int \varphi^2(x)\, dx, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

*and it does not converge to zero as $t \to 0$. Therefore, $L^2$-continuity does not hold.*

Also, the reverse does not hold.

**Example 9** *Poisson process (which violates the Kolmogorov continuity criterion) is $L^2$-continuous, but its paths are not continuous.* □

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