# Generalized stochastic processes in algebras of generalized functions: independence, stationarity and SPDEs ☆

Snežana Gordić[a], Michael Oberguggenberger[b], Stevan Pilipović[c,*], Dora Seleši[c]

[a]*Faculty of Education, University of Novi Sad, Podgorička 4, 25000 Sombor, Serbia*
[b]*Institute for Basic Science in Engineering Science, University of Innsbruck, Technikerstrasse 13, 6020 Innsbruck, Austria*
[c]*Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia*

**Abstract**

Stochastic processes are regarded in the framework of Colombeau-type algebras of generalized functions. Colombeau stochastic processes with independent values and stationary Colombeau stochastic processes are studied with special attention paid to the property of translational invariance of generalized functions. Processes with stationary increments are characterized via stationarity of their gradient. Gaussian stationary solutions are analyzed for linear stochastic partial differential equations with generalized constant coefficients in the framework of Colombeau stochastic processes.

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## 1. Introduction

Generalized stochastic processes appear as solutions to stochastic partial differential equations (SPDEs) that possess no classical solution. Depending on the type of continuity imposed on the process, one can distinguish between different types of generalized stochastic processes. One possibility was developed by Gel'fand and Vilenkin in [1] and has been studied by many authors thereafter.

Stochastic processes with values in Colombeau algebras of generalized functions, or shortly Colombeau stochastic processes (CSPs), were considered in [2, 3, 4, 5, 6, 7, 8, 9, 10, 11], but none of these papers addressed the question of probabilistic properties of CSPs.

In [12], we have started to study some probabilistic properties of CSPs (their measurability, the representation of their generalized characteristic function and the structure of their generalized correlation function). Also, we established the notion of the point values of CSPs in compactly supported generalized points. In this paper, which relies on the results of [5] and [12], we continue to study the probabilistic properties of CSPs with emphasis on their independence, stationarity and the stationarity of their increments.

As an application of the obtained results, we investigate the solutions to a class of SPDEs in the framework of stationary Gaussian Colombeau stochastic processes (GCSPs).

*Corresponding author
*Email addresses:* `michael.oberguggenberger@uibk.ac.at` (Michael Oberguggenberger), `pilipovic@dmi.uns.ac.rs` (Stevan Pilipović )
*URL:* `snezana.gordic@pef.uns.ac.rs` (Snežana Gordić), `dora@dmi.uns.ac.rs` (Dora Seleši)

The paper is organized as follows: after a short introduction of basic notions in Section 2, we turn to Section 3, define CSPs with independent values and give a characterization of such processes via their generalized correlation function in the classical Colombeau algebra of generalized numbers. In Section 4 we study the properties of stationary CSPs, distinguishing between strict stationarity and weak stationarity. We prove that the generalized expectation of a stationary CSP is a generalized constant and we provide a special form of its generalized correlation function. Stationarity of the increments can be defined via stationarity of the gradient of the process. The most prominent example is the one of Brownian motion which has stationary increments. Its first derivative, the white noise process, is stationary. Section 5 uses techniques of the Fourier transform to solve a class of SPDEs. Necessary conditions for the existence of a stationary Gaussian solution to $P(D)u = f$, where $P(D)$ is a differential operator with constant coefficients and $f$ is a stationary GCSP are given. As an illustration, we consider the equation $(1-\Delta)u = a + b\cdot\partial^k w$, where $w$ is spatial white noise on $\mathbb{R}^d$ considered in the framework of GCSPs, $a, b$ are constants, $\Delta$ the Laplace operator and $\partial^k w = \partial^{k_1}_{x_1}\partial^{k_2}_{x_2}\dots\partial^{k_d}_{x_d} w(x,\omega)$ the $k$th derivative of white noise for $k\in\mathbb{N}_0^d$.

## 2. Basic notions

Since this paper is a direct successor of [5] and [12], we refer to the notations used therein and most of all to the Appendix provided in [12] as a complete reference to the Colombeau algebra $\mathcal{G}(\Omega)$ of generalized functions. By $\Omega$ we denote an open set in $\mathbb{R}^d$ and $K \Subset \Omega$ a compact subset. Let $\mathcal{C}^k(\Omega)$ denote the space of $k$-times continuously differentiable functions and $\mathcal{S}'(\Omega)$ the space of tempered distributions. The Fourier transform on $\mathcal{S}(\mathbb{R}^d)$ is defined as $\mathcal{F}(\phi(x))(\xi) = \hat{\phi}(\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d}\phi(x)e^{-ix\cdot\xi}dx$.

We fix a sequence of mollifiers $\varphi_n \in \mathcal{S}(\mathbb{R}^d), n\in\mathbb{N}$, of the form $\varphi_n(x) = n^d\varphi(nx),\ x\in\mathbb{R}^d,\ n\in\mathbb{N}$, where $\int\varphi(x)\,dx = 1$, $\int x^m\varphi(x)\,dx = 0$, $m\in\mathbb{N}$, and $\varphi$ is positive definite.

Let $(\mathfrak{O}, \mathcal{U}, P)$ be a probability space. $L^p(\mathfrak{O})$, $p\geq 1$, denotes the space of random variables with finite $p$th moments. We recall, [12], the definition of CSPs with values in $L^p(\mathfrak{O})$.

**Definition 2.1.** *Let $p\in[1,\infty]$ and $k\in\mathbb{N}\cup\{\infty\}$. Let $\mathcal{E}^k_{L^p}(\mathfrak{O},\Omega) = (\mathcal{C}^k_{L^p}(\mathfrak{O},\Omega))^{\mathbb{N}}$ be the set of sequences $(u_n(\omega,x))_n$, $\omega\in\mathfrak{O}$, $x\in\Omega$, $n\in\mathbb{N}$, such that the mapping $x\mapsto u_n(\omega,x)$ is in $\mathcal{C}^k(\Omega)$ for almost all (a.a.) $\omega\in\mathfrak{O}$, and for every $x\in\Omega$, $u_n(\cdot,x)$ is in $L^p(\mathfrak{O})$. Define:*

$$\begin{aligned}\mathcal{E}^k_{M,L^p}(\mathfrak{O},\Omega) &= \Big\{(u_n)_n\in\mathcal{E}^k_{L^p}(\mathfrak{O},\Omega)\,:\,(\forall K\Subset\Omega)(\forall\alpha\in\mathbb{N}_0^d,\ |\alpha|\leq k)\\ &\qquad(\exists a\in\mathbb{N})\left(\sup_{x\in K}\|\partial^\alpha u_n(\cdot,x)\|_{L^p} = \mathcal{O}(n^a)\right)\Big\},\end{aligned}$$

$$\begin{aligned}\mathcal{N}^k_{L^p}(\mathfrak{O},\Omega) &= \Big\{(u_n)_n\in\mathcal{E}^k_{L^p}(\mathfrak{O},\Omega)\,:\,(\forall K\Subset\Omega)(\forall\alpha\in\mathbb{N}_0^d,\ |\alpha|\leq k)\\ &\qquad(\forall b\in\mathbb{N})\left(\sup_{x\in K}\|\partial^\alpha u_n(\cdot,x)\|_{L^p} = \mathcal{O}(n^{-b})\right)\Big\}.\end{aligned}$$

*Elements of the vector spaces $\mathcal{E}^k_{M,L^p}(\mathfrak{O},\Omega)$ and $\mathcal{N}^k_{L^p}(\mathfrak{O},\Omega)$ are called moderate and negligible sequences of functions with values in $L^p(\mathfrak{O})$, respectively. The elements of the quotient space*

$$\mathcal{G}^k_{L^p}(\mathfrak{O},\Omega) = \mathcal{E}^k_{M,L^p}(\mathfrak{O},\Omega)/\mathcal{N}^k_{L^p}(\mathfrak{O},\Omega)$$

*are called CSPs over $\Omega$ with values in $L^p(\mathfrak{O})$.*

In the previous definition we require pathwise continuity almost everywhere (a.e.) and pathwise differentiability $k$ times, but in general the mappings $x \mapsto u_n(\cdot, x)$ do not have to be continuous or differentiable with respect to the $L^p$-norm. Note that pathwise $C^k$-continuity for a.a. $\omega \in \mathfrak{O}$ can easily be modified to obtain $C^k$-continuity for every $\omega \in \mathfrak{O}$.

For the case $k = \infty$, in the above definition we will omit the superscript $\infty$ and use the notation $\mathcal{E}_{M,L^p}(\mathfrak{O}, \Omega)$, $\mathcal{N}_{L^p}(\mathfrak{O}, \Omega)$ and $\mathcal{G}_{L^p}(\mathfrak{O}, \Omega)$.

## 3. Colombeau stochastic processes with independent values

Denote the complement of the diagonal by $Q_\Omega = \{(x, y) \in \Omega \times \Omega : \ x \neq y\}$ and the diagonal by $D_\Omega = \Omega \times \Omega \setminus Q = \{(x, y) \in \Omega \times \Omega : \ x = y\}$. In the case $\Omega = \mathbb{R}^d$, we use the notation $Q = \{(x, y) \in \mathbb{R}^{2d} : x \neq y\}$ and $D = \mathbb{R}^{2d} \setminus Q$.

**Definition 3.1.** *A CSP $u$ over $\Omega$ with values in $L^p(\mathfrak{O})$ has independent values if it has a representative $(u_n)_n$ such that the following conditions hold:*

*(1) for every $n \in \mathbb{N}$, $u_n(\omega, x)$ and $u_n(\omega, y)$ are independent random variables for $(x, y) \in K$, $K \Subset Q_\Omega$, i.e. for every $n \in \mathbb{N}$,*

$$P\{u_n(\omega, x) \in B_1 \cap u_n(\omega, y) \in B_2\} = P\{u_n(\omega, x) \in B_1\} P\{u_n(\omega, y) \in B_2\}$$

*for all $B_1, B_2 \in \mathcal{B}(\mathbb{R})$ and $(x, y) \in K$, $K \Subset Q_\Omega$,*

*(2) for $n \neq m$, $u_n(\omega, x)$ and $u_m(\omega, y)$ are independent random variables for every $x, y \in \Omega$, i.e. for $n \neq m$,*

$$P\{u_n(\omega, x) \in B_1 \cap u_m(\omega, y) \in B_2\} = P\{u_n(\omega, x) \in B_1\} P\{u_m(\omega, y) \in B_2\}$$

*for all $B_1, B_2 \in \mathcal{B}(\mathbb{R})$ and $x, y \in \Omega$.*

**Remark 1.** Note that if $(u_n)_n$ is a representative of a CSP satisfying the conditions of Definition 3.1 and $(N_n)_n$ a negligible CSP with non-independent values, then $(u_n)_n + (N_n)_n$ is a representative of the same equivalence class (of the same CSP) which does not have independent values. Thus, not all representatives are with independent values. In the sequel we will call the representatives that satisfy the conditions of Definition 3.1 shortly *IV-representatives.*

**Remark 2.** The notion of CSP with independent values are dependent on existence of an special representative. We note that it is not possible to give a characterization of this notion independent of representatives.

The Colombeau algebra of compactly supported generalized constants $\mathcal{R}_c$ is endowed with the topology generated by sharp open balls; see Appendix A.2. In [12] we proved the measurability of CSPs, thus inverse images of sharp open balls are always in $\mathcal{U}$.

**Theorem 1.** *Let $u$ be a CSP over $\Omega$ with values in $L^p(\mathfrak{O})$ and let $u$ have independent values. Then*

$$P\{u(\omega, \tilde{x}) \in O_1 \cap u(\omega, \tilde{y}) \in O_2\} = P\{u(\omega, \tilde{x}) \in O_1\} P\{u(\omega, \tilde{y}) \in O_2\} \tag{1}$$

*for all open balls $O_1, O_2$ in $\mathcal{R}_c$ and $(\tilde{x}, \tilde{y}) \in (\tilde{Q}_\Omega)_c$.*

Proof. Let $u$ have independent values and let $(u_n)_n$ be the corresponding IV-representative as stated in Definition 3.1. Let

$$\begin{aligned} O_i &= L((c_{i;n})_n, k_i) = \{[(z_n)_n] \in \mathcal{R}_c : \limsup_{n\to\infty} |c_{i;n} - z_n|^{(\log n)^{-1}} < k_i\} \\ &= \{[(z_n)_n] \in \mathcal{R}_c : |c_{i;n} - z_n| < n^{\log k_i} \text{ for all } n > n_0 \text{ for some } n_0 \in \mathbb{N}\}, \end{aligned}$$

$i = 1, 2$, be two open balls in $\mathcal{R}_c$ and $(\tilde{x}, \tilde{y}) = ((x_n)_n, (y_n)_n) \in (\tilde{Q}_\Omega)_c$. There exists a compact set $K \Subset Q_\Omega$ such that $(x_m, y_m) \in K$ for all $m \in \mathbb{N}$. Since $u$ has independent values and $x_m \neq y_m$, it follows that $u_m(\omega, x_m)$ and $u_m(\omega, y_m)$ are independent random variables for every $m \in \mathbb{N}$, $u_n(\omega, x_n)$ and $u_m(\omega, x_m)$ are independent random variables for $n \neq m$, and $u_n(\omega, y_n)$ and $u_m(\omega, y_m)$ are independent random variables for $n \neq m$. Therefore, the events $\{\omega \in \mathfrak{O} : u_m(\omega, x_m) \in B_1\}$ and $\{\omega \in \mathfrak{O} : u_m(\omega, y_m) \in B_2\}$ are independent for every $m \in \mathbb{N}$, as well as the events $\{\omega \in \mathfrak{O} : u_n(\omega, x_n) \in B_1\}$ and $\{\omega \in \mathfrak{O} : u_m(\omega, x_m) \in B_2\}$ for $n \neq m$ and the events $\{\omega \in \mathfrak{O} : u_n(\omega, y_n) \in B_1\}$ and $\{\omega \in \mathfrak{O} : u_m(\omega, y_m) \in B_2\}$ for $n \neq m$.

We have

$$\begin{aligned} A^{\tilde{x}} &= \{\omega \in \mathfrak{O} : u(\omega, \tilde{x}) \in O_1\} \\ &= \{\omega \in \mathfrak{O} : \limsup_{n\to\infty} |c_{1;n} - u_n(\omega, x_n)|^{(\log n)^{-1}} < k_1\} \\ &= \{\omega \in \mathfrak{O} : |c_{1;n} - u_n(\omega, x_n)| < n^{\log k_1} \text{ for all } n > n_0 \text{ for some } n_0 \in \mathbb{N}\} \\ &= \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{\omega \in \mathfrak{O} : |c_{1;m} - u_m(\omega, x_m)| < m^{\log k_1}\} \\ &= \bigcup_{n=1}^{\infty} A_n^{\tilde{x}}, \end{aligned}$$

where $A_n^{\tilde{x}} = \bigcap_{m\geq n} \{\omega \in \mathfrak{O} : |c_{1;m} - u_m(\omega, x_m)| < m^{\log k_1}\} = \bigcap_{m\geq n} I_m^{x_m}$. It holds $A_1^{\tilde{x}} \subset A_2^{\tilde{x}} \subset A_3^{\tilde{x}} \subset \ldots$ and by continuity of the probability measure we obtain

$$P(A^{\tilde{x}}) = P\left(\bigcup_{n=1}^{\infty} A_n^{\tilde{x}}\right) = \lim_{n\to\infty} P(A_n^{\tilde{x}}) = \lim_{n\to\infty} P\left(\bigcap_{m\geq n} I_m^{x_m}\right).$$

Put $A_n^k = \bigcap_{m\geq n}^k I_m^{x_m}$, $k \geq n$. Then $\bigcap_{m\geq n} I_m^{x_m} = \bigcap_{k=n}^{\infty} A_n^k$. It holds $A_n^n \supset A_n^{n+1} \supset A_n^{n+2} \supset \ldots$ and hence

$$\begin{aligned} P(A^{\tilde{x}}) &= \lim_{n\to\infty} P\left(\bigcap_{m\geq n} I_m^{x_m}\right) = \lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty} A_n^k\right) = \lim_{n\to\infty} \lim_{k\to\infty} P(A_n^k) \\ &= \lim_{n\to\infty} \lim_{k\to\infty} P\left(\bigcap_{m=n}^{k} I_m^{x_m}\right) = \lim_{n\to\infty} \lim_{k\to\infty} \prod_{m=n}^{k} P(I_m^{x_m}), \end{aligned} \tag{2}$$

where in the last step we used the independence of the events $\{\omega \in \mathfrak{O} : |c_{1;m} - u_m(\omega, x_m)| < m^{\log k_1}\}$ and $\{\omega \in \mathfrak{O} : |c_{1;n} - u_n(\omega, x_n)| < n^{\log k_1}\}$ for $m \neq n$.

Analogously, we have

$$B^{\tilde{y}} = \{\omega \in \mathfrak{O} : u(\omega, \tilde{y}) \in O_2\} = \bigcup_{n=1}^{\infty} B_n^{\tilde{y}},$$

$$B_n^{\tilde{y}} = \bigcap_{m\geq n} \{\omega \in \mathfrak{O} : |c_{2;m} - u_m(\omega, y_m)| < m^{\log k_2}\} = \bigcap_{m\geq n} J_m^{y_m},$$

$$B_n^k = \bigcap_{m\ge n}^{k} J_m^{y_m}, \quad k \ge n,$$

and

$$P(B^{\tilde y}) = \lim_{n\to\infty} P(B_n^{\tilde y}) = \lim_{n\to\infty} P\left(\bigcap_{m\ge n} J_m^{y_m}\right) = \lim_{n\to\infty}\lim_{k\to\infty}\prod_{m=n}^{k} P(J_m^{y_m}). \qquad (3)$$

We have

$$A^{\tilde x}\cap B^{\tilde y} = \left(\bigcup_{n\in\mathbb{N}} A_n^{\tilde x}\right)\cap\left(\bigcup_{l\in\mathbb{N}} B_l^{\tilde y}\right) = \bigcup_{n,l\in\mathbb{N}}(A_n^{\tilde x}\cap B_l^{\tilde y}).$$

Since $A_n^{\tilde x}\subset A_{n+1}^{\tilde x}$, $n\in\mathbb{N}$, and $B_l^{\tilde y}\subset B_{l+1}^{\tilde y}$, $l\in\mathbb{N}$, we have $A_n^{\tilde x}\cap B_l^{\tilde y}\subset A_{n+1}^{\tilde x}\cap B_{l+1}^{\tilde y}$, $n,l\in\mathbb{N}$, and therefore

$$P(A^{\tilde x}\cap B^{\tilde y}) = P\left(\bigcup_{n,l\in\mathbb{N}}(A_n^{\tilde x}\cap B_l^{\tilde y})\right) = P\left(\bigcup_{n\in\mathbb{N}}(A_n^{\tilde x}\cap B_n^{\tilde y})\right) = \lim_{n\to\infty} P(A_n^{\tilde x}\cap B_n^{\tilde y}).$$

Since

$$\begin{aligned} A_n^{\tilde x}\cap B_n^{\tilde y} &= \left(\bigcap_{m\ge n} I_m^{x_m}\right)\cap\left(\bigcap_{l\ge n} J_l^{y_l}\right) = \bigcap_{m,l\ge n}(I_m^{x_m}\cap J_l^{y_l}) \\ &= \bigcap_{k=n}^{\infty}\bigcap_{m,l\ge n}^{k}(I_m^{x_m}\cap J_l^{y_l}) = \bigcap_{k=n}^{\infty} C_{n;k}, \end{aligned}$$

and $C_{n;k}\supset C_{n,k+1}$, $k\ge n$, we obtain

$$\begin{aligned} P(A^{\tilde x}\cap B^{\tilde y}) &= \lim_{n\to\infty} P(A_n^{\tilde x}\cap B_n^{\tilde y}) = \lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty} C_{n;k}\right) = \lim_{n\to\infty}\lim_{k\to\infty} P(C_{n;k}) \\ &= \lim_{n\to\infty}\lim_{k\to\infty} P\left(\bigcap_{m,l\ge n}^{k}(I_m^{x_m}\cap J_l^{y_l})\right) = \lim_{n\to\infty}\lim_{k\to\infty}\prod_{m,l\ge n}^{k} P(I_m^{x_m}\cap J_l^{y_l}) \\ &= \lim_{n\to\infty}\lim_{k\to\infty}\prod_{m,l\ge n}^{k} P(I_m^{x_m})P(J_l^{y_l}), \qquad (4) \end{aligned}$$

where in the last step we used the independence of the events $\{\omega\in\mathfrak{O} : |c_{1;m} - u_m(\omega,x_m)| < m^{\log k_1}\}$ and $\{\omega\in\mathfrak{O} : |c_{2;m} - u_m(\omega,y_m)| < m^{\log k_2}\}$ .

Now, from (2), (3) and (4) we have

$$\begin{aligned} P(A^{\tilde x}\cap B^{\tilde y}) &= \lim_{n\to\infty}\lim_{k\to\infty}\prod_{m,l\ge n}^{k} P(I_m^{x_m})P(J_l^{y_l}) \\ &= \left(\lim_{n\to\infty}\lim_{k\to\infty}\prod_{m,l\ge n}^{k} P(I_m^{x_m})\right)\left(\lim_{n\to\infty}\lim_{k\to\infty}\prod_{m,l\ge n}^{k} P(J_l^{y_l})\right) \\ &= P(A^{\tilde x})P(B^{\tilde y}). \end{aligned}$$

□

The notions of generalized expectation and generalized correlation function of CSPs $u$ with values in $L^2(\mathfrak{O})$ are recalled in Appendix A.4.

**Proposition 3.1.** *Let $u$ be a CSP over $\Omega$ with values in $L^2(\mathfrak{O})$ and let $u$ have independent values. Then the generalized correlation function $B(\tilde{x},\tilde{y})$ is supported by the diagonal, i.e. $B(\tilde{x},\tilde{y}) = 0$ for all $(\tilde{x},\tilde{y}) \in (\tilde{Q}_\Omega)_c$.*

Proof. Let $(u_n)_n$ be an IV-representative and suppose that the representative $(B_n)_n$ of its generalized correlation function is determined by this same $(u_n)_n$. Without restriction of generality we may assume that all generalized expectations are zero, thus we have $E(u_n(\cdot,x_n)) = N_n$, $|N_n| = \mathcal{O}(n^{-k})$ for all $k > 0$. Let $(\tilde{x},\tilde{y}) \in (\tilde{Q}_\Omega)_c$ be arbitrary and choose its representatives such that $x_m \neq y_m$ for all pairs $(x_m, y_m) \in K$, $m \in \mathbb{N}$, $K \Subset Q_\Omega$. Hence, by independence at different points we obtain

$$B_n(x_n,y_n) = E(u_n(\cdot,x_n)u_n(\cdot,y_n)) = E(u_n(\cdot,x_n))E(u_n(\cdot,y_n)) = N_n M_n,$$

$|N_n| = \mathcal{O}(n^{-k})$, $|M_n| = \mathcal{O}(n^{-k})$. Thus, $B(\tilde{x},\tilde{y}) = 0$ in $\mathcal{R}_c$. □

**Corollary 3.1.** *Let $u$ and $B$ be as in Proposition 3.1. If $B$ is associated to $F \in \mathcal{D}'(\Omega\times\Omega)$, then $B$ is associated to a generalized function which has a representative of the form*

$$B_n^*(x,y) = \int_\Omega \sum_{j,k\in\mathbb{N}_0} R_{j,k}(s)\varphi_n^{(j)}(x-s)\varphi_n^{(k)}(y-s)ds, \quad x,y\in\Omega, \tag{5}$$

*where for every $n \in \mathbb{N}$ only a finite number of continuous functions $R_{j,k}$ are different from zero on any compact subset of $\Omega$.*

Proof. From Proposition 3.1 it follows that $B(\tilde{x},\tilde{y}) = 0$ for all $(\tilde{x},\tilde{y}) \in (\tilde{Q}_\Omega)_c$. Then the result of [12] (Theorem 2 (c) Section 5.1) imply that $B$ is associated to a generalized function which has a representative of the form (5). □

## 4. Stationary Colombeau stochastic processes

**Definition 4.1.** *A CSP $u$ over $\Omega$ with values in $L^p(\mathfrak{O})$ is called stationary if it has a representative $(u_n)_n$ such that for every $n \in \mathbb{N}$, for arbitrary $x_1,\ldots,x_m \in \Omega$ and for every $h \in \mathbb{R}^d$ such that $x_1+h,\ldots,x_m+h \in \Omega$, the random variables*

$$(u_n(\cdot,x_1),\ldots,u_n(\cdot,x_m)) \quad \textit{and} \quad (u_n(\cdot,x_1+h),\ldots,u_n(\cdot,x_m+h)) \tag{6}$$

*are identically distributed.*

**Remark 3.** Notice that it is not possible to give a characterization of stationary CSP independent of representatives.

**Theorem 2.** *The generalized expectation $m \in \mathcal{G}(\Omega)$ of a stationary CSP over $\Omega \subseteq \mathbb{R}^d$ is a generalized constant $m \in \mathcal{R}_c$.*

Proof. If $u$ is stationary, taking a representative $(u_n)_n$ which satisfies (6) and calculating $m_{u_n}(x+h) - m_{u_n}(x) = E(u_n(\cdot,x+h) - E(u_n(\cdot,x)) = 0$, we immediately obtain that $m_{u_n}(x) = c_n$, $x \in \Omega$ (if a smooth function is translation invariant, then it has to be a constant), that is, $m_u = [(m_{u_n})_n]$ is a constant in $\mathcal{R}_c$. □

**Remark 4.** Similarly as in Remark 1 and Example 4, not all representatives have to be stationary, but all of them have constant expectations.

In the sequel, we will assume that $\Omega$ is a centrally symmetric convex set. This will imply that $\Omega - \Omega = \{z \in \mathbb{R}^2 : z = x - y, x, y \in \Omega\} \cong 2\Omega$.

**Theorem 3.** *Let $\Omega \subseteq \mathbb{R}^d$ be a centrally symmetric convex open set and let $u$ be a stationary CSP over $2\Omega$ with values in $L^2(\mathfrak{O})$. If $B = [(B_n)_n] \in \mathcal{G}(\Omega \times \Omega)$ is the generalized correlation function of $u$, then there exists a positive-definite generalized function $B^* = [(B_n^*)_n] \in \mathcal{G}(2\Omega)$ such that*

$$B_n(x,y) = B_n^*(x-y), \quad x,y \in \Omega, \, n \in \mathbb{N}.$$

For the notion of positive-definite generalized functions see the Appendix A.4 in [12].

Proof. Without loss of generality we may assume that the generalized expectation of $u$ is zero. Since $u$ is stationary, it follows that

$$\begin{aligned} B_n(x,y) &= E(u_n(\cdot,x)u_n(\cdot,y)) = E(u_n(\cdot,x+h)u_n(\cdot,y+h)) \\ &= B_n(x+h,y+h), \;\; n \in \mathbb{N}, \end{aligned} \tag{7}$$

for every $x, y \in \Omega$ and every $h \in \mathbb{R}^d$ such that $x+h, y+h \in \Omega$. Thus, the representative of its generalized correlation function is translation invariant. Putting $h = -y$ in (7), we obtain $B_n(x,y) = B_n(x-y,0)$, $n \in \mathbb{N}$. Define $B_n^*(x-y) = B_n(x-y,0)$, $n \in \mathbb{N}$. Let us show that $B^* = [(B_n^*)_n]$ is a positive-definite generalized function. Let $K \Subset \Omega$. Let $a > 0$ and $\zeta_1, \dots, \zeta_m \in \mathbb{R}$ be arbitrary. Then we have

$$\inf_{x_k,x_j \in K} \sum_{k,j=1}^{m} (B_n^*(x_k - x_j) + n^{-a})\zeta_k\zeta_j = \inf_{x_k,x_j \in K} \sum_{k,j=1}^{m} (B_n(x_k - x_j, 0) + n^{-a})\zeta_k\zeta_j$$

$$= \inf_{x_k,x_j \in K} \sum_{k,j=1}^{m} (B_n(x_k, x_j) + n^{-a})\zeta_k\zeta_j \geq 0,$$

for all $n \geq n_0, n_0 \in \mathbb{N}$, since $B$ is a translation-invariant positive-definite generalized function. Therefore, $B^*$ is positive definite. □

**Corollary 4.1.** *Let $\Omega \subseteq \mathbb{R}^d$ be a centrally symmetric convex open set and let $u$ be a stationary CSP over $2\Omega$ with values in $L^2(\mathfrak{O})$. If $B = [(B_n)_n] \in \mathcal{G}(\Omega \times \Omega)$ is the generalized correlation function of $u$, then there exists a positive-definite generalized function $B^* = [(B_n^*)_n] \in \mathcal{G}(2\Omega)$ such that*

$$B(\tilde{x},\tilde{y}) = B^*(\tilde{x} - \tilde{y}), \quad \tilde{x}, \tilde{y} \in \tilde{\Omega}_c.$$

**Corollary 4.2.** *Let $\Omega \subseteq \mathbb{R}^d$ be a centrally symmetric convex open set and let $u$ be a stationary CSP over $2\Omega$ with independent values in $L^2(\mathfrak{O})$. Then the generalized function $B^*$ from Corollary 3.1 satisfies $B^*(\tilde{z}) = 0$ for every $\tilde{z} \in \tilde{\Omega}_c$, $\tilde{z} \neq \tilde{0}$.*

Proof. Since $u$ has independent values, from Proposition 3.1, it follows that $B(\tilde{x},\tilde{y}) = 0$ for all $(\tilde{x},\tilde{y}) \in (\tilde{Q}_\Omega)_c$. Since $u$ is stationary, from Corollary 4.1 it follows that $B(\tilde{x},\tilde{y}) = B^*(\tilde{x} - \tilde{y})$. Put $\tilde{z} = \tilde{x} - \tilde{y}$, for $(\tilde{x},\tilde{y}) \in (\tilde{Q}_\Omega)_c$. Clearly, $\tilde{z} \neq \tilde{0}$ and $B^*(\tilde{z}) = 0$. □

**Example 1.** *Let $m$ be a generalized constant and $B$ a positive-definite generalized function. It is known from [4] and [5] that it is possible to construct Gaussian CSPs (GCSPs) such that $m$ is the generalized expectation of $u$ and $B$ is the generalized correlation function of $u$. (For the definition and basic properties properties of GCSPs see the Appendix A.5.) Observe that process $u$ is stationary.*

*4.1. Weakly stationary CSPs.*

**Definition 4.2.** *Let $\Omega$ be a centrally symmetric convex open set in $\mathbb{R}^d$. A CSP $u$ over $2\Omega$ with values in $L^2(\mathfrak{O})$ is called weakly stationary if its expectation $m_u \in \mathcal{G}(\Omega)$ and correlation function $B_u \in \mathcal{G}(\Omega\times\Omega)$ are translation invariant, i.e. $[m_u(x+h)] = [m_u(x)]$ for all $h \in \mathbb{R}$ such that $x, x+h \in \Omega$, and*

$$[B_u(x,y)] = [B^*(x-y)], \quad x, y \in \Omega,$$

*for some positive-definite generalized function $B^* \in \mathcal{G}(2\Omega)$.*

**Remark 5.** Unlike the stationary CSP, the weakly stationary CSP is defined independently of representatives.

In [13] (Theorem 6, p.798), it has been shown that a generalized function $f \in \mathcal{G}(\mathbb{R}^d)$ invariant under all translations is a generalized constant. In the following theorem we show that this fact also holds for $f \in \mathcal{G}(\Omega)$, where $\Omega$ is an open convex subset of $\mathbb{R}^d$.

**Theorem 4.** *Assume that $\Omega$ is an open convex set in $\mathbb{R}^d$ and that for every $K \Subset \Omega$ and every $h \in \mathbb{R}$ such that $t \in K$ implies $t + h \in \Omega$, the following holds:*

$$(\forall p \in \mathbb{N})(\exists n_p \in \mathbb{N})(\forall n \in \mathbb{N})(n \geq n_p \Rightarrow \sup_{t\in K} n^p|f_n(t+h) - f_n(t)| \leq 1). \tag{8}$$

*Then $[(f_n)_n]$ is a generalized constant on $\Omega$.*

**Remark 6.** $[(f_n)_n]$ is a generalized constant on $\Omega$ means that there exists $(r_n)_n \in \mathbb{C}^{\mathbb{N}}$ such that for every $K \Subset \Omega$ and every $p > 0$ there exists $n_p > 0$ such that

$$\sup_{x\in K} n^{p-2}|f_n(x) - r_n| \leq 1, \;\; n > n_p. \tag{9}$$

Proof. Assume first that $\Omega$ is a bounded convex set and that $K \Subset \Omega$ , where $m > 0$ be chosen so that $K \Subset \Omega_{3m}$; $\Omega_{3m}$ is the set of $t \in \Omega$ such that $d(t, \partial\Omega) > 3m$. We know that $\overline{\Omega_m} \supset \overline{\Omega_{2m}} \supset \overline{\Omega_{3m}}$ are compact in $\Omega$ and that they are all convex. We will show that there exists a generalized constant $[(r_n)_n]$ such that for every $p > 0$ there exists $n_p > 0$ such that (9) holds.

Let $p \in \mathbb{N}$. Put

$$F_{l,p} = \{h \in \overline{B(0,m)} : (n \geq l) \Rightarrow (\sup_{t\in\overline{\Omega_m}} n^p|f_n(t+h) - f_n(t)| \leq 1)\}.$$

Then $F_{l,p}$ are closed sets and

$$\bigcup_{l\in\mathbb{N}} F_{l,p} = \overline{B(0,m)}.$$

By the Baire theorem, there exist $h_0 \in B(0,m)$ and $c \in (0,m)$ such that $B(h_0,c) \subset F_{l_0,p}$, that is,

$$\sup_{t\in\Omega_m, h\in B(h_0,c)} n^p|f_n(t+h) - f_n(t)| \leq 1, \;\; n \geq l_0. \tag{10}$$

We will show that

$$\sup_{t\in\Omega_{2m}, h\in B(0,c)} n^p|f_n(t+h) - f_n(t)| \leq 2, \;\; n \geq l_0. \tag{11}$$

Note, if $|\omega| < m$, then $\Omega_{2m} - \omega \in \Omega_m$. Every $h \in B(0,c)$ can be written as $h = h_1 - h_0, h_1 \in B(h_0, c)$. Thus, for $t \in \Omega_m$, we write

$$\begin{aligned} |f_n(t+h) - f_n(t)| &\leq |f_n(t+h_1-h_0) - f_n(t+h_1)| + |f_n(t+h_1) - f_n(t)| \\ &= |f_n(v) - f_n(v+h_0)| + |f_n(t+h_1) - f_n(t)|, \end{aligned}$$

where $t + h_1 - h_0 = v$. Since $v \in \Omega_m$, by (10),

$$\begin{aligned} &\sup_{t\in\Omega_{2m}, h\in B(0,c)} n^p |f_n(t+h) - f_n(t)| \\ &\leq \sup_{v\in\Omega_m, h\in B(h_0,c)} n^p|f_n(v) - f_n(v+h)| + \sup_{t\in\Omega_m, h_1\in B(h_0,c)} n^p|f_n(t+h_1) - f_n(t)| \\ &\leq 2, \quad n \geq l_0. \end{aligned}$$

This proves (11).

For the moment, consider points of $\mathbb{R}^d$ as vectors $\vec{h} \equiv h$. Any $\vec{h} \in \mathbb{R}^d$, so that $|\vec{h}| < m$, can be written as a sum of vectors $\vec{h}_i$ with the same direction as $\vec{h}$ so that $\vec{h} = \sum_{i=1}^{w} \vec{h}_i$, where $w \leq [m/c] + 1$ and $|\vec{h}_i| < c, i = 1, \ldots, w$. Returning to the "point" notation we have that for any $t \in \Omega_{3m}$ and $h \in B(0,m)$,

$$\begin{aligned} |f_n(t+h) - f_n(t)| &\leq |f_n(t+h) - f_n(t + \sum_{i\leq w-1} h_i)| \\ &+|f_n(t + \sum_{i\leq w-1} h_i) - f_n(t+\sum_{i\leq w-2} h_i)| + \cdots + |f_n(t+h_1) - f_n(t)| \\ &\leq \left(\frac{2m}{c} + 2\right) n^{-p}, \quad n \geq l_0. \end{aligned} \tag{12}$$

The last estimate follows from the fact that

$$t + \sum_{i\leq j-1} h_i = s \in \Omega_{2m}, \text{ for any } j = 2, \ldots, w.$$

So the summands become $|f_n(s+h_j) - f_n(s)|$; we have to use (11) and obtain (12). Possibly enlarging $l_0$ and denoting it by $n'_p$, we obtain

$$\sup_{t\in\Omega_{3m}} n^{p-1}|f_n(t+h) - f_n(t)| \leq 1, \ n > n'_p. \tag{13}$$

Now we will use the fact that $K$ can be covered by a finite set of balls with radius less than $m$ and all lying inside $\Omega_{3m}$. Denote them by $B_1, \ldots, B_j$ with centers at $t_1, \ldots, t_j$ and put

$$r_n = f_n(t_1), \quad n \in \mathbb{N},$$

where we assume that $t_1 \in K$. We note that every $t \in K$ can be connected to $t_1$ by a finite (at most $2j-1$) number of segments connecting points belonging to the intersections of two balls and the centers of the balls. Points of the intersections will be denoted by

$$s_1 \in B_1 \cap B_2, \ldots, s_{j-1} \in B_{j-1} \cap B_j.$$

Let $t \in B_j$. (If $t \in B_k, k < j$, the procedure is similar.) We write

$$|f_n(t) - f_n(t_1)| \leq$$

$$|f_n(t) - f(t_j)| + |f_n(t_j) - f_n(s_{j-1})| + |f_n(s_{j-1}) - f_n(t_{j-1})| + \cdots + |f_n(s_1) - f_n(t_1)|.$$

Since

$$d(t,t_j),\ d(t_j, s_{j-1}),\ d(s_{j-1}, t_{j-1}), \ldots, d(s_1, t_1) < m$$

we may apply, for each absolute value on the right hand side, the same procedure as above and in this way, obtain

$$\sup_{x\in K} n^{p-1}|f_n(x) - r_n| \leq C, \quad n > n'_p.$$

Again increasing $n'_p$ to $n_p$, we obtain (9).

If $\Omega$ is an unbounded open set and $K \Subset \Omega$, then, since the convex hull of $K$ is also compact, there exists an open bounded convex set $\Omega_0$ such that $K \Subset \Omega_0 \subset \Omega$. Repeating the above given proof for $K$ and $\Omega_0$ we obtain the complete proof of the theorem. □

From Theorem 4 we obtain the following result.

**Corollary 4.3.** *The generalized expectation $m \in \mathcal{G}(\Omega)$ of a weakly stationary CSP over a centrally symemetric convex open set $\Omega \subseteq \mathbb{R}^d$ is a generalized constant $m \in \mathcal{R}_c$.*

**Remark 7.** Clearly, stationarity implies weak stationarity of a process (Theorem 2 and Theorem 3). The converse is not true in general: weak stationarity is defined only via the first two moments. However, since GCSPs are completely determined via their expectation and correlation (Theorem 7), it follows that every weakly stationary GCSP is also stationary. Also, derivatives of a (weakly) stationary CSP are (weakly) stationary.

*4.2. CSPs with stationary increments.*

Following [1] we introduce the notion of CSP with stationary increments.

**Definition 4.3.** *A CSP $u$ over $\Omega$ with values in $L^p(\mathfrak{O})$ has stationary (resp. weakly stationary) increments if the derivative of the process $\nabla u$ is stationary (resp. weakly stationary).*

Thus, the study of processes with stationary or independent increments reduces to the study of their derivative process. This again reduces to the study of the derivatives of the generalized expectation and the generalized correlation function, i.e. to checking if $\nabla m(x) = \nabla E(u(\cdot,x)) = E(\nabla u(\cdot,x))$ corresponds to the expectation of a stationary process and if $\nabla_x \cdot \nabla_y B(x,y) = \nabla_x \cdot \nabla_y E(u(\cdot,x)u(\cdot,y)) = E(\nabla_x u(\cdot,x) \cdot \nabla_y u(\cdot,y))$ corresponds to the correlation function of a stationary process.

**Example 2.** *Let $d = 1$ and $\delta(x-y) \in \mathcal{G}(\mathbb{R})$ with the sequence of representatives $\int_{\mathbb{R}} \varphi_n(s-x)\varphi_n(s-y)ds$, $n \in \mathbb{N}$, be the generalized correlation function of white noise (see Example 5). White noise is a stationary CSP. The correlation function $B_n(x,y) = min\{x,y\} * \varphi_n(x)\varphi(y)$, corresponds to the representative of the generalized correlation function of Brownian motion (see Example 5) and $\partial_x\partial_y B_n(x,y) = \int_{\mathbb{R}} \varphi_n(s-x)\varphi_n(s-y)ds$. Thus Brownian motion has stationary increments as expected.*

## 5. Stationary solutions to some classes of SPDEs

We present a method for solving a class of SPDEs in the framework of stationary GCSPs over $\mathbb{R}^d$ with values in $L^2(\mathfrak{O})$. Since we will use the Fourier transform, we need to switch to tempered Colombeau generalized functions; see Appendix A.3.

**Definition 5.1.** *Set*

$$\mathcal{E}_{\tau,L^2}(\mathfrak{O},\mathbb{R}^d) = \Big\{(u_n)_n \in \mathcal{E}_{L^2}(\mathfrak{O},\mathbb{R}^d) \,:\, (\forall\alpha\in\mathbb{N}_0^d)(\exists N\in\mathbb{N}) \Big( \sup_{x\in\mathbb{R}^d} \|\partial^\alpha u_n(\cdot,x)\|_{L^2}(1+|x|)^{-N} = \mathcal{O}(n^N)\Big)\Big\},$$

$$\mathcal{N}_{\tau,L^2}(\mathfrak{O},\mathbb{R}^d) = \Big\{(u_n)_n \in \mathcal{E}_{L^2}(\mathfrak{O},\mathbb{R}^d) \,:\, (\forall\alpha\in\mathbb{N}_0^d)(\exists N\in\mathbb{N})(\forall b\in\mathbb{N}) \Big( \sup_{x\in\mathbb{R}^d} \|\partial^\alpha u_n(\cdot,x)\|_{L^2}(1+|x|)^{-N} = \mathcal{O}(n^{-b})\Big)\Big\}.$$

*Elements of the vector spaces $\mathcal{E}_{\tau,L^2}(\mathfrak{O},\mathbb{R}^d)$ and $\mathcal{N}_{\tau,L^2}(\mathfrak{O},\mathbb{R}^d)$ are called moderate and negligible sequences of functions with values in $L^2(\mathfrak{O})$, respectively. The elements of the quotient space*

$$\mathcal{G}_{\tau,L^2}(\mathfrak{O},\mathbb{R}^d) = \mathcal{E}_{\tau,L^2}(\mathfrak{O},\mathbb{R}^d)/\mathcal{N}_{\tau,L^2}(\mathfrak{O},\mathbb{R}^d)$$

*are called tempered CSPs over $\mathbb{R}^d$ with values in $L^2(\mathfrak{O})$.*

The whole theory of Colombeau generalized stochastic processes over $\mathbb{R}^d$ can be adapted word by word, with the change of negligible sets, to tempered Colombeau stochastic processes.

*5.1. Matching the expectation and correlation*

Let $P(D)$ be a differential operator of order $k$ with generalized constant coefficients, $P(D) = \sum_{|\alpha|\le k} \tilde{a}_\alpha D_x^\alpha$, where $\tilde{a}_\alpha \in \mathcal{R}_c$, $D_x^\alpha = (-i)^{|\alpha|}\partial_{x_1}^{\alpha_1}\dots\partial_{x_d}^{\alpha_d}$, $\alpha\in\mathbb{N}_0^d$. Its symbol is $P(\xi) = \sum_{|\alpha|\le k}\tilde{a}_\alpha \xi^\alpha$, $\xi\in\mathbb{R}^d$. Consider the equation

$$P(D)u(\omega,x) = f(\omega,x), \;\; \omega\in\mathfrak{O},\, x\in\mathbb{R}^d, \tag{14}$$

where $f = [(f_n)_n]$ is a weakly stationary tempered GCSP over $\mathbb{R}^d$ with values in $L^2(\mathfrak{O})$ with generalized expectation $\tilde{m}_f = [(m_{f_n})_n] \in \mathcal{R}_c$ (it is a constant due to Corollary 4.3) and generalized correlation function $B_f = [(B_{f_n})_n] \in \mathcal{G}(\mathbb{R}^{2d})$. Recall that a weakly stationary GCSP is also stationary. We interpret (14) as a family of equations

$$P_n(D)u_n(\omega,x) = f_n(\omega,x), \;\; \omega\in\mathfrak{O}, x\in\mathbb{R}^d, \; n\in\mathbb{N}, \tag{15}$$

in $\mathcal{E}^2_{\tau,L^2}(\mathfrak{O},\mathbb{R}^d)$; where $P_n(D) = \sum_{|\alpha|\le k}(a_\alpha)_n D_x^\alpha$, see [14].

Since Gaussian processes are completely determined by their expectation and correlation, we will match the expectations and correlations on the left hand side with the corresponding ones on the right hand side. It is the same technique as used in [5]. Also, due to Corollary 4.3 the expectation of any stationary solution $u$ will have to be a generalized constant, while its correlation will have to be of the form $B_u^*(x-y)$ due to Theorem 3.

**Theorem 5.** *Let $f = [(f_n)_n] \in \mathcal{G}_{\tau,L^2}(\mathfrak{O},\mathbb{R}^d)$ be a weakly stationary tempered GCSP with generalized expectation $\tilde{m}_f = [(m_{f_n})_n]$ and generalized correlation function $B_f = [(B_{f_n})_n]$.*

(a) *The generalized expectation* $\tilde{m}_u = [(m_{u_n})_n] \in \mathcal{R}_c$ *of a weakly stationary solution to equation* (14) *satisfies*

$$\tilde{m}_u = \begin{cases} \frac{\tilde{m}_f}{\tilde{a}_0}, & \text{if } \tilde{a}_0 \neq \tilde{0}, \\ \text{arbitrary}, & \text{if } \tilde{a}_0 = \tilde{0} \text{ and } \tilde{m}_f = \tilde{0}, \\ \text{does not exist}, & \text{if } \tilde{a}_0 = \tilde{0} \text{ and } \tilde{m}_f \neq \tilde{0}. \end{cases} \tag{16}$$

*Especially, if* $\tilde{a}_0 = \tilde{0}$ *and* $\tilde{m}_f \neq \tilde{0}$*, then equation* (14) *has no weakly stationary solutions in* $\mathcal{G}_{\tau,L^2}(\mathfrak{O}, \mathbb{R}^d)$*.*

(b) *The generalized correlation function* $[(B_{u_n})_n] \in \mathcal{G}_\tau(\mathbb{R}^{2d})$ *of a weakly stationary solution to equation* (14) *satisfies*

$$P_n(D)P_n(-D)B_{u_n}(z) = B_{f_n}(z), \;\; z = x - y \in \mathbb{R}^d, \, n \in \mathbb{N}. \tag{17}$$

*Especially, if there exists an open set* $S \subset \mathbb{R}^d$ *such that* $\hat{B}_{f_n}(\xi) > 0$*, for* $\xi \in S, n \in \mathbb{N}$ *and* $P_n(\xi)P_n(-\xi) < 0$*, for* $\xi \in S, n \in \mathbb{N}$ *for all representatives of the coefficients* $(a_\alpha)_n$*, then* $B_{u_n}$ *cannot be a positive-definite function.*

(c) *Let* $|P_n(\xi)| \geq Cn^{-r}(1 + |\xi|)^k$*,* $n \in \mathbb{N}$*,* $\xi \in \mathbb{R}^d$*, for some* $C > 0$*,* $r > 0$*,* $k > 0$*, for some representative of the coefficients* $(a_\alpha)_n$*. Then* (14) *has a weakly stationary solution* $u = [(u_n)_n] \in \mathcal{G}_{\tau,L^2}(\mathfrak{O}, \mathbb{R}^d)$ *and its correlation function satisfies*

$$P_n(\xi)P_n(-\xi)\hat{B}_{u_n}(\xi) = \hat{B}_{f_n}(\xi), \;\; \xi \in \mathbb{R}^d, \, n \in \mathbb{N}. \tag{18}$$

Proof. (a) If $u = [(u_n)_n] \in \mathcal{G}_{\tau,L^2}(\mathfrak{O}, \mathbb{R}^d)$ is a solution of (14), then $(u_n)_n \in \mathcal{E}_{\tau,L^2}(\mathfrak{O}, \mathbb{R}^d)$ is a solution of the family of equations (15). Taking expectations on both sides of equation (15) and using the fact that stationary processes have constant expectations, we obtain

$$P_n(D)m_{u_n} = m_{f_n}, \, n \in \mathbb{N}. \tag{19}$$

Since $m_{u_n}$ is a constant, $P_n(D)m_{u_n} = (a_0)_n m_{u_n}$ if $(a_0)_n \neq 0$, while $P_n(D)m_{u_n} = 0$ for $(a_0)_n = 0$. This means that (19) will have no solutions if $(a_0)_n = 0$ and $m_{f_n} \neq 0$. If $(a_0)_n = 0$ and $m_{f_n} = 0$, then $m_{u_n}$ can be taken as an arbitrary constant from $\mathcal{E}_M$. Finally, if $(a_0)_n \neq 0$, then from $P(D)m_{u_n} = (a_0)_n m_{u_n} = m_{f_n}$ we obtain $m_{u_n} = \frac{m_{f_n}}{(a_0)_n}$. Thus, (16) follows.

(b) Taking expectations on both sides in the equation

$$P_n(D_x)P_n(D_y)u_n(\omega, x)u_n(\omega, y) = f_n(\omega, x)f_n(\omega, y), \;\; \omega \in \mathfrak{O}, \, x, y \in \mathbb{R}^d, \, n \in \mathbb{N},$$

we obtain

$$P_n(D_x)P_n(D_y)B_{u_n}(x, y) = B_{f_n}(x, y), \quad x, y \in \mathbb{R}^d, n \in \mathbb{N}. \tag{20}$$

Since we seek for a stationary solution process $u$, Theorem 3 implies that $B_{u_n}(x, y) = B_{u_n}(x - y)$, $x, y \in \mathbb{R}^d$, $n \in \mathbb{N}$. Therefore, we may rewrite equation (20) in the form (17) and $[(B_{u_n})_n] \in \mathcal{G}_\tau(\mathbb{R}^{2d})$.

Applying the Fourier transform to (17), we obtain (18).

Since $[(B_{f_n})_n]$ is a generalized correlation function of $f = [(f_n)_n]$ and thus positive definite, $\hat{B}_{f_n}$ is a positive distribution for all $n \in \mathbb{N}$. From (18) it follows that $P_n(\xi)P_n(-\xi)$ must be non-negative in order that $\hat{B}_{u_n}(\xi)$ can be a positive distribution. By the Bochner theorem [1], $B_{u_n}(x)$ will be a positive-definite function.

(c) First we fix $\omega \in \mathfrak{O}$. Colombeau stochastic processes possess smooth regular paths on the representative level, which enables one to construct pathwise solutions i.e. solutions for any fixed realization $\omega \in \mathfrak{O}$.

Applying the Fourier transform to (15) we obtain $P_n(\xi)\hat{u}_n(\omega,\xi) = \hat{f}_n(\omega,\xi)$, and since the polynomial $P_n(\xi)$ has no real zeros, we obtain $\hat{u}_n(\omega,\xi) = \frac{1}{P_n(\xi)}\hat{f}_n(\omega,\xi)$, i.e.,

$$u_n(\omega,x) = S_n(x) * f_n(\omega,x), \quad \omega \in \mathfrak{O}, x \in \mathbb{R}^d, n \in \mathbb{N}, \tag{21}$$

as the solution to (15), where $S_n(x) = \mathcal{F}^{-1}\left(\frac{1}{P_n(\xi)}\right)(x)$.

Clearly we have $|P_n(\xi)| \leq Cn^s(1+|\xi|)^k$ for some $C > 0$ and $s > 0$. Then from the assumption $|P_n(\xi)| \geq Cn^{-r}(1+|\xi|)^k$ we have that $\frac{1}{P_n(\xi)}$ is in $\mathcal{O}_M(\mathbb{R}^d)$ and $S_n = \mathcal{F}^{-1}\left(\frac{1}{P_n(\xi)}\right)$ is in $\mathcal{O}'_c(\mathbb{R}^d)$. The sequence $(u_n)_n$ is moderate, since $S_n \in \mathcal{O}'_c(\mathbb{R}^d)$; see [15].

We continue to investigate the stationarity of the solution. Since $P_n(\xi) \neq 0$ implies $\tilde{a}_0 \neq \tilde{0}$, the expectation of $u_n$ is given by $m_{u_n} = \frac{m_{f_n}}{(a_0)_n}$ as stated in (a). Indeed, from (21) one can also derive by Fubini's theorem that

$$\begin{aligned} m_{u_n} &= S_n * E(f_n) = m_{f_n} S_n * 1 = m_{f_n} \int_{\mathbb{R}^d} S_n(x)dx \\ &= m_{f_n}\hat{S_n}(0) = m_{f_n}\frac{1}{P_n(0)} = \frac{m_{f_n}}{(a_0)_n}. \end{aligned}$$

From (21) we obtain

$$u_n(\omega,x)u_n(\omega,y) = S_n(x) * f_n(\omega,x) \cdot S_n(y) * f_n(\omega,y)$$

and taking expectations and applying Fubini's theorem we get

$$E(u_n(\cdot,x)u_n(\cdot,y)) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} S_n(\xi)S_n(\eta)E(f_n(\cdot,x-\xi)f_n(\cdot,y-\eta))d\xi\, d\eta$$

i.e.,

$$B_{u_n}(x,y) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} S_n(\xi)S_n(\eta)B_{f_n}(x-\xi,y-\eta)d\xi\, d\eta.$$

Since $f$ is stationary, $B_{f_n}(x-\xi,y-\eta) = B_{f_n}(x-\xi-(y-\eta))$ and we may apply the change of variables $\sigma = \xi-\eta$, $\tau = \eta$, to obtain

$$\begin{aligned} B_{u_n}(x-y) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} S_n(\xi)S_n(\eta)B_{f_n}(x-\xi-(y-\eta))d\xi\, d\eta \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} S_n(\sigma+\tau)S_n(\tau)B_{f_n}(x-y-\sigma)d\sigma\, d\tau \\ &= \int_{\mathbb{R}^d} (\check{S_n} * S_n)(\sigma) * B_{f_n}(x-y-\sigma)d\sigma \\ &= (\check{S_n} * S_n * B_{f_n})(x-y), \end{aligned}$$

where $\check{S_n}(\tau) = S_n(-\tau)$.

Taking $z = x-y$ we obtain $B_{u_n}(z) = (\check{S_n} * S_n * B_{f_n})(z)$. This is in compliance with (17) in (b). Taking the Fourier transform we obtain (18). According to the assumption in (c), $P_n(\xi)P_n(-\xi)$ is positive and bounded away from zero. It follows that

$$\hat{B}_{u_n}(\xi) = \frac{\hat{B}_{f_n}(\xi)}{P_n(\xi)P_n(-\xi)}, \; n \in \mathbb{N},$$

is a positive distribution and hence $[(B_{u_n})_n]$ is a positive-definite generalized function.

Thus, $u_n$ given by (21) is weakly stationary. $\square$

**Remark 8.** Let $P_n(\xi)P_n(-\xi) \geq 0$, $\xi \in \mathbb{R}^d, n \in \mathbb{N}$ for some representative of the coefficients $(a_\alpha)_n$ and let $N = \{\xi \in \mathbb{R}^d : P_n(\xi) = 0\}$, $V = \{\xi \in \mathbb{R}^d : P_n(\xi)P_n(-\xi) = 0\}, n \in \mathbb{N}$. (The sets $N$ and $V$ are assumed to be the same for all $n \in \mathbb{N}$.) Assume that $P\{\omega \in \mathfrak{O} : \hat{f}_n(\omega, \xi) = 0, \xi \in N\} = 1$, and $\hat{B}_{f_n}(\xi) = 0$, $\xi \in V, n \in \mathbb{N}$. Then, for the existence of a solution one needs to consider the problem of division with $P(\xi)P(-\xi)$ which is highly non-trivial. It is an old and classical result of Lojasiewicz, cf. Hörmander [16]. In the case of Colombeau generalized functions, this has been solved in [14] (see also [17] for the general question of extending distributions out of a set $\Omega$). We leave this question for another paper.

Let us continue with the same notation. If there exists a stationary CSP $u = [(u_n)_n] \in \mathcal{G}_{\tau,L^2}(\mathfrak{O}, \mathbb{R}^d)$ as a solution to (14), then from Theorem 5 it follows that its generalized expectation and generalized correlation function are given by

$$\tilde{m}_u = \frac{\tilde{m}_f}{\tilde{a}_0} \tag{22}$$

and

$$P_n(\xi)P_n(-\xi)\hat{B}_{u_n}(\xi) = \hat{B}_{f_n}(\xi), \xi \notin V, n \in \mathbb{N}. \tag{23}$$

We will illustrate in Example 3 that in some special cases one may still find a weakly stationary Gaussian solution. Note that the solution does not have to be unique.

**Example 3.** *We illustrate the case when the sets of points $\xi$ for which $P(\xi)P(-\xi) = 0$ and for which $\hat{B}_{f_n}(\xi) = 0$ coincide, $n \in \mathbb{N}$. Let $d = 1$ and consider the equation*

$$\left(1 + \frac{d^2}{dx^2}\right) u = f,$$

*where $f$ is a stationary Gaussian CSP with zero expectation and generalized correlation function*

$$B_{f_n}(x - y) = \varphi_n(x - y) + 2\varphi_n^{(2)}(x - y) + \varphi_n^{(4)}(x - y), \quad x, y \in \mathbb{R}, n \in \mathbb{N}.$$

*Then $P(\xi) = 1 - \xi^2$ and $P(\xi)P(-\xi) = (1 - \xi^2)^2$. Clearly, $V = \{-1, 1\}$. Applying the Fourier transform to the representative of the generalized correlation function of $f$ we obtain $\hat{B}_{f_n}(\xi) = (1 - 2\xi^2 + \xi^4)\hat{\varphi}_n(\xi) = (1 - \xi^2)^2\hat{\varphi}_n(\xi)$, $\xi \in \mathbb{R}$, $n \in \mathbb{N}$. Thus, we have*

$$(1 - \xi^2)^2\hat{B}_{u_n}(\xi) = (1 - \xi^2)^2\hat{\varphi}_n(\xi), \quad \xi \in \mathbb{R} \setminus \{-1, 1\},\ n \in \mathbb{N}. \tag{24}$$

*Since $\hat{\varphi}_n(\xi)$ is continuous on $\mathbb{R}$, we may extend representatives of the generalized correlation function of the solution to be $\hat{B}_{u_n}(\xi) = \hat{\varphi}_n(\xi)$, $\xi \in \mathbb{R}$, $n \in \mathbb{N}$, i.e. $B_{u_n}(x - y) = \varphi_n(x - y)$, $x, y \in \mathbb{R}$, $n \in \mathbb{N}$. Thus, $u$ is a GCSP with zero generalized expectation and generalized correlation function associated to the Dirac delta. This means that the white noise GCSP is a solution to the given equation. This solution is not unique. Note that*

$$\hat{B}_{v_n}(\xi) = \hat{B}_{u_n}(\xi) + \delta(\xi - 1) + \delta(\xi + 1),$$

*also satisfies equation (24). Therefore, a GCSP $v = [(v_n)_n]$ with zero generalized expectation and generalized correlation function given by*

$$B_{v_n}(x - y) = B_{u_n}(x - y) + 2\cos(x - y),$$

*is a solution to the given equation as well.*

*5.2. The stationary Klein–Gordon equation driven by higher order derivatives of white noise.*

We will illustrate the method described above on the equation

$$(\tilde{1} - \Delta_x)u(\omega, x) = \tilde{c} + \tilde{f} \cdot \partial_x^k w(\omega, x), \;\; \omega \in \mathfrak{O},\, x \in \mathbb{R}^d, \tag{25}$$

where $\tilde{1} = (1, 1, 1, \ldots)$, $\tilde{c}, \tilde{f} \in \mathcal{R}_c$ are generalized constants and $w = [(w_n)_n]$ is the white noise GCSP with zero expectation and generalized correlation function $B_{w_n}(x, y) = \varphi_n(x-y)$, $x, y \in \mathbb{R}^d$, $n \in \mathbb{N}$. In [5] it was shown that all derivatives of a GCSP are also GCSPs. Therefore, $g(\omega, x) = \tilde{c} + \tilde{f} \cdot \partial_x^k w(\omega, x)$ is a stationary GCSP with expectation $[(m_{g_n})_n] = [(c_n)_n] = \tilde{c} \in \mathcal{R}_c$ and generalized correlation function $[(B_{g_n}(x, y))_n] = [(c_n^2 + f_n^2 \partial_x^k \partial_y^k \varphi_n(x-y))_n]$, $x, y \in \mathbb{R}^d$, $n \in \mathbb{N}$.

Equation (19) reduces to

$$(1 - \Delta_x)m_{u_n} = c_n, \;\; x \in \mathbb{R}^d,\, n \in \mathbb{N},$$

and this implies $m_{u_n} = c_n$. Therefore, $[(m_{u_n})_n] = [(c_n)_n] = \tilde{c} \in \mathcal{R}_c$.

Equation (20) reduces to

$$(1 - \Delta_x)(1 - \Delta_y)B_{u_n}(x - y) = c_n^2 + f_n^2\, \partial_x^k \partial_y^k \varphi_n(x - y), \;\; x, y \in \mathbb{R}^d,\, n \in \mathbb{N}.$$

i.e. after the change of variables $x - y = z$

$$(1 - \Delta)^2 B_{u_n}(z) = c_n^2 + f_n^2 (-1)^k \partial_z^{2k} \varphi_n(z), \;\; z \in \mathbb{R}^d, n \in \mathbb{N}. \tag{26}$$

Applying the Fourier transform to (26) we obtain

$$(1 + \|\xi\|^2)^2 \hat{B}_{u_n}(\xi) = c_n^2 (2\pi)^{d/2} \delta(\xi) + f_n^2 \xi^{2k} \hat{\varphi}_n(\xi), \;\; \xi \in \mathbb{R}^d,\, n \in \mathbb{N}.$$

Clearly, the condition of Theorem 5 (c) holds and the right hand side is also positive. Now we have

$$\begin{aligned} B_{u_n}(z) = {} & c_n^2 (2\pi)^{-d/2} * \mathcal{F}^{-1}\left(\frac{1}{(1 + \|\xi\|^2)^2}\right)(z) \\ & + f_n^2 (2\pi)^{-d/2} \varphi_n * \mathcal{F}^{-1}\left(\frac{\xi^{2k}}{(1 + \|\xi\|^2)^2}\right)(z), \;\; z \in \mathbb{R}^d, \end{aligned}$$

which can be expressed as

$$B_{u_n}(z) = c_n^2 (2\pi)^{-d} * b^{*2}(z) + f_n^2 (2\pi)^{-d} (-1)^k \left(\partial_z^k b\right)^{*2} * \varphi_n(z), \;\; z \in \mathbb{R}^d,$$

where

$$b(z) = \mathcal{F}^{-1}\left(\frac{1}{1 + \|\xi\|^2}\right)(z) = 2\pi^{-d/2} \|z\|^{1-d/2} K_{d/2-1}(\|z\|), \;\; z \in \mathbb{R}^d$$

is the fundamental solution of $1 - \Delta_x$ vanishing at infinity [18, p. 128], expressed in terms of the modified Bessel function $K_{d/2-1}$.

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## Appendix A. Notes on Colombeau stochastic processes

*Appendix A.1. Colombeau algebra*

Let

$$\mathcal{E}_M(\Omega) = \{(u_n)_n \in \mathcal{E}(\Omega) : (\forall K \Subset \Omega)(\forall \alpha \in \mathbb{N}_0^d)(\exists a \in \mathbb{N})(\sup_{x\in K} |\partial^\alpha u_n(x)| = \mathcal{O}(n^a))\},$$

$$\mathcal{N}(\Omega) = \{(u_n)_n \in \mathcal{E}(\Omega) : (\forall K \Subset \Omega)(\forall \alpha \in \mathbb{N}_0^d)(\forall b \in \mathbb{N})(\sup_{x\in K} |\partial^\alpha u_n(x)| = \mathcal{O}(n^{-b}))\}.$$

The space of moderate functions $\mathcal{E}_M(\Omega)$ is the largest differential subalgebra of $\mathcal{E}(\Omega)$ in which the set of negligible functions $\mathcal{N}(\Omega)$ is a differential ideal. The Colombeau algebra is defined as $\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$. If $(u_n)_n \in \mathcal{E}_M(\Omega)$ is a representative of $u \in \mathcal{G}(\Omega)$, we write $u = [(u_n)_n]$.

*Appendix A.2. Generalized constants and sharp topology*

Let $\Omega_M = \{(x_n)_n \in \Omega^{\mathbb{N}} : (\exists a \in \mathbb{N})(|x_n| = \mathcal{O}(n^a))\}$, and define the equivalence relation $\sim$ by $(x_n)_n \sim (y_n)_n \Leftrightarrow (\forall m \in \mathbb{N})(|x_n - y_n| = \mathcal{O}(n^{-m}))$. The quotient space $\tilde{\Omega} = \Omega_M/\sim$ is called the set of generalized points. The set $\tilde{\Omega}_c = \{\tilde{x} = [(x_n)_n] \in \tilde{\Omega} : (\exists K \Subset \Omega)(x_n \in K)\}$ is called the set of compactly supported generalized points; see [19]. For $\Omega = \mathbb{R}$ we write $\tilde{\mathbb{R}}_c = \mathcal{R}_c$. If $u = [(u_n)_n] \in \mathcal{G}(\Omega)$ and $\tilde{x} \in \tilde{\Omega}_c$, then $u(\tilde{x}) = [(u_n(x_n))_n] \in \mathcal{R}_c$ and it is called the point value of $u$ at the compactly supported generalized point $\tilde{x} \in \tilde{\Omega}_c$; see [19].

**Theorem 6.** *[19] Let $u = [(u_n)_n] \in \mathcal{G}(\Omega)$. Then $u = 0$ in $\mathcal{G}(\Omega)$ if and only if $u(\tilde{x}) = 0$ in $\mathcal{R}_c$ for all $\tilde{x} \in \tilde{\Omega}_c$.*

The mapping $\tilde{d}_c : \mathcal{R}_c \times \mathcal{R}_c \to \mathbb{R}$,

$$\tilde{d}_c([(f_n)_n], [(g_n)_n]) \mapsto \|(f_n - g_n)_n\| = \limsup_{n\to\infty} |f_n - g_n|^{(\log n)^{-1}}$$

is an ultrametric on $\mathcal{R}_c$. The topology defined by $\tilde{d}_c$ is called the sharp topology on $\mathcal{R}_c$. Sharp open balls in $\mathcal{R}_c$ are of the form

$$L((f_n)_n, k)) = \{[(g_n)_n] \in \mathcal{R}_c : \limsup_{n\to\infty} |f_n - g_n|^{(\log n)^{-1}} < k\}.$$

*Appendix A.3. Tempered Colombeau generalized functions*

Let $\mathcal{E}_\tau(\mathbb{R}^d)$ be the algebra of all sequences $(f_n)_n \in \mathcal{E}^{\mathbb{N}}$ with the property that for every $\alpha \in \mathbb{N}_0^d$ there exists $N \in \mathbb{N}$ such that $\sup_{x\in\mathbb{R}^d} |f_n^{(\alpha)}(x)|(1+|x|)^{-N} = O(n^N)$. The set of negligible elements $\mathcal{N}_\tau(\mathbb{R}^d)$ is the algebra of all sequences $(f_n)_n \in \mathcal{E}^{\mathbb{N}}$ with the property that for every $\alpha \in \mathbb{N}_0^d$ there exists N such that for every $a > 0$ $\sup_{x\in\mathbb{R}^d} |f_n^{(\alpha)}(x)|(1+|x|)^{-N} = O(n^{-a})$. It is well-known that $\mathcal{E}_\tau(\mathbb{R}^d) \cap \mathcal{N}(\mathbb{R}^d)$ is a proper subset of $\mathcal{N}_\tau(\mathbb{R}^d)$ which implies that the canonical mapping $\mathcal{E}_\tau(\mathbb{R}^d)/\mathcal{N}_\tau(\mathbb{R}^d) = \mathcal{G}_\tau(\mathbb{R}^d) \to \mathcal{G}(\mathbb{R}^d)$ is not injective.

*Appendix A.4. Generalized expectation and correlation function*

Let $u = [(u_n)_n] \in \mathcal{G}_{L^2}(\mathfrak{O}, \Omega)$ be a CSP. The generalized expectation of $u$ is an element $m$ of $\mathcal{G}(\Omega)$ with representative is $m_{u_n}(x) = E(u_n(\cdot, x))$, $x \in \Omega$, $n \in \mathbb{N}$. The generalized correlation function of $u$ is an element $B$ of $\mathcal{G}(\Omega \times \Omega)$ with the representative $B_{u_n}(x, y) = E(u_n(\cdot, x)u_n(\cdot, y))$, $x, y \in \Omega$, $n \in \mathbb{N}$. We proved in [5] that $\partial^\alpha m_{u_n}(x) = m_{\partial^\alpha u_n}(x)$, $x \in \Omega$, $n \in \mathbb{N}$ and $\partial_x^k \partial_y^k B_{u_n}(x, y) = B_{\partial^k u_n}(x, y)$, $x, y \in \Omega$, $n \in \mathbb{N}$. The generalized correlation function $B = [(B_{u_n}(x, y))_n]$ is positive definite in the Colombeau sense, i.e. it has a representative consisting of bilinear positive-definite functions; see [20].

*Appendix A.5. Gaussian Colombeau stochastic processes*

Following [5] and [1], an element $u \in \mathcal{G}_{L^2}(\mathfrak{O}, \Omega.$ is said to be a Gaussian Colombeau stochastic process (GCSP), if there exists a representative $(u_n)_n$ and $n_0 \in \mathbb{N}$ such that for every $n > n_0$ and arbitrary $x_1, \dots, x_r \in \Omega \subset \mathbb{R}^d$, the probability that $X_n = (u_n(x_1, \omega), \dots, u_n(x_r, \omega)) \in B$, where $B$ is a Borel set in $\mathbb{R}^r$, is

$$P(X_n \in B) = \left(\frac{\det A_n}{(2\pi)^d}\right)^{1/2} \int_B \exp\left(-\frac{1}{2}\, s^T A_n s\right) ds, \;\; n > n_0,$$

where $A_n, n \in \mathbb{N}$, stands for a sequence of non-degenerate positive-definite matrices and $s^T A_n s = \sum_{i=1}^r \sum_{j=1}^r a_{ijn} s_i s_j$, $n > n_0$. We will call $(u_n)_n$ a Gaussian representative of $u$.

**Example 4.** *Let $(c_n)_n \in \mathcal{N}$ be negligible sequence and $s(\omega, x)$ any non-Gaussian stochastic process. Then $c_n s(\omega, x)$ is a non-Gaussian negligible sequence. If $(u_n)_n$ is a Gaussian representative of a GCSP, then $(u_n + c_n s)_n$ is a non-Gaussian representative of the same GCSP. Therefore, not all representatives of a GCSP are Gaussian.*

Let $u$ be a GCSP with Gaussian representative $(u_n)_n$ and let $(B_{u_n})_n$ be a representative of its generalized correlation function. Then, by [5], $A_n = (B_{u_n}(x_i, x_j))^{-1}$, $n \in \mathbb{N}$, for all $x_1, \dots, x_d \in \mathbb{R}$. Also we know by [5] that partial derivatives of a GCSP are again GCSPs.

**Theorem 7.** *[5] Let $u = [(u_n)_n] \in \mathcal{G}_{L^2}(\mathfrak{O}, \Omega)$ be a Colombeau stochastic process with generalized expectation $m = [(m_{u_n}(x))_n] \in \mathcal{G}(\Omega)$ and generalized correlation function $B = [(B_{u_n}(x, y))_n] \in \mathcal{G}(\Omega \times \Omega)$. There exists a GCSP with the given generalized expectation and generalized correlation function.*

**Example 5.** *White noise $w$ is a GCSP with zero expectation and with a representative of its correlation function $B_w(x, y) = [(\varphi_n(x - y))_n]$ that is associated to the Dirac delta $\delta(x - y)$ supported on the diagonal. Another choice to represent white noise is by the correlation function $B_w(x, y) = [(\int_{\mathbb{R}} \varphi_n(s - x)\varphi_n(s - y) ds)_n]$.*

*Brownian motion is the GCSP with zero expectation and with a representative of its correlation function $B_n(x, y) = \min\{s, t\} * \varphi_n(x)\varphi_n(y) = \int_{\mathbb{R}} \int_{\mathbb{R}} \min\{s, t\}\varphi_n(s - x)\varphi_n(t - y)\, ds\, dt$, $x, y \in \mathbb{R}$, $n \in \mathbb{N}$.*

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